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Insights in Characterizing Asymptotic Behavior for Quasipolynomials with Two Delays

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Abstract— **This paper proposes an analytical method to characterize the behavior of multiple critical roots of a retarded system with two delays. Expressing locally the related characteristic function, as a Weierstrass polynomial, we derive several results to analyze the stability behavior of such characteristic roots with respect to small variations on the delay parameters. The proposed results are illustrated by considering several numerical examples.**

Index Terms— **Delay Systems, Weierstrass Polynomial, Newton-Diagram, Puiseux Series.**

I. Introduction

It is well recognized that one of the main issues in dynamical systems are those that concern to the stability analysis (see, for instance, [1]). In particular, for the case of Linear Time-Invariant (LTI) delay systems, because of the infinite dimensional state space, such an analysis is not a simple task, even in the case of fixed-delays [2]. Furthermore, it is well known that time-delay systems (and in consequence, quasi-polynomials) have always infinitely many solutions (see, for further details, [3] and the references therein), however in general, we will only be interested in analyzing the behavior of a critical zero of finite multiplicity.

In [4], for a general *retarded* LTI delay system with multiple commensurate delays, the stability behavior of the solutions has been explored in detail. More precisely, they have first, fully characterized the stability properties of such systems by proposing conditions to find the set of critical delay values, at which the system's characteristic quasi-polynomial has critical zeros on the imaginary axis. Secondly, considering the delay as a variable parameter and by adopting an operator based-approach (see, for instance, [5]) they have expanded the solutions of the quasi-polynomial in terms of a Taylor (or Puiseux) series, allowing them to analyze the solutions behavior as the delay varies around a critical delay value.

Multiple delays can also be presented in a more general form, as non-commensurate delays, that is, all delays are assuming to be independent of each other. In this case, the stability of the related quasi-polynomial becomes more complex and is less studied. For the case of solutions of multiplicity two, in [6] the authors have proposed two sectors to analyze the qualitative properties of these roots when delays are subject to small deviations and restricted to such sectors. In this vein, it is worth to mention that unlike the case of a single parameter, in the multiparameter case there exist some *singular* and *unexpected* behaviors (see, the motivating examples section) which must be taken into account (see, for instance [7]), in order that the problem is well-posed. In other words, the Puiseux type arguments cannot be extended straightforwardly from one parameter to multiparameter case.

Based on the above arguments, the main goal of this paper is threefold. First, for multiple imaginary roots, give conditions to guarantee the existence of a convergent Puiseux (or Taylor) series. Second, extend the use of the well known Newton diagram to the case of two parameters and give a methodology to express the solutions as a Puiseux series. Finally, to characterize the solutions according to their geometric behavior.

The remaining part of the paper is organized as follows: Section II introduces some preliminary results, motivating examples and the problem formulation. Section III is devoted to the main results; specifically we present a method to compute the Weierstrass Polynomial, an algorithm to extend the Newton diagram procedure, necessary conditions to derive the generalized Puiseux series and the splitting properties around multiple roots are described. Finally, Section IV includes some numerical examples illustrating the proposed results. The contribution ends with some concluding remarks.

Notations: In the sequel, the following notations will be adopted: $\mathbb{C}$ is the set of complex numbers, $\boldsymbol{i} := \sqrt{-1}$. Next, $\mathbb{R}_+$ denotes the set of positive real values. The *order* of a power series $f(\boldsymbol{x}) = \sum_{i,j} a_{i,j} x_1^i x_2^j$ will be denoted by $\operatorname{ord}(f)$ and defined as the smallest number $n = i + j$ such that $a_{i,j} \neq 0$. The ring of complex formal power series is denote by $\mathbb{C}[[\boldsymbol{x}]]$, with subring $\mathbb{C}\{\boldsymbol{x}\}$ of convergent power series. Finally, given two polynomials $f(z) = \sum_{j=0}^{n} a_{n-j} z^j$ and $g(z) = \sum_{j=0}^{m} b_{m-j} z^j$, the *resultant* of f, g is defined as

$$\mathcal{R}(f,g) := \det \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & \\ & a_0 & a_1 & \cdots & \cdots & a_n & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & b_m & & & \\ & b_0 & b_1 & \cdots & \cdots & b_m & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \cdots & \cdots & b_m \end{bmatrix}.$$

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II. Preliminaries

A. Retarded Linear Time-Invariant Systems

Consider a retarded LTI system with h_a-delays τ_k, described in the state-space form as:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{h_a} A_k x(t-\tau_k), \qquad \tau_k \geq 0, \tag{1}$$

or by the differential-difference equation

$$y^{(n)}(t) + \sum_{\ell=0}^{n-1}\sum_{k=0}^{h_a} a_{k\ell} y^{(\ell)}(t-\tau_k) = 0, \qquad \tau_k \geq 0, \tag{2}$$

with characteristic function given by the quasi-polynomial:

$$f(s,\tau) = \sum_{k=0}^{h_a} p_k(s) e^{-\tau_k s}, \qquad \tau_k \geq 0, \tag{3}$$

where the polynomials p_k are given by

$$p_0(s) = s^n + \sum_{\ell=0}^{n-1} a_{0\ell} s^\ell, \; p_k(s) = \sum_{\ell=0}^{n-1} a_{k\ell} s^\ell, \quad k = 1, \ldots, h_a.$$

B. Local Representation of Analytic Functions

It it possible to reduce the analytic properties of $f(x,y)$ to algebraic ones. To this purpose, let us consider the following result (for further details see [8]).

Theorem 1 (Weierstrass Preparation Theorem): Let $f(z,\boldsymbol{x})$ be an analytic function vanishing at the singular point $z_0 \in \mathbb{C}$, $\boldsymbol{x}_0 \in \mathbb{C}^n$, where $z = z_0$ is an $m-$multiple root of the equation $f(z,\boldsymbol{x}) = 0$, i.e.,

$$f(z_0,\boldsymbol{x}_0) = \frac{\partial f}{\partial z} = \cdots = \frac{\partial^{m-1} f}{\partial z^{m-1}} = 0, \quad \frac{\partial^m f}{\partial z^m} \neq 0.$$

where derivatives are evaluated at $(z_0,\boldsymbol{x}_0)$. Then, there exist a neighborhood $U_0 \subset \mathbb{C}^{n+1}$ of the point $(z_0,\boldsymbol{x}_0) \in \mathbb{C}^{n+1}$ in which the function $f(z,\boldsymbol{x})$ can be expressed as

$$f(z,\boldsymbol{x}) = W(z,\boldsymbol{x})\, b(z,\boldsymbol{x}), \tag{4}$$

where $W(z,\boldsymbol{x})$ is given by

$$(z-z_0)^m + w_{m-1}(\boldsymbol{x})(z-z_0)^{m-1} + \cdots + w_0(\boldsymbol{x}),$$

and $w_0(\boldsymbol{x})$,…,$w_{m-1}(\boldsymbol{x})$, $b(z,\boldsymbol{x})$ are analytic functions uniquely defined by the function $f(z,\boldsymbol{x})$, and $w_i(\boldsymbol{x}_0) = 0$, $b(z_0,\boldsymbol{x}_0) \neq 0$.

