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On Well-posedness & Approximation of Composite Systems*

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***Abstract*— In this paper we consider some issues in modeling and approximation of composite control systems defined by coupled partial differential equations and ordinary differential equations. Although these systems are motivated by applications to thermal management systems, the fundamental issues occur in many hybrid systems of subsystems. We establish a well-posedness result and provide a short discussion of how the problem formulation can impact the choice of a specific numerical approximation. In particular, the form of the coupling can impact the choice of finite volume, finite element or higher order schemes and their convergence properties. Examples are given to illustrate the ideas.**

I. Introduction and Motivation

This paper is concerned with approximation of composite (and hybrid) control systems. The motivation comes from modeling and control of vapor compression systems (VCS) (see [1], [2], [3], [4], [5], [6]). As illustrated in Figure 1, VCSs are composite systems of interconnected components (compressor, HX1, expansion valve, HX2 and actuators). The component models for the thermal fluid systems that define the heat exchangers (HX1, HX2) are partial differential equations (PDEs) while the other two components (actuator/compressor and expansion valve) are typically modeled by (finite dimensional) ordinary differential equations (ODEs) or by empirical maps. Thus, when these component models are connected together to produce a VCS system, this system is modeled by interconnected ODE-PDE-ODE-PDE equations. In particular, outputs of the ODEs that govern the actuator dynamics drive the PDEs that govern the HX1 dynamics through the boundary conditions. In turn, outputs at the boundary of the PDEs that govern the HX1 dynamics, drive the ODEs that describe the expansion valve dynamics and this pattern repeats as one moves around the CVS loop.

These boundary inputs and boundary outputs are interactions defined at the boundary of the spatial domains and hence are not defined by bounded (continuous) operators. The counter example on pages 144–145 in the Chen and Grimmer paper [7] illustrates that even the simplest "unbounded composition" of two well-posed infinite dimensional linear systems may fail to be well-posed. Thus, establishing the well-posedness of coupled infinite and finite dimensional component models requires care. In addition, there are "non-standard" approximation issues that need to be considered when the composite system model is to be used for optimization, control and design. Thus, problems

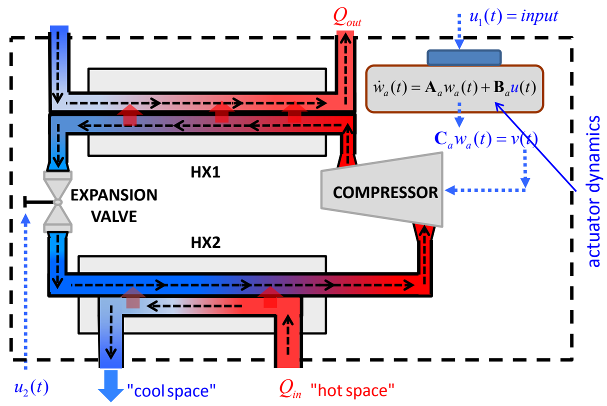

Fig. 1. A Basic VCS

related to well-posedness, developing convergent approximations that also preserve basic properties such as stability (stabilizability, controllability) and numerical algorithms for optimization and control of composite infinite dimensional systems need to be addressed. In this paper we focus on a well-posedness problem, but we also provide a short discussion of some issues that arise in the development of numerical schemes for approximating composite systems.

Example 1.1: The following simple example from [8] illustrates that composing control systems in certain ways can produce systems with undesirable properties. Let

$$\dot{y}(t) = a_0 y(t) + b_0 v(t) \tag{1}$$

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{2}$$

Note that if $b_0 \neq 0$, then both (1) and (2) are controllable. If one connects the systems by setting

$$v(t) = Hz(t) = \begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix},$$

then the composite system is given by

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_0 & -b_0 h_1 & -b_0 h_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t). \tag{3}$$

If $a_0 = b_0 = 1$ and $H = \begin{bmatrix} h_1 & h_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$, then the

