23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

# On the state space realization of 2D (2,2)-periodic image behaviors

JC Aleixo