Remark 1: The holomorphic function

$$W(z,\boldsymbol{x}) = z^m + w_{m-1}(\boldsymbol{x}) z^{m-1} + \cdots + w_0(\boldsymbol{x}), \tag{5}$$

is known as the *Weierstrass polynomial* (for further details on Weierstrass polynomials, see, for instance, [9]).

Remark 2: It can be seen from Theorem 1, that since $b(z,\boldsymbol{x})$ is an holomorphic non vanishing function at $(0,\boldsymbol{0})$, then, there must exist some neighborhood $U \subset \mathbb{C}^n$ at which $b(z,\boldsymbol{x})$ preserves the same property. Hence, based on this observation we can ensure that the roots behavior of a given quasi-polynomial f in the neighborhood U will be completely described by the roots behavior of $W(\boldsymbol{x},x)$.

C. Newton Diagram Method

It is well known that solutions of the equation $f(x,y) = 0$ can be compute term by term by means of the *Newton Diagram Method*. Thus, in order to use such a procedure, let us introduce the following notation (for more details, see, for instance, [10]). Let $f(x,y)$ be a *pseudo-polynomial* in y, i.e.,

$$f(x,y) = \sum_{k=0}^{n} a_k(x) y^k, \tag{6}$$

where the corresponding coefficients are given by,

$$a_k(x) = x^{\rho_k} \sum_{r=0}^{\infty} a_{rk} x^{r/q}, \tag{7}$$

$a_{rk} \in \mathbb{C}$, x and y are complex variables, ρ_k are non-negative rational numbers, q is an arbitrary natural number, $a_n(x) \not\equiv 0$, and $a_0(x) \not\equiv 0$.

Since by simple translation, any point on a curve can be moved to the origin, we will consider expansions of the solution of (6) $f(x,y) = 0$ around the origin, in the following form

$$y(x) = y_{\epsilon_1} x^{\epsilon_1} + y_{\epsilon_2} x^{\epsilon_2} + y_{\epsilon_3} x^{\epsilon_3} + \cdots, \tag{8}$$

where $\epsilon_1 < \epsilon_2 < \epsilon_3 < \cdots$ and $y_{\epsilon_1} \neq 0$. To determine the possible values of ϵ_1, y_{ϵ_1}, ϵ_2, y_{ϵ_2}, …, it is necessary to consider the *Newton's diagram*.

Definition 2.1 (Newton's Diagram and Polygon): Given a pseudo-polynomial of the form (6) with coefficients given by (7), plot k versus ρ_k for $k = 0, 1, \ldots, n$ (if $a_k(\cdot) \equiv 0$, the corresponding point is disregarded). Denote each of these points by $\pi_k = (k, \rho_k)$ and let

$$\Pi = \{\pi_k \,:\, a_k(\cdot) \neq 0\},$$

be the set of all plotted points. Then, the set Π will be called the *Newton diagram*, and the *Newton polygon* associated with $f(x,y)$ will be given by the lower boundary of the convex hull of the set Π.

For a given $f(x,y)$, Fig.1 simply illustrates Definition 2.1.

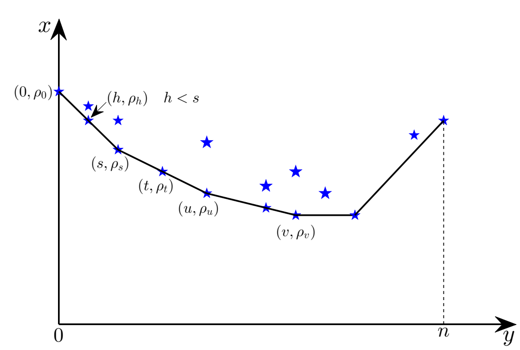


Fig. 1. The Newton Diagram for $f(x,y)$.

Thus, the leading terms of the solutions will have exponents given by the slopes $\gamma := \epsilon$. Its coefficients will be given by the nonzero solutions of the polynomial equation

$$\mathcal{P}(y_\epsilon) := \sum_i a_{0,i} y_\epsilon^i = 0, \tag{9}$$

where the sum runs over the terms satisfying $\rho_k + \gamma k = \nu$ with constant $\nu \in \mathbb{Q}$. For equations $f(x,y) = 0$, the *Newton Diagram Method* can be formalized by the following theorem (see [11]).

Theorem 2 (Puiseux Theorem): The equation $f(x,y) = 0$, with f given in formal power series such that $f(0,0) = 0$, posses at least one solution in power series of the form:

$$x = t^q, \quad y = \sum_{i=1}^{\infty} c_i t^i, \quad q \in \mathbb{N}.$$

D. Generalized Puiseux Series and Cones

Theorem 2 allows to find fractional power series solutions, known as Puiseux series, in the case of algebraic equations. In this vein, when we deal with singularities of greater dimension, we must use a ring of multivariable fractional power series. In [12] it is defined the fractional power series ring that contains the solutions of algebraic hypersurfaces. This can be achieved through formal power series defined in a geometric way, by taking infinite power series

$$\sum_{i=1}^{\infty} c_{\boldsymbol{a_i}} \boldsymbol{x}^{\boldsymbol{a}_i/d}, \quad \text{where } \boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n},$$

where the exponents $\boldsymbol{a}$ are taken from a fixed convex cones with structure related to its Newton polytopes [13].

Definition 2.2: A convex polyhedral cone is a set of the form

$$\sigma = \left\{ \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \in \mathbb{R}_+ \right\},$$

where $M = \{v_1, \ldots, v_m\}$ is a finite set of vectors in $\mathbb{R}^n$. We will use fractional iterated power series of several variables as *Generalized Puiseux Power Series* (see [14], [15]), denoted by $K_{\boldsymbol{x},d}$. This series can be constructed by induction, taking as a base the univariable case $K_{\boldsymbol{x}_1,d}$ and then, proceed with the field of power series in $x_1^{1/d}$ with power series coefficients in $x_2^{1/d} \cdots x_n^{1/d}$ such that

$$K_{\boldsymbol{x},d} = \mathbb{C}\left(\left(x_1^{1/d}\right)\right) \cdots \left(\left(x_n^{1/d}\right)\right).$$

E. Motivating Examples

Although we can reduce the analysis of a given entire function f to the study of an algebraic function P_f, in this section we aim to point out some difficulties that arise in regarding multiparameter functions. In order to illustrate such arguments, let us consider the following examples.