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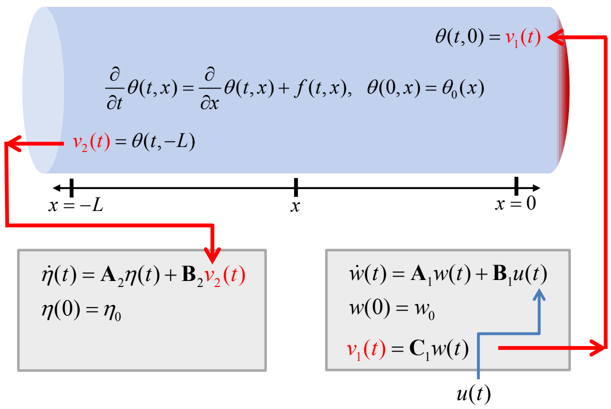

Fig. 2. A Simple Hybrid System

composite system

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

is not controllable. In fact this composite system is not even stabilizable!

Observe that the composite system above has the form

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} a_0 & F \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

where $F = b_0 H$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

A similar structure occurs in modeling of thermal-fluid systems and other multi-discipline infinite dimensional systems. Even in the case where all the component models are linear, establishing well-posedness of interconnected systems can be nontrivial. We focus on a simple interconnected linear system of the type depicted in Figure 2 to illustrate the issues.

II. An ODE Driving a PDE

Consider a problem where an ODE "drives" a PDE through the boundary. If $\theta(t,s)$ represents the convection of temperature in a fluid moving from right to left, then the simplest linear model is given by

$$\theta_t(t,s) = \theta_s(t,s), \quad -L < s < 0, \tag{4}$$
$$\theta(t,0) = v_1(t). \tag{5}$$

As noted in [9], [10], [11], a change of variables $\omega(t,s) = \theta(t,s) - v_1(t)$ produces

$$\omega_t(t,s) = \omega_s(t,s) - \dot{v}_1(t), \quad -L < s < 0, \tag{6}$$
$$\omega(t,0) = 0. \tag{7}$$

Let

$$\dot{w}(t) = \mathbf{A}_a w(t) + \mathbf{B}_a u(t) \tag{8}$$

with $\mathbf{A}_a$ a $n \times n$ and $\mathbf{B}_a$ an $n \times m$ matrix, respectively. Here, (8) represent actuator dynamics with output

$$v_1(t) = \mathbf{C} w(t). \tag{9}$$

Since

$$\dot{v}_1(t) = \mathbf{C}\dot{w}(t) = \mathbf{C}\left[\mathbf{A}_a w(t) + \mathbf{B}_a u(t)\right],$$

the composite system becomes

$$\omega_t(t,s) = \omega_s(t,s) + [-\mathbf{C}\mathbf{A}_a] w(t) + [-\mathbf{C}\mathbf{B}_a] u(t) \tag{10}$$
$$\dot{w}(t) = \mathbf{A}_a w(t) + \mathbf{B}_a u(t). \tag{11}$$

Let $X = L^2(-L,0)$, $Y = \mathbb{R}^n$ and define the operators $F : \mathbb{R}^n \to X$, $B_1 : \mathbb{R}^m \to X$ by $[Fw](s) \equiv [-\mathbf{C}\mathbf{A}_a] w$ (a constant function of s) and $B_1 = [-\mathbf{C}\mathbf{B}_a]$, respectively. The abstract version of this system can be written as

$$\frac{d}{dt}\begin{bmatrix} \omega(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} A_0 & F \\ 0 & \mathbf{A}_a \end{bmatrix}\begin{bmatrix} \omega(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ \mathbf{B}_a \end{bmatrix} u(t), \tag{12}$$

where A_0 is defined on the domain

$$D(A_0) = H_R^1(-L,0) = \left\{\phi(\cdot) \in H^1(-L,0) : \varphi(0) = 0\right\}$$

by

$$A_0\phi(\cdot) = \phi'(\cdot).$$

Note that F is a bounded linear operator on $\mathbb{R}^n$, $[A_0]^{-1}$ exists and

$$\left([A_0]^{-1}\phi(\cdot)\right)(s) = \int_0^s \phi(\tau)d\tau$$

is bounded (compact). Moreover, if we set $F_R \triangleq [A_0]^{-1}F$, then

$$F_R w = [A_0]^{-1}Fw = \int_0^s [-\mathbf{C}\mathbf{A}_a] w d\tau$$
$$= (s)[-\mathbf{C}\mathbf{A}_a] w \in D(A_0)$$

and F_R maps all of $\mathbb{R}^n$ into $D(A_0)$. Observe that (at least formally) $\mathscr{A}$ can now be factored as the product

$$\mathscr{A} = \begin{bmatrix} A_0 & F \\ 0 & \mathbf{A}_a \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & \mathbf{A}_a \end{bmatrix}\begin{bmatrix} I_{L^2} & F_R \\ 0 & \mathbf{I}_n \end{bmatrix}.$$

We will be able to apply part (a) of Theorem 5 and conclude that the composite system (12) is well-posed (see also [9], [10], [11]).

III. Product Space Formulations

The previous two examples fall into a class of problems with the following structure. Assume that X and Y are Hilbert spaces and let $Z = X \times Y$ be the product space with standard inner product. Assume that $A_0 : D(A_0) \subseteq X \to X$ generates a C_0-semigroup on X and likewise that $A_1 : D(A_1) \subseteq Y \to Y$ generates a C_0-semigroup on Y. Let $F : D(F) \subseteq Y \to X$ and consider the composite operator $\mathscr{A} : D(\mathscr{A}) \subseteq Z \to Z$ defined by

$$\mathscr{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_0 & F \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_0 x + Fy \\ A_1 y \end{bmatrix}. \tag{13}$$

Let $\mathscr{A}_D : D(\mathscr{A}_D) \subseteq Z \to Z$ be the diagonal operator defined by

$$\mathscr{A}_D = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix},$$

with domain $D(\mathscr{A}_D) = D(A_0) \times D(A_1)$. We assume that $D(A_1) \subseteq D(F)$. Moreover, even one assumes that F is bounded one must impose certain restrictions on these operators to precisely define the composite operator $\mathscr{A}$ and its domain $D(\mathscr{A})$ (see [12] and [13]).

The results by Vinter and Zabczyk provide a simple test of well-posedness for connected systems. The following results follow from Theorem 1 in [13] and the modifications in [14] and [7] .

Theorem 1: Let $\mathscr{A} : D(\mathscr{A}) \subseteq Z \to Z$ generate a C_0-semigroup on the Hilbert space Z and assume $\mathscr{P}$ is a bounded linear operator on Z. If $\mathscr{P}$ is invertible and $\mathscr{F} = \mathscr{I} - \mathscr{P}$ satisfies $\mathscr{F}(Z) \subseteq D(\mathscr{A})$, then,

(1A) The operator $\mathscr{A}\mathscr{P}$ with domain

$$D(\mathscr{A}\mathscr{P}) = \{z \in Z : \mathscr{P}z \in D(\mathscr{A})\}$$

generates a C_0-semigroup on Z.

(1B) The operator $\mathscr{P}\mathscr{A}$ with domain $D(\mathscr{P}\mathscr{A}) = D(\mathscr{A})$ generates a C_0-semigroup on Z.

Theorem 2: Let $Z = X \times Y$ be a product space of Hilbert spaces and $\mathscr{A}_D : D(\mathscr{A}) \subseteq Z \to Z$ be a "diagonal operator"

$$\mathscr{A}_D = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix},$$

where A_0, A_1 with domains $D(A_0) \subseteq X =$ and $D(A_1) \subseteq Y$, respectively. Assume $\hat{F} : Y \to X$ is a bounded linear operator and A_0 and A_1 generate a C_0-semigroups on X and Y, respectively. If $\mathscr{P}$ is the bounded linear operator defined by

$$\mathscr{P} = \begin{bmatrix} I & \hat{F} \\ 0 & I \end{bmatrix},$$

then,

(2A) The operator $\mathscr{A}\mathscr{P}$ with domain

$$D(\mathscr{A}\mathscr{P}) = \{z \in Z : \mathscr{P}z \in D(\mathscr{A})\}$$

generates a C_0-semigroup on Z.