Example 2.1: Consider the following polynomial

$$P(z, \epsilon_1, \epsilon_2) = z^2 + 3\epsilon_1 z + 2\left(\epsilon_1^2 + 2\epsilon_2^2\right), \tag{10}$$

where ϵ_1 and ϵ_2 are considered as perturbation parameters. It is clear to see, that for $\epsilon_1 = \epsilon_2 = 0$, $z = 0$ is a root of multiplicity two. In this case, the solutions $z_{1,2}(\boldsymbol{\epsilon})$ are not analytic at $\boldsymbol{\epsilon} := (\epsilon_1, \epsilon_2) = (0,0) = \boldsymbol{0}$. Furthermore, $z_{1,2}(\boldsymbol{\epsilon})$ does not have a *unique* representation as a power series which is convergent in some punctured neighborhood of the origin. In order to illustrate this assertion, let us consider the region $|\epsilon_1| < |\epsilon_2|$, in this region the solutions admit the following representation

$$\begin{aligned} z_{1,2}(\boldsymbol{\epsilon}) \quad = \quad & -\frac{1}{2}(3\epsilon_1 \pm \boldsymbol{i}4\epsilon_2) + \frac{1}{16}\epsilon_1\left(\pm\boldsymbol{i}\frac{\epsilon_1}{\epsilon_2} \pm \frac{\boldsymbol{i}}{64}\left(\frac{\epsilon_1}{\epsilon_2}\right)^3 + \right. \\ & \left. \pm\frac{\boldsymbol{i}}{2048}\left(\frac{\epsilon_1}{\epsilon_2}\right)^5 + \mathcal{O}\left(\left(\frac{\epsilon_1}{\epsilon_2}\right)^5\right)\right). \end{aligned}$$

Now, if instead of the previous region, we consider the region $|\epsilon_2| < |\epsilon_1|$, then for $k \in \{1,2\}$ the solutions admit the following representation

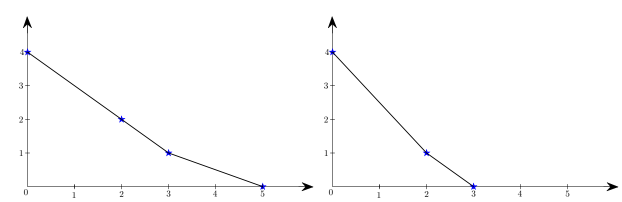
$$z_k(\boldsymbol{\epsilon}) = -2^{k-1}\epsilon_1 + (-1)^k 4\epsilon_2\left(\frac{\epsilon_2}{\epsilon_1} + 4\left(\frac{\epsilon_2}{\epsilon_1}\right)^3 + 32\left(\frac{\epsilon_2}{\epsilon_1}\right)^5 + \mathcal{O}\left(\left(\frac{\epsilon_1}{\epsilon_2}\right)^5\right)\right).$$

The above arguments clearly have shown that some further considerations must be taken into account in the case of multiparameter functions. Next, as mentioned in previous sections, in the single parameter case, the Newton diagram is a powerful tool to analyze the asymptotic behavior for the solutions of pseudo-polynomials. However, in order to be able to apply such a procedure to the multiparameter case, some special situations must be taken into consideration. In order to motivate the above arguments, let us consider the following.

Example 2.2: Consider the polynomial

$$P(z, \boldsymbol{\epsilon}) := z^5 + \left(\epsilon_1\epsilon_2^3 + \epsilon_1^2\epsilon_2^2\right)z^3 + \left(\epsilon_1^2\epsilon_2^2 + \epsilon_1^3\epsilon_2\right)z^2 + \left(\epsilon_1^4\epsilon_2\right). \tag{11}$$

Clearly, $z = 0$ is a $5-$multiple root at $\boldsymbol{\epsilon} = (0,0)$. Now, let us form the Newton diagram with respect to ϵ_1, obtaining $\Pi = \{(0,4), (2,2), (3,1), (5,0)\}$, illustrated in Fig.2-(a).



(a) Newton polygon with respect to ϵ_1.

(b) Newton polygon with respect to y_2.

Fig. 2. Newton polygons for $P(z, \boldsymbol{\epsilon})$ in Example 11.

The slope $\beta_0 = 1$ determines $3-$solutions with respect to ϵ_1, and coefficients that are solutions of the polynomial

$$\mathcal{P}(\xi) = \epsilon_2 + \epsilon_2^2\xi^2 + \epsilon_2^3\xi^3 = 0.$$

In this case, it is clear that the solutions cannot be easily computed. In order to compute solutions by applying the Newton procedure, we seek for a monic polynomial. Thus, with the aim of overcoming such a difficulty, let us consider the change of variables (blowing-ups)

$$\zeta; = \xi \quad \epsilon_1 = v_1\epsilon_2 \quad \epsilon_2 = v_2,$$

and

$$v_1 = y_1 v_2 \quad v_2 = y_2.$$

These change of variables enable us to avoid horizontal segments in the subsequent steps of the process. The resulting polynomial $P'(y_1, y_2)$ posses the same Newton polygon, and the segment with $\beta_0 = 1$ has a monic polynomial

$$y_2^{-5}\mathcal{P}'(\xi, y_2) = y_2^4 + y_2\xi^2 + \xi^3 = 0.$$

Applying Newton procedure, (see figure 2-(b)) to the above equation, we derive the fractional power series solution of P':

$$\zeta_1(y_1, y_2) = -y_2 y_1 + o(y_1 y_2),$$
$$\zeta_{2,3}(y_1, y_2) = \pm \boldsymbol{i} y_2^{2/3} y_1 + o\left(y_1 y_2^{1/3}\right).$$

F. Problem Formulation

The present work is focused on computing the first approximation of the solution of quasi-polynomials around multiple imaginary roots. In this vein, we will focus in the following problems:

(i) compute an approximation of the associated Weierstrass polynomial;
(ii) extend the Newton diagram procedure to Weierstrass polynomial of several variables;
(iii) obtain conditions that allow obtaining Puiseux series solutions
$$s(\tau_1, \tau_2) = c(\tau_2^{1/d})\tau_1^{\beta} + o(\tau_1^{1/d}\tau_2^{1/d}),$$
where $\beta = \alpha/d$ and $\alpha \in \mathbb{N}$;
(iv) give conditions on $f(s, \tau)$ which describes the splitting properties of its solutions $s(\tau_1, \tau_2)$: Regular Splitting, Completely Regular Splitting and Non Regular Splitting.

III. Main Results

A. Computation of Weierstrass Polynomial

In [8], the authors propose a method to compute the Weierstrass polynomial for an holomorphic function. This method is based on its partial derivatives and combinatorial factors related in a recursive way. For the case of holomorphic function, $f(z, \boldsymbol{x})$ of complex variables with $\boldsymbol{x} = (x_1, x_2)$ and $z = 0$ a m-multiple root at $(x_1, x_2) = (0, 0)$ the computation is given as follows. The coefficients w_i (5) are analytic, $w_i(0, 0) = 0$ and can expressed as convergent power series:

$$w_i(x_1, x_2) = \sum_{h_1+h_2=1}^{\infty} \frac{1}{h_1! h_2!} w_{i,\boldsymbol{h}} x_1^{h_1} x_2^{h_2},$$

where $\boldsymbol{h} = (h_1, h_2)$. It is not difficult to see that the coefficients $w_{i,\boldsymbol{h}}$ can be computed by means of the following partial derivatives

$$w_{i,\boldsymbol{h}} = \left.\frac{\partial^{h_1+h_2} w_i}{\partial x_2^{h_1} \partial x_1^{h_2}}\right|_{(0,0)}.$$

Since the analytic function locally satisfy $f = Wb$, thus its partial derivatives satisfy the following recursive relations

$$w_{i,\boldsymbol{h}} = \sum_{j=0}^{i} \alpha_{ij} F_{j,\boldsymbol{h}}, \tag{12}$$
$$F_{j,\boldsymbol{h}} = f_{j,\boldsymbol{h}} - \sum_{k=0}^{j} \sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}} c\left(j, k; \boldsymbol{h}', \boldsymbol{h}''\right) w_{k,\boldsymbol{h}'} b_{j-k,\boldsymbol{h}''},$$

with $\boldsymbol{h}' \neq \boldsymbol{0}$, $\boldsymbol{h}'' \neq \boldsymbol{0}$ and constant coefficients:

$$\alpha_{jj} := \frac{m!}{j! f_{m,\boldsymbol{0}}}, \quad \alpha_{ij} := -\frac{m!}{f_{m,\boldsymbol{0}}} \sum_{k=j}^{i-1} \frac{f_{m+i-k,\boldsymbol{0}} \alpha_{kj}}{(m+i-k)!},$$
$$c\left(j, k; \boldsymbol{h}_1, \boldsymbol{h}_2\right) := \frac{j!}{(j-k)!} \prod_{s=1}^{2} \frac{(h_s' + h_s'')!}{h_s'! h_s''!},$$

and for $\boldsymbol{h}' \neq 0$, $k' = k + m$, $b_{k,\boldsymbol{h}}$ is given by

$$\frac{k!}{(m+k)!}\left[f_{k',\boldsymbol{h}} - \sum_{j=0}^{m-1} \sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}} c\left(k', j; \boldsymbol{h}', \boldsymbol{h}''\right) w_{j,\boldsymbol{h}'} b_{k'-j,\boldsymbol{h}''}\right].$$

Since we are only interested in the leading terms of w_i, namely a first approximation of the Weierstrass polynomial, we adopt the following notation.

Definition 3.1: Let the natural numbers $n_i^{(j)}$, for $i \in \{0, 1, \cdots, m-1\}$ and $j = 1, 2$, denote the first nonzero partial derivative in (z, x_1, x_2) of f, such that the following conditions hold

$$f(0,0,0) = \frac{\partial^i f}{\partial z^i} = \cdots = \frac{\partial^{i+n_i^{(j)}-1} f}{\partial z^i \partial \tau_j^{n_i^{(j)}-1}} = 0, \quad \frac{\partial^{i+n_i^{(j)}} f}{\partial z^i \partial \tau_j^{n_i^{(j)}}} \neq 0,$$

with derivatives evaluated at $(0, 0, 0)$. For $n_i^{(j)} = \infty$ we have derivatives

$$\frac{\partial^i f}{\partial z^i} = \cdots = \frac{\partial^{i+n_i'-1} f}{\partial z^i \partial \tau_2^{n_i'-1}} = 0, \quad \frac{\partial^{i+n_i'} f}{\partial z^i \partial \tau_2^{n_i'}} \neq 0,$$

evaluated at $(z, \boldsymbol{x}) = (0, 0, 1)$, with $n_i' \in \mathbb{Z}_{\geq 0}$.

Leading terms of coefficients w_i can be easy found up to the $n_i^{(j)}$ and n_i' derivatives, as a first observation we give the following result.

Proposition 1: Suppose that the Weierstrass polynomial has first nonzero partial derivative, such that

$$n_i^{(j)} > n_{i+1}^{(j)}, \quad 0 \leq i < m \text{ and } j = 1, 2.$$

Then, the leading terms of $w_i(\boldsymbol{x})$ are given by

$$w_i(x_1, x_2) = \alpha_{i,i} f_{i,(n_i^{(1)},0)} x_1^{n_i^{(1)}} + \alpha_{i,i} f_{i,(0,n_i^{(2)})} x_2^{n_i^{(2)}} + \cdots.$$

If $n_i^{(j)} = \infty$, we have

$$w_i(x_1, x_2) = \alpha_{i,i} f_{i,(n_i', \eta)} x_1^{n_i'} x_2^{\eta} + \cdots .$$

Remark 3: There may be a case in which

$$f_{i,(h_1,h_2)}\big|_{(0,0,0)} = 0 \quad \forall\, h_1, h_2 \in \mathbb{N}.$$

Since w_i are analytic functions, this is equivalent to $w_i(\boldsymbol{x}) \equiv 0$ for $i \leq i \leq \kappa - 1$. Thus, according to Theorem 1 f has the following local structure:

$$z^{\kappa}\left[z^{m-\kappa} + w_{m-\kappa}(\boldsymbol{x}) z^{m-\kappa-1} + \cdots + w_{\kappa}(\boldsymbol{x})\right] b(z, \boldsymbol{x}).$$

Thus, there are κ-invariant solutions $z = 0$ for all $\boldsymbol{x}$. If such number κ does not exist (i.e., if such situation does not occur), then κ will be simply defined as $\kappa := 0$.

B. The Newton Diagram Method for Two Parameters

Consider the monic pseudo-polynomial f given as

$$f(z, \boldsymbol{x}) := z^m + a_{m-1}(x_1, x_2) z^{m-1} + \cdots + a_0(x_1, x_2), \quad (13)$$

with $a_i(\boldsymbol{x}) \in \mathbb{C}\left[[\boldsymbol{x}]\right]$, such that $f(0, \boldsymbol{0}) = 0$. The equation $f = 0$ can be solved by applying the Newton diagram method, this is done by taking into account just one variable, say x_1, and proceeding iteratively. We take the point π_k as the order of a_k in x_1, taking x_2 as an element of $\mathbb{C}\left((x_2)\right)$. For such a purpose, the following definition will be useful

$$\rho_k := \operatorname{ord}_{x_1}\left(a_k(x_1, x_2)\right) = \operatorname{ord}\left(a_k(x_1, 1)\right). \quad (14)$$

Then, the *Newton Polygon* of $f(z, \boldsymbol{x})$, with respect to x_1, is defined by the lower boundary of the convex hull of the points $(k, \rho_k) \in \Pi$ (see, Definition 2.1). In order to apply the the Newton diagram procedure, according to Section II-D, the solution z will take the following structure

$$z(x_1, x_2) = \sum_i c_i(x_2) x_1^{i/d},$$

where the coefficient $c_i(x_2)$, is in general, given by a single parameter Puiseux series in x_2.