(2B) If the operator $\hat{F}A_1$ has a bounded linear extension to all of Y, then $\mathscr{P}\mathscr{A}$ with domain $D(\mathscr{P}\mathscr{A}) = D(\mathscr{A})$ generates a C_0-semigroup on Z.

If $F : D(F) = Y \to X$ is a bounded linear operator (as in the previous two examples), then (13) can be written as

$$\mathscr{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_0 x \\ A_1 y \end{bmatrix} + \begin{bmatrix} Fy \\ 0 \end{bmatrix},$$

with $D(\mathscr{A}) = D(A_0) \times D(A_1)$. The assumption that F is bounded implies by standard perturbation theory that $\mathscr{A} : D(\mathscr{A}) \subseteq Z \to Z$ generates a C_0-semigroup on Z. When F is not bounded one needs another approach to well-posedness. This case arises when the output to a PDE occurs at the boundary and this output drives an ODE. In the next section we use the simple model problem to illustrate this point.

IV. A PDE Driving an ODE

Let the spatial domain be $(-L, 0)$ and again assume the convection of the temperature is from right to left so that

$$\theta_t(t,s) = \theta_s(t,s), \quad -L < s < 0, \tag{14}$$

$$\theta(t,0) = 0. \tag{15}$$

Given an initial condition $\varphi(\cdot) \in L^2(-L,0)$

$$\theta(0,s) = \varphi(s), \quad -L < s < 0, \tag{16}$$

uniquely determines the solution $T(t,s)$. Consider a simple scalar ODE system

$$\dot{\eta}(t) = a_1\eta(t) + b_1\xi(t) + b_2 u(t), \tag{17}$$

$$\eta(0) = \eta_0, \tag{18}$$

and note that both systems are well-posed. Now assume that the input to the ODE system is given as the output to the PDE system at the left boundary so that

$$\xi(t) = \theta(t, -L) \tag{19}$$

The resulting composite system becomes

$$\dot{\eta}(t) = a_1\eta(t) + b_1\theta(t,-L) + b_2 u(t) \tag{20}$$

$$\theta_t(t,s) = \theta_s(t,s), \quad -L < s < 0. \tag{21}$$

As before let $X = \mathbb{R}$, $Y = L^2(-L,0)$ and define A_1 on the domain

$$D(A_1) = H^1_R(-L,0) = \left\{\varphi(\cdot) \in H^1 : \varphi(0) = 0\right\} \tag{22}$$

by

$$A_1\phi(\cdot) = \varphi'(\cdot) \tag{23}$$

and $F : H^1(-L,0) \to \mathbb{R}$ by

$$F\varphi(\cdot) = b_1\varphi(-L). \tag{24}$$

The abstract version of the composite system (20)-(21) has the form

$$\frac{d}{dt}\begin{bmatrix} \eta(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} a_1 & F \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} \eta(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} b_2 \\ 0 \end{bmatrix} u(t).$$

Observe that, unlike in Section II, the operator F defined by (24) is not a bounded linear operator on $X = L^2(-L,0)$. Moreover, if the coefficient $a_1 \neq 0$ and we set $F_R \triangleq [a_1]^{-1}F$, then (again formally)

$$\mathscr{A} = \begin{bmatrix} a_1 & F \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} I & F_R \\ 0 & I \end{bmatrix}. \tag{25}$$

However, again F_R is not a bounded linear operator on $X = L^2(-L,0)$ so Zabczyk's framework does not apply directly to this problem.

To set up a framework to address this question we follow the approach in [13]. Let

$$\mathscr{A}_D = \begin{bmatrix} a_1 & 0 \\ 0 & A_1 \end{bmatrix}$$

be the diagonal operator with domain $D(\mathscr{A}) = \mathbb{R} \times D(A_1)$. It is well known that there is a function $g \in H^1(-L,0)$ such that if $\varphi(\cdot) \in H^1$ and $\varphi(0) = 0$, then

$$b_1\varphi(-L) = \int_{-L}^{0} g(s)\varphi(s)ds + \int_{-L}^{0} g'(s)\varphi'(s)ds.$$