1) First Step into the Newton Procedure: Let us suppose that we have determined the Newton diagram of the Weierstrass polynomial (13) of f. Since we are dealing with a monic polynomials, the Newton polygon has a finite number of segments, each one with a corresponding set of points $\Pi^{(\ell)}$ and rational numbers $\beta_\ell \geq 0$ satisfying

$$\beta_0 > \beta_1 > \cdots > \beta_r.$$

Therefore, the segments are presented in two possible ways. The first one corresponds to a Newton polygon with a horizontal segment with $\beta_i = 0$, and the second one where $\beta_j > 0$ (for $i \neq j$). In this vein, for $0 \leq \ell < m$, the Newton Diagram Π is given as the set $\Pi = \Pi' \cup \Pi''$:

$$\{(0, \rho_0), \ldots, (\ell, 0)\} \bigcup \{(\ell, 0), \ldots, (k, \rho_k), \ldots, (m, 0)\}.$$

Lets take at the first step of the process a horizontal segment with slope $\beta_r = 0$. We have the next two propositions:

Proposition 2: Let $f(z, \boldsymbol{x})$ be a pseudo-polynomial with the same structure as (13). Suppose that at least one coefficient $a_i(\boldsymbol{x})$ posses order $\rho_i = 0$. Then, the equation $\mathcal{P}(x_1, \xi) = 0$ (9) of the corresponding horizontal segment has solutions $c_k(x_2^{1/d})$ in the form of Puiseux series.

Now, at the first step of the process, the case with negative slope is considered.

Proposition 3: Assume that f has the same structure than (13) and assume that the first Newton diagram posses a segment with negative slope. Then, there exist a change of variables $(z, x_1, x_2) \mapsto (\zeta, y_1, y_2)$ such that the polynomial $\mathcal{P}(y_2, \xi)$ has Puiseux series solutions $c_k(y_2^{1/d})$.

Hence, applying to f the change of variables $z = \zeta$, $x_1 = y_1^{a_1}$ and $x_2 = y_2^{a_2}$ we get $\tilde{f}(\zeta, y_1, y_2)$, which can be solved. Therefore, solutions $z(x_1, x_2)$ of $f = 0$ are obtained by applying the inverse change of variables to solutions ζ. The following theorem allow us to use the iterative Newton procedure described above.

Theorem 3 (See [14].): The iteration of the classical Newton Procedure for one variable gives rise to representation of all the roots of the equation (13) by generalized Puiseux series with terms $\boldsymbol{x}^{\boldsymbol{a}/d}$, $d \in \mathbb{Z}_+$, such that $\boldsymbol{a}$ belong to n-dimensional, lex-positive strictly convex polyhedral cone.

C. Newton Polygon Algorithm

Let us consider the points $\pi_\ell = (\ell, \rho_\ell) \in \Pi$ to get the Newton polygon, obtaining a finite number of segments with slopes $-\beta_r$. Now, taking as a basis the Newton procedure introduced in Section II-C, we propose the Algorithm 1.

Algorithm 1 Auxiliary Puiseux Series Expansion

Let $f(s, \boldsymbol{\tau})$ have a critical pair such that $s^* = \boldsymbol{i}\omega^*$ is a m-multiple root at $\boldsymbol{\tau} = (\tau_1^*, \tau_1^*)$. Consider the initial values as $r := 0$, $i_{-1} := \kappa$, $j := \kappa$ and $k := \rho_\kappa$.

1) Set $\mathcal{E}_r := \left\{\frac{\ell - k}{j - i} : (i, \ell) \in \Pi, \text{ and } i > j\right\}$;
2) Let $\beta_r := \max \mathcal{E}_j$ and $\Pi^{(r)} := \left\{(i, \ell) \in \Pi : \beta_j \equiv \frac{k - \ell}{j - i}\right\}$;
3) Set $(i_r, \ell_r) \in \Pi^{(r)}$ such that $i_r \geq i$, $\forall (i, \ell) \in \Pi^{(r)}$;
4) Set $j := i_r$, $m_r := i_r - i_{r-1}$ and $r = r + 1$;
5) If $j < m$ go to step 1. Otherwise the algorithm ends.

In the Algorithm 1, κ is defined according to Remark 3. In order to used the iterated Newton diagram procedure we need to find solutions of the equation $\mathcal{P}$, which can be solved by the usual Newton diagram method. For this purpose the input Π of algorithm, has points π_ℓ determined by equation $\mathcal{P} = 0$.

D. Puiseux Series for Quasi-Polynomials with two delays

Since any critical solution $(s^*, \tau_1^*, \tau_2^*)$ can always be translated to the origin by appropriate shifts $s \mapsto s - s^*$, $\tau_1 \mapsto \tau_1 - \tau_1^*$, $\tau_2 \mapsto \tau_2 - \tau_2^*$, hereinafter we will assume that $(s^*, \tau_1^*, \tau_2^*) = (0, 0, 0)$.

Proposition 4: Let the quasi-polynomial

$$f(s,\boldsymbol{\tau}) = p_0(s) + p_1(s)e^{-s\tau_1} + p_1(s)e^{-s\tau_2}, \qquad (15)$$

with $s = 0$ a m-multiple root at $\boldsymbol{\tau} := (\tau_1, \tau_2) = (0,0)$ and local representation $f(s,\boldsymbol{\tau}) = W(\boldsymbol{\tau})b(s,\boldsymbol{\tau})$. Assume that $n_i^{(j)} = 0$ for $i = 0, 1, \ldots, k$. Then, the $k+1$ coefficients of the Weierstrass polynomial W satisfy $w_{m-i}(\boldsymbol{\tau}) \equiv 0,\ i \in \{0, 1, \ldots, k\}$.

Proposition 5: Let the quasi-polynomial $f(s,\boldsymbol{\tau})$ have a m-multiple roots $s = 0$ at $\boldsymbol{\tau} = (0,0)$, with associated Weierstrass polynomial W. Assume that

$$\mathcal{R}\left(W, \frac{\partial W}{\partial s}\right) = \tau_1^{a_1}\tau_2^{a_2}\mathcal{U}(\tau_1,\tau_2) \quad \text{such that } \mathcal{U}(0,0) \neq 0,$$

where $(a_1, a_2) \in \mathbb{Z}^2_{\geq 0} \setminus \{\mathbf{0}\}$, $\mathcal{U} \in \mathbb{C}\{\tau_1, \tau_2\}$. Then, $f = 0$ posses $m-$solutions given by a generalized Puiseux series.

The following result gives conditions to have a regular Newton diagram.

Proposition 6: Let $W(s,\boldsymbol{\tau})$ be the Weierstrass polynomial of a given quasi-polynomial $f(s,\boldsymbol{\tau})$. Assume that for a given $\ell-$segment of the Newton diagram, be $\beta_\ell > 0$ its slope with corresponding points $\Pi^{(\ell)} = \{(k_1, \rho_{k_1}), (k_2, \rho_{k_2}), \ldots, (k_s, \rho_{k_s})\}$. Then, the equation $\mathcal{P}$ can be solved without any change of variables if the leading terms of the coefficients w_{k_i} satisfy

$$w_{k_i,(\rho_{k_i},\eta_{k_i})} \neq 0 \text{ whenever } \eta_{k_i} > \eta_{k_s}, \quad i < s.$$

Finally, using iterated Newton diagram procedure together with the Weierstrass polynomial of the quasi-polynomial $f(s,\boldsymbol{\tau})$, we can find the leading terms of the power series solutions.

Proposition 7: Let $s^* = \boldsymbol{i}\omega^*$ be a $m-$multiple root of $f(s,\boldsymbol{\tau})$ at $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$. Assume that $\kappa = 0$ and r, β_j, (i_j, ℓ_j), m_j and $\Pi^{(j)}$, for $j = 0, 1, \ldots, r-1$ are given by the Algorithm 1. Then, at $\boldsymbol{\tau} = \boldsymbol{\tau}^*$ the $m-$zeros of $f(s,\boldsymbol{\tau})$ can be expanded as

$$s_{j\sigma}(\boldsymbol{\tau}) = \boldsymbol{i}\omega^* + c_{j\sigma}(\tau_2)\left(\tau_1 - \tau_1^*\right)^{\beta_j} + o\left(|\tau_1 - \tau_1^*|^{\beta_j}|\tau_2 - \tau_2^*|^{\beta'_j}\right),$$

for $j = 0, 1, \ldots, r-1$, $\sigma = 0, \ldots, m_j$ and $m = m_0 + \cdots + m_{r-1}$. For $\beta_j > 0$ the coefficients $c_{j\sigma}(x_2)$ are roots of the polynomial:

$$\mathcal{P}_j(\xi, x_2) = \sum_{k=i_{j-1}}^{i_j} w_{k,(n_k^{(1)}, n'_k)}\tau_2^{n'_k}\xi^{k-i_{j-1}}, \quad \left(k, n_k^{(1)}\right) \in \Pi^{(j)},$$

when $\beta_{r-1} = 0$, the coefficients are given by the solution of

$$\mathcal{P}_j(\xi, x_2) = \sum_{k=i_{j-1}}^{i_j} w_{k,(0,n_k^{(2)})}\tau_2^{n_k^{(2)}}\xi^{k-i_{j-1}}, \quad (k,0) \in \Pi^{(r-1)}$$

where $n_k^{(1)}$, $n_k^{(2)}$, n'_k are given by the first nonzero partial derivatives of Definition 3.1; the constant terms $w_{k,(n,\eta)} \in \mathbb{C}$ are computed using (12).

E. Splitting Properties

The main goal of this subsection is to explore some qualitative properties of the solutions $s(\boldsymbol{\tau})$ of the quasi-polynomial f around the $m-$multiple critical pair $(0, \mathbf{0})$. The solution curve $\mathcal{C} \in \mathbb{C}^3$, defined by the equation $f = 0$, is composed of m generalized Puiseux series solutions. These solutions can be arranged in $r-$branches, given as

$$s_{j\sigma}(\boldsymbol{\tau}) = c_{j\sigma}\tau_1^{\beta_j}\ \tau_2^{\beta'_j} + o\left(|\tau_1|^{\beta_j}|\tau_2|^{\beta'_j}\right), \qquad (16)$$

with $\sigma = 0, \ldots, m'_j - 1$ and $j = 0, \ldots, r-1$. The leading terms have exponents defined by $\beta_j = p/m_j$, $\beta'_j = q/m'_j$, with $p, q \in \mathbb{Z}_{\geq 0}$. Each branch has multiplicity $m_j = m'_0 + m'_1 + \cdots + m'_\ell$, for some $\ell \in \mathbb{Z}_{\geq 0}$ such that $m = m_0 + m_1 + \cdots + m_{r-1}$. In this vein, we have the following definition.

Definition 3.2: We say that there is a *Completely Regular Splitting* (CRS) property of the solution $s^* = 0$ at $\boldsymbol{\tau}^* = \mathbf{0}$ if $c_{j\sigma} \neq 0$, and $p \cdot q \leq 1$, $\forall j$. For the *Regular Splitting* (RS) property, some of the coefficients $c_{j\sigma}$ for which $m_j = 1$ may be equal to zero. In the remaining cases of the coefficient $c_{j\sigma}$ we say that *Non Regular Splitting* (NRS) property is present. We have the following proposition.

Proposition 8: Let $f(s,\boldsymbol{\tau})$ be a quasi-polynomial with $s^* = \boldsymbol{i}\omega$ a $m-$multiple root at $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$ such that satisfy Proposition 5. Suppose that β_j, m_j, for $j = 0, 1, \cdots, r-1$ are given by Algorithm 1. Then, the following relations holds:

1) if $(m_j \cdot \beta_j) \cdot \left(m'_j \cdot \beta'_j\right) \leq 1\ \forall j \in \{0, 1, \ldots, r-1\}$, then the solution $(\boldsymbol{i}\omega, \boldsymbol{\tau}^*)$ has the CRS property;
2) if the pairs (m_k, β_k) that do not fulfill 1), satisfy $\beta_k \geq m_k \equiv 1$, then the solution $(\boldsymbol{i}\omega, \boldsymbol{\tau}^*)$ has the RS property;
3) for the remaining cases of β_j, the solution $(\boldsymbol{i}\omega, \boldsymbol{\tau}^*)$ has the NRS property.

Remark 4: It should be mentioned that the Definition 3.2 takes into account the behavior of $m - \kappa$ solutions, without considering the $\kappa-$invariant solutions.

IV. Numerical Examples

In order to illustrate the proposed approach, let us consider the following numerical examples.

Example 4.1: Consider the following quasi-polynomial

$$f(s,\boldsymbol{\tau}) = \left(s^2 - 2s + 1\right) - 2e^{-s\tau_1} + 2\pi s e^{-s\tau_2} + e^{-2s\tau_2}, \quad (17)$$

with $\boldsymbol{\tau}^* = (1, \pi)$, we have a triple root at $s = 0$. In order to apply the proposed results, let us consider $\tilde{f}(s,\boldsymbol{\tau}) := f(s, \tau_1 + 1, \tau_2 + \pi)$. Now, according to the Weierstrass Preparation Theorem, the local behavior around the solution $\mathbf{0}$ of $\tilde{f}$ is captured by the solutions of

$$W(s,\boldsymbol{\tau}) = s^3 + w_2(\boldsymbol{\tau})s^2 + w_1(\boldsymbol{\tau})s + w_0(\boldsymbol{\tau}).$$

Considering Definition 3.1, we take the first partial derivatives of $\tilde{f}$. We have that $n_0^{(1)} = n_0^{(2)} = \infty$, for $h_1 = (1,0), h_2 = (0,1)$ we get the natural numbers $n_1^{(j)}$ determined by

$$f_{1,h_j} = (-1)^{j+1}2 \;\Rightarrow n_1^{(j)} = 1, \quad j = 1, 2.$$

Similarly, for $n_2^{(j)}$ we have that

$$f_{2,h_1} = -4, \quad f_{2,h_2} = 4\pi \quad \Rightarrow n_2^{(j)} = 1.$$

Applying Proposition 4 we have that $w_0 \equiv 0$. Proposition 1 allows to find the leading terms given by

$$w_1(\boldsymbol{\tau}) = \frac{6}{1-\pi^3}\tau_1 - \frac{6}{1-\pi^3}\tau_2 + \cdots,$$
$$w_2(\boldsymbol{\tau}) = \frac{-3\left(3+4\pi^3(\pi-1)\right)}{2\left(\pi^3-1\right)^2}\tau_1 + \frac{3\left(4\pi-1\right)}{2\left(\pi^3-1\right)}\tau_2 + \cdots.$$

Next, according to (14), the order with respect to τ_1 of $w_i(\boldsymbol{\tau})$ define the set of points $\Pi'' = \{(1,0),(2,0),(3,0)\}$. Hence, applying Algorithm 1, we derive the results summarized in Table I.