Following the approach in [13] (see page 531), one defines the bounded linear operator $E : L^2(-L,0) \to \mathbb{R}$ by

$$E(\varphi(\cdot)) = \int_{-L}^{0} g(s)\left[\int_{0}^{s}\varphi(\tau)d\tau\right]ds + \int_{-L}^{0} g'(s)\varphi(s)ds.$$

Note that if $\varphi(\cdot) \in D(A_1)$, then

$$\begin{aligned} E(\varphi'(\cdot)) &= \int_{-L}^{0} g(s)\left[\int_{0}^{s}\varphi'(\tau)d\tau\right]ds \\ &+ \int_{-L}^{0} g'(s)\varphi'(s)ds \\ &= \int_{-L}^{0} g(s)\left[\varphi(s) - \varphi(0)\right]ds \\ &+ \int_{-L}^{0} g'(s)\varphi'(s)ds \\ &= \int_{-L}^{0} g(s)\varphi(s)ds + \int_{-L}^{0} g'(s)\varphi'(s)ds \\ &= b_1\varphi(-r). \end{aligned}$$

Thus, we have a factorization of $\mathscr{A}$ into

$$\mathscr{A} = \begin{bmatrix} I & E \\ 0 & I \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & A_1 \end{bmatrix}. \tag{26}$$

The operator

$$\mathscr{P} = \begin{bmatrix} I & E \\ 0 & I \end{bmatrix}$$

is a bounded linear operator and

$$\mathscr{A} = \mathscr{P}\mathscr{A}_D.$$

Moreover, $\mathscr{P}$ is invertible and since $\mathbb{R}$ is finite dimensional $E : L^2(-L,0) \to \mathbb{R}$ maps all of $L^2(-L,0)$ into the domain of a_1. It now follows from Theorem 2-(2B) that $\mathscr{A} = \mathscr{P}\mathscr{A}_D$ generates a generates a C_0-semigroup on Z. Combining these observations we one has the following result.

Theorem 3: The operator

$$\mathscr{A}\begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} a_1 & F \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} a_1\eta + b_1\varphi(-L) \\ \varphi(\cdot) \end{bmatrix} \tag{27}$$

with domain

$$D(\mathscr{A}) = \left\{\begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} : \varphi(\cdot) \in H^1,\ \varphi(0) = \eta\right\} \tag{28}$$

generates a C_0-semigroup on $Z = \mathbb{R} \times L^2(-L,0)$.

The proof above makes use of the approach in Vinter's paper [12] and ideas in Zabczyk's paper [13]. However, what is interesting here is that the problem arises from the analysis of a system where the output of a simple PDE drives an ODE. The Vinter and Zabczyk work was motivated by delay differential equations and there is an interesting connection between the two problems.

A. Connection to Delay Differential Equations

Consider the ordinary delay differential equation

$$\dot{z}(t) = a_1 z(t) + b_1 z(t-L), \tag{29}$$

with initial data

$$z(0) = \eta,\ \ z(s) = \varphi(s),\ \ -L < s < 0. \tag{30}$$

It is well known (see [15]) that (29) - (30) defines a dynamical system on the state space $R \times L^2(-L,0)$ and the generator of this dynamical system is the operator defined by

$$\mathscr{A}\begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} a_1\eta + b_1\varphi(-L) \\ \varphi'(\cdot) \end{bmatrix} \tag{31}$$

with domain

$$D(\mathscr{A}) = \left\{\begin{bmatrix} \eta \\ \varphi(\cdot) \end{bmatrix} : \varphi(\cdot) \in H^1(-L,0), \varphi(0) = \eta\right\} \tag{32}$$

Observe that the operator defined by (27) - (28) in the previous Theorem is exactly the same. Thus, the original interconnected PDE-ODE system is equivalent to an ordinary delay differential equation.