TABLE I

RESULTS SUMMARY FOR THE QUASI-POLYNOMIAL (17).

Initial Data	Algorithm Output
$m = 3$, $\kappa = 1$. $\rho_1 = 0$	$r = 1$, $m_0 = 2$, $\beta_0 = 0$
$\Pi = \{(1,0),(2,0),(3,0)\}$	$\Pi^{(0)} = \{(1,0),(2,0),(3,0)\}$

From Table I we can observe that the Newton polygon has an horizontal slope with $\beta_0 = 0$. Thus, by considering Proposition 7, we need solve the polynomial equation

$$\mathcal{P}_0(\xi, \tau_2) := \xi^2 + w_{2,(0,1)}\tau_2\xi + w_{1,(0,1)}\tau_2 = 0,$$

and by Proposition 6 we know that there is no need of change of variables. Taking as input $\Pi = \{(0,1),(1,1),(2,0)\}$, we obtain Table II from the Newton polygon algorithm.

TABLE II

RESULTS SUMMARY FOR THE QUASI-POLYNOMIAL (17).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} :\ \mathcal{P}_j(\xi) = 0\}$
$m' = 2$, $\kappa = 0$, $\rho_0 = 1$	$r = 1$, $m'_0 = 2$ $\beta'_0 = 1/2$	$\mathcal{P}_0(\xi) := \xi^2 + \frac{6}{1-\pi^3}$
$\Pi = \{(0,1),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,1),(2,0)\}$	$\left\{c_{0,\sigma} = \pm\sqrt{\frac{6}{1-\pi^3}}\right\}$

Since $\kappa = 1$, there is an invariant root at $s = 0$. The leading terms of solutions are given by

$$s_{0,\sigma}(\boldsymbol{\tau}) = \pm\sqrt{\frac{6}{1-\pi^3}}(\tau_2-\pi)^{1/2} + o\left(|\tau_1-1|^{\frac{1}{2}}|\tau_2-\pi|^{\frac{1}{2}}\right),$$

where $\sigma \in \{0,1\}$. We can see that the solutions $(0,1,\pi)$ splits in one invariant roots s_0 and two solutions such that $\beta_0 \cdot m_1 = 1$, thus by Proposition 8, we know that the solution possess the CRS property.

Example 4.2: Lets consider the quasi-polynomial $f(s,\boldsymbol{\tau}) = p_0(s) + p_1(s)e^{-s\tau_1} + p_2(s)e^{-s\tau_2}$ where

$$p_0(s) = s^5 + s^4 + \frac{4+\pi}{2}s^3 + 2s^2 + \frac{2+\pi}{2}s + 2, \quad \text{(18a)}$$
$$p_1(s) = 1, \quad p_2(s) = 2s^4 + 4s^2 + 2. \quad \text{(18b)}$$

For $\boldsymbol{\tau}^* = (\pi, 1)$, f has a double root at $s = \boldsymbol{i}$. With the aim of applying the proposed methodology, let us shift from $(\boldsymbol{i}, \pi, 1)$ to the origin, obtaining $\tilde{f}'$. We find that the first nonzero partial derivatives of the quasi-polynomial at $(0,0,0)$ are given by

$$f_{0,(1,0)} = \boldsymbol{i} \quad \Rightarrow n_0^{(1)} = 1, \quad f_{0,(0,n)} = 0 \Rightarrow n_0^{(2)} = \infty,$$
$$f_{1,(1,0)} = 1 - \boldsymbol{i}\pi \Rightarrow n_1^{(1)} = 1, \quad f_{1,(0,n)} = 0 \Rightarrow n_1^{(2)} = \infty.$$

Hence, by Proposition 1, we have that $\rho_j = n_j^{(1)}$ for $j = 1, 2$. The Newton diagram is given by $\Pi = \{(0,1),(1,1),(2,0)\}$. Table III summarizes the results deriving from Algorithm 1. Since coefficient w_1 is not over the Newton polygon, following Proposition 1, we compute the leading term of $w_i(\boldsymbol{\tau})$, as

$$w_0(\boldsymbol{\tau}) = \frac{-2\boldsymbol{i}}{(8+\pi^2) + \boldsymbol{i}(8-3\pi) + 16e^{-\boldsymbol{i}}}\tau_1 + \cdots.$$

TABLE III

RESULTS SUMMARY FOR THE QUASI-POLYNOMIAL (18).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} :\ \mathcal{P}_j(\xi) = 0\}$
$m = 2$, $\kappa = 0$. $\rho_0 = 1$	$r = 1$, $m_0 = 2$, $\beta_0 = 1/2$	$\mathcal{P}_0(\xi) := \xi^2 + w_{0,(1,0)}$
$\Pi = \{(0,1),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,1),(2,0)\}$	$\left\{c_{0,\sigma} = \pm\sqrt{w_{0,(1,0)}}\right\}$

From the algorithm output, we get a segment with slope $\beta_0 = 1/2$. According to Proposition 7 we need to solve:

$$\mathcal{P}(\xi) = \xi^2 - \frac{2\boldsymbol{i}}{(8+\pi^2) + \boldsymbol{i}(8-3\pi) + 16e^{-\boldsymbol{i}}} = 0,$$

for $\sigma \in \{0,1\}$, the solutions are given by

$$s_{0,\sigma}(\boldsymbol{\tau}) = \boldsymbol{i} \pm \frac{\sqrt{2}\boldsymbol{i}^{3/2}}{\sqrt{(8+\pi^2)+\boldsymbol{i}(8-3\pi)+16e^{-\boldsymbol{i}}}}(\tau_1-\pi)^{1/2} + o\left(|\tau_1-\pi|^{\frac{1}{2}}|\tau_2-1|^{\frac{1}{2}}\right).$$

Since $\beta_0 \cdot m_0 = 1$, Proposition 8 implies that the solution $(\boldsymbol{i}, \pi, 1)$ has the CRS property. This behavior is illustrated in Figure 3.

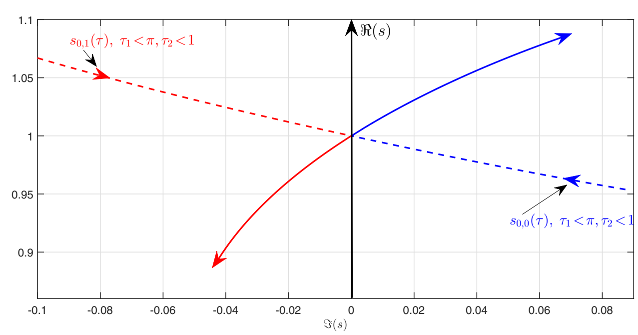


Fig. 3. Root locus of quasi-polynomial $f(s,\boldsymbol{\tau})$ (18) around $(\boldsymbol{i}, \pi, 1)$.