Remark 4: These types of problems were considered in the papers [9], [10], [11] where issues of well-posedness and numerical approximations were discussed when the actuator dynamics are included. Vinter [12] and Zabczyk [16] were among the first to observe that including finite dimensional actuator dynamics in boundary control problems produces special composite systems defined on product spaces and this structure could be exploited to establish well-posedness of the composite system. However, if one employs more complex (infinite dimensional) models of the actuators components, then additional issues occur. Recently this structure has also been exploited to develop numerical approximations for specific classes of actuator dynamics (see [17], [18]). As noted above, when the output to the PDE system drives a finite dimensional ODE through boundary outputs the composite system is more complex and connecting such systems needs to be done with care. In particular, these types of interconnected systems lead to the case where the operator $F : D(F) \to X$ is not a bounded linear operator. This is clearly illustrated in Section IV above. This situation occurs in a VCS system when the output to a PDE system drives an ODE system as, for example, when the output to the HX1 (a PDE system) drives the expansion valve (typically modeled by an ODE system).

Observe that if $[A_0]^{-1}$ exists, then $\mathscr{A}$ can be factored (again formally) as a product

$$\mathscr{A} = \begin{bmatrix} A_0 & F \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} I_X & [A_0]^{-1}F \\ 0 & I_Y \end{bmatrix}. \tag{33}$$

If one now includes control inputs, say with a finite number of controllers, then $\mathscr{B} : \mathbb{R}^m \to Z$ of the form

$$\mathscr{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{34}$$

Thus, the control system becomes

$$z(t) = \mathscr{A} z(t) + \mathscr{B} u(t). \tag{35}$$

In view of the examples above if we consider only the case where $\mathscr{B} : \mathbb{R}^m \to Z$ is a bounded (hence compact) linear operator, then well-posedness of the control system is determined by the coupled system operator $\mathscr{A}$. Thus, we focus on the system operator.

However, as illustrated in Section IV, we need to consider the case where the connection operator $F : D(F) \subseteq Y \to X$ is not bounded. We are interested in two fundamental questions:

1) When is the composite system well-posed?
2) How do we approximate the composite system to ensure convergence and dual convergence for control design?

In the following section we provide a simple test for certain cases.

V. Well-Posedness

One could apply standard (additive) perturbation results to address well-posedness (see Chapter 3 in Kato [19]) by requiring some type of $\mathscr{A}$ - boundedness on the operator F. Another approach is to impose some special structure on the system to take advantage of Zabczyk's results. This approach was used in [17], [18] where the composite system included actuator dynamics with delays.

If one assumes that $0 \in \rho(A_0)$ so that $[A_0]^{-1}$ is defined and bounded on all of X, then

$$\mathscr{A} = \begin{bmatrix} A_0 & F \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} I & F_R \\ 0 & I \end{bmatrix}$$

where

$$F_R = [A_0]^{-1} F. \tag{36}$$

Likewise, if $0 \in \rho(A_1)$ so that $[A_1]^{-1}$ is defined and bounded, then

$$\mathscr{A} = \begin{bmatrix} A_0 & F \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} I & F_L \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}$$

where

$$F_L = F[A_1]^{-1}. \tag{37}$$

The problem now becomes a multiplicative perturbation problem with a long history (see [20]) and suitable for applying Zabczyk's theorems 1 and 2 above. As a special case for bounded operators $F : D(F) \to X$, we have the following result.

Theorem 5: If (a) $0 \in \rho(A_0)$ and $F_R = [A_0]^{-1} F$ is bounded or (b) $0 \in \rho(A_1)$ and $F_L = F[A_1]^{-1}$ has a bounded extension to all of Y, then $\mathscr{A}$ generates a C_0-semigroup on Z.

Proof: Consider case (a) and set $F_R = [A_0]^{-1} F$. It follows that

$$\mathscr{P} = \begin{bmatrix} I & F_R \\ 0 & I \end{bmatrix}$$

is bounded and defined on all of Y. Moreover, $\mathscr{P}$ is invertible with inverse

$$\mathscr{P}^{-1} = \begin{bmatrix} I & -F_R \\ 0 & I \end{bmatrix}.$$

The operator

$$\mathscr{F} = \mathscr{I}_Z - \mathscr{P} = \begin{bmatrix} 0 & F_R \\ 0 & 0 \end{bmatrix}$$

clearly maps all of Z into the domain of $\mathscr{A}_D$ since

$$\mathscr{F} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} [A_0]^{-1} F y \\ 0 \end{bmatrix} \in D(\mathscr{A}_D).$$

Applying Theorem 1-(1A) it follows that

$$\mathscr{A} = \mathscr{A}_D \mathscr{P} = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} I & F_R \\ 0 & I \end{bmatrix}$$

generates a C_0-semigroup on Z.

Now consider case (b) and set $F_L = F[A_1]^{-1}$. It follows that

$$\mathscr{Q} = \begin{bmatrix} I & F_L \\ 0 & I \end{bmatrix}$$

and

$$\mathscr{A} = \mathscr{Q} \mathscr{A}_D = \begin{bmatrix} I & F_L \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}$$

Since $D(A_1) \subseteq D(F)$ it follows that $F_L = F[A_1]^{-1}$ has a bounded extension to all of Y. The results now follow from Theorem 2-(2B). $\square$

VI. Implications for Numerical Discretization

As illustrated above, one must exercise care in how one connects components into system level models to ensure well-posedness. This issue needs to be considered when developing numerical approximations for the composite system.

One may approach the problem of approximating an operator of the form

$$\mathscr{A} = \begin{bmatrix} A_0 & F \\ 0 & A_1 \end{bmatrix}$$

by first approximating the individual component operators, say

$$A_0^N \to A_0, \;\; A_1^N \to A_1 \;\; \text{and} \;\; F^N \to F.$$

Then, an approximation of the composite operator is constructed by

$$\mathscr{A}^N = \begin{bmatrix} A_0^N & F^N \\ 0 & A_1^N \end{bmatrix}.$$

Observe that when the coupling operator F is unbounded additional care is required in constructing " good" approximations. This process is very common and may be described by composing discretizations of the component models.

However, the boundary conditions for the coupled systems can be lost or too many boundary conditions can be imposed with a corresponding loss of well-posedness. In particular, connecting approximations of the components to obtain a system level approximation can produce discretized models that are not consistent with the system level dynamics. In particular, coupling a discretization of the hyperbolic operator A_1 to the ODE system will not necessarily produce a convergent numerical scheme.

One implication of the analysis in the previous section is the observation that approximation methods developed for delay equations might be useful in the development of convergent approximations of coupled PDE-ODE systems. As observed in Section IV one can decompose the system operator into

$$\mathscr{A} = \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}$$

and construct convergent approximation schemes for the (possibly unbounded) operators

$$A_0^N \to A_0, \;\; A_1^N \to A_1$$

and the bounded operators

$$E^N \to E.$$

VII. Conclusions and Next Steps

There are two key issues to be considered when connecting hybrid ODE-ODE models into a system level model. The first is that establishing well-posedness of such interconnected systems is not always obvious. In particular, it is important to identify the correct boundary conditions. Establishing well-posedness allows for a proper composition of discretized models and can enable the development of higher order schemes. Initial results along this line may be found in the papers [17], [18], [21]. The basic ideas are to use combined finite element-finite volume methods from [22] and [23] to construct higher order "DG" type schemes (see [22]). Details about these schemes can be found in [21], [24].

Finally, we have conducted several computational experiments on the simple examples discussed here and these numerical results match the conclusions above. We are now in the process of extending these ideas and computational algorithms to more realistic systems and these results will appear in a forthcoming full paper.

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References

[1] X.-D. He, S. Liu, H. H. Asada, and H. Itoh, "Multivariable control of vapor compression systems," *HVAC&R Research*, vol. 4, no. 3, pp. 205–230, 1998.

[2] H. Pangborn, A. G. Alleyne, and N. Wu, "A comparison between finite volume and switched moving boundary approaches for dynamic vapor compression system modeling," *International Journal of Refrigeration*, vol. 53, pp. 101–114, 2015.

[3] B. P. Rasmussen and A. G. Alleyne, "Control-oriented modeling of transcritical vapor compression systems," *Journal of dynamic systems, measurement, and control*, vol. 126, no. 1, pp. 54–64, 2004.

[4] B. P. Rasmussen, "Dynamic modeling for vapor compression systemspart i: Literature review," *HVAC&R Research*, vol. 18, no. 5, pp. 934–955, 2012.