Example 4.3: For our final example, let us consider a special case of a model of population dynamics (see [16]), with two distributed delays such that the characteristic functions is given by

$$f(s,\tau_1,\tau_2) = s - a\frac{1-e^{-s\tau_1}}{s} - c\frac{1-e^{-s\tau_2}}{s}, \qquad (19)$$

with positive parameters $a = -0.214104$, $b = -0.996801$. At $\boldsymbol{\tau}^* \approx (3.84003026849, 10.44866732901)$, f posses a double root at $s^* = \boldsymbol{i}$. We make a change of variable in order to shift the critical point to the origin, obtaining $\tilde{f}$. In order to compute the associated Weierstrass polynomial $W(s,\boldsymbol{\tau}) = s^2 + w_1(\boldsymbol{\tau}) + w_0(\boldsymbol{\tau})$, we consider as a first step, the first nonzero partial derivatives given in Definition 3.1,

$$f_{0,(1,0)} \neq 0 \Rightarrow n_0^{(1)} = 1, \quad f_{0,(0,1)} \neq 0 \Rightarrow n_0^{(2)} = 1,$$

$$f_{1,(1,0)} \neq 0 \Rightarrow n_1^{(1)} = 1, \quad f_{1,(0,1)} \neq 0 \Rightarrow n_1^{(2)} = 1.$$

Next, by means of Proposition 1, we are able to compute the first terms $w_i(\boldsymbol{\tau})$ of W, as follows:

$$w_0(\boldsymbol{\tau}) = \alpha\tau_1 + \beta\tau_2 + \cdots,$$

$$w_1(\boldsymbol{\tau}) = \gamma\tau_1 + \delta\tau_2 + \cdots,$$

with $\alpha = (-0.0231801 + 0.0439373\boldsymbol{i})$, $\beta = (0.00397688 - 0.0245988\boldsymbol{i})$, $\gamma = (0.0105549 + 0.0292551\boldsymbol{i})$, $\delta = (-0.0359858 + 0.150344\boldsymbol{i})$. Applying Algorithm 1, to the Newton diagram $\Pi = \Pi' \bigcup \Pi'' = \{(0,0),(0,1),(1,0),(1,1),(2,0)\}$, we derive the results summarized in Table IV.

TABLE IV
RESULTS SUMMARY FOR THE QUASI-POLYNOMIAL (19).

Initial Data	Algorithm Output
$m = 2$, $\kappa = 0$. $\rho_0 = 0$	$r = 1$, $m_0 = 2$, $\beta_0 = 0$
$\Pi = \Pi' \bigcup \Pi''$	$\Pi^{(0)} = \{(0,0),(1,0),(2,0)\}$

By Proposition 7 the solutions of the polynomial

$$\mathcal{P}(\xi) = \xi^2 + w_{1,(0,1)}\xi + w_{0,(1,0)}$$

determine the leading terms of the solutions. Taking into consideration the Newton diagram $\Pi^{(0)}$, we obtain Table V.

TABLE V
RESULTS SUMMARY FOR THE QUASI-POLYNOMIAL (19).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} :\ \mathcal{P}_j(\xi) = 0\}$
$m = 2$, $\kappa = 0$. $\rho_0 = 0$	$r = 1$, $m_0 = 2$, $\beta_0 = 0$	$\mathcal{P}_0(\xi) := \xi^2 + w_{0,(0,1)}$
$\Pi = \{(0,1),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,1),(2,0)\}$	$\left\{c_{0,\sigma} = \pm\sqrt{w_{0,(0,1)}}\right\}$

Thus, for $\sigma = \{0,1\}$, the solutions are given by

$$s_{0,\sigma}(\boldsymbol{\tau}) = \pm(0.1202 - 0.1023\boldsymbol{i})(\tau_2 - \tau_2^*)^{1/2} + o\left(|\tau_1 - \tau_1^*|^{\frac{1}{2}}|\tau_2 - \tau_2^*|^{\frac{1}{2}}\right),$$

by Proposition 8 the solutions posses the CRS splitting property.

V. Concluding Remarks

In this paper, we have considered some issues concerning the asymptotic behavior of multiple critical roots for quasi-polynomials with two delays. By means of the Weierstrass polynomial, the proposed approach is based on an iterative Newton diagram method which can be effectively applied to find the leading terms of power series solutions expressed as a generalized Puiseux series. Finally, the splitting properties of a given solution have been described by means of CRS, RS and NRS properties in order to get some insights about its geometric behavior.

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References

[1] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*. New York: Springer, 1993.
[2] W. Michiels and S.-I. Niculescu, *Stability, Control, and Computation for Time-Delay Systems. An Eigenvalue-Based Approach (2. ed.)*, ser. Advances in design and control. Philadelphia: SIAM, 2014, vol. 27.
[3] K. Gu, J. Chen, and V. L. Kharitonov, *Stability of time-delay systems*. New York: Springer Science & Business Media, 2003.
[4] J. Chen, P. Fu, S.-I. Niculescu, and Z. Guan, "An eigenvalue perturbation approach to stability analysis, part II: when will zeros of time-delay systems cross imaginary axis?" *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 5583–5605, 2010.
[5] ——, "An eigenvalue perturbation approach to stability analysis, part I: eigenvalue series of matrix operators," *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 5564–5582, 2010.
[6] D. A. Irofti, K. Gu, I. Boussaada, and S.-I. Niculescu, "Some insights into the migration of double imaginary roots under small deviation of two parameters," *Automatica*, vol. 88, pp. 91–97, 2018.
[7] A. A. Monforte and M. Kauers, "Formal Laurent series in several variables," *Expositiones Mathematicae*, vol. 31, no. 4, pp. 350–367, 2013.
[8] A. A. Mailybaev and S. S. Grigoryan, "On the Weierstrass Preparation Theorem." *Matematicheskie Zametki*, vol. 69, no. 2, pp. 194–199, 2001.
[9] L. Hormander, *An introduction to complex analysis in several variables*. Amsterdam: North-Holland Co., 1973, vol. 7.
[10] M. Vainberg and V. Trenogin, *Theory of branching of solutions of non-linear equations*. Leyden: Noordhoff International Pub., 1974.
[11] C. T. C. Wall, *Singular points of plane curves*. New York: Cambridge University Press, 2004, vol. 63.
[12] J. McDonald, "Fiber polytopes and fractional power series," *Journal of Pure and Applied Algebra*, vol. 104, no. 2, pp. 213 – 233, 1995.
[13] G. M. Ziegler, *Lectures on polytopes*. Springer Science & Business Media, 2012, vol. 152.
[14] M. Soto and J. L. Vicente, "The Newton Procedure for several variables," *Linear Algebra and its Applications*, vol. 435, no. 2, pp. 255–269, 2011.
[15] B. H. Neumann, "On Ordered Division Rings," *Transactions of American Mathematical Society*, no. 66, pp. 202–252, 1949.
[16] D. Irofti, K. Gu, I. Boussaada, and S.-I. Niculescu, "Some insights into the migration of double imaginary roots under small deviation of two parameters," *Automatica*, vol. 88, pp. 91–97, 2018.