[5] J. Catano, T. Zhang, J. T. Wen, M. K. Jensen, and Y. Peles, "Vapor compression refrigeration cycle for electronics cooling–part i: Dynamic modeling and experimental validation," *International Journal of Heat and Mass Transfer*, vol. 66, pp. 911–921, 2013.

[6] C. R. Laughman, H. Qiao, V. Aute, and R. Radermacher, "A comparison of transient heat pump cycle models using alternative flow descriptions," *Science and Technology for the Built Environment*, vol. 21, no. 5, pp. 666–680, 2015.

[7] G. Chen and R. Grimmer, "Semigroups and integral equations," *The Journal of Integral Equations*, pp. 133–154, 1980.

[8] A. Bacciotti and G. Beccari, "A version of hautus' test for tandem connection of linear systems," *Automatic Control, IEEE Transactions on*, vol. 40, no. 1, pp. 111–114, 1995.

[9] J. A. Burns and L. Zietsman, "On the inclusion of actuator dynamics in boundary control of distributed parameter systems," *IFAC Proceedings Volumes*, vol. 45, no. 19, pp. 138–142, 2012.

[10] ——, "Control of PDE systems with delays," in *Control of Systems Governed by Partial Differential Equations*, vol. 1, no. 1, 2013, pp. 79–84.

[11] ——, "Control of a thermal fluid heat exchanger with actuator dynamics," in *55th IEEE Conference on Decision and Control (CDC), Las Vegas, NV*. IEEE, 2016, pp. 3131–3136.

[12] R. B. Vinter, "Semigroups on product spaces with application to initial value problems with non-local boundary conditions," in *Control of Distributed Parameter Systems: Proceeding of the Second IFAC Symposium, Coventry Great Britain, 28 June-1 July, 1977*, vol. 2, 1978, pp. 91–98.

[13] J. Zabczyk, "On decomposition of generators," *SIAM Journal on Control and Optimization*, vol. 16, no. 4, pp. 523–534, 1978.

[14] ——, "Erratum: On decomposition of generators," *SIAM Journal on Control and Optimization*, vol. 18, no. 3, pp. 325–325, 1980.

[15] J. Burns, T. Herdman, and H. Stech, "Linear functional differential equations as semigroups on product spaces," *SIAM Journal on Mathematical Analysis*, vol. 14, no. 1, pp. 98–116, 1983.

[16] J. Zabczyk, "A semigroup approach to boundary value control," in *Control of Distributed Parameter Systems: Proceeding of the Second IFAC Symposium, Coventry Great Britain, 28 June-1 July, 1977*, 1978, pp. 99–107.

[17] J. A. Burns, T. L. Herdman, and L. Zietsman, "Approximating parabolic boundary control problems with delayed actuator dynamics," in *American Control Conference (ACC), 2013*, 2013, p. to appear.

[18] J. A. Burns and E. M. Cliff, "Numerical methods for optimal control of heat exchangers," in *2014 American Control Conference*, 2014, pp. 1649–1654.

[19] T. Kato, *Perturbation Theory for Linear Operators*. New York: Springer-Verlag, 1980.

[20] K. Gustafson and G. Lumer, "Multiplicative perturbation of semigroup generators," *Pacific Journal of Mathematics*, vol. 41, no. 3, pp. 731–742, 1972.

[21] J. A. Burns, E. M. Cliff, and T. L. Herdman, "Identification of dynamical systems with structured uncertainty," *Inverse Problems in Science and Engineering*, pp. 1–42, 2017.

[22] P. Deuring, R. Eymard, and M. Mildner, "L^2-Stability Independent of Diffusion for a Finite Element–Finite Volume Discretization of a Linear Convection-Diffusion Equation," *SIAM Journal on Numerical Analysis*, vol. 53, no. 1, pp. 508–526, 2015.

[23] P. Deuring and M. Mildner, "Stability of a combined finite element-finite volume discretization of convection-diffusion equations," *Numerical Methods for Partial Differential Equations*, vol. 28, no. 2, pp. 402–424, 2012.

[24] J. A. Burns and E. M. Cliff, "Control of hyperbolic pde systems with actuator dynamics," in *IEEE 53rd Conference on Decision and Control*, 2014, pp. 2864–2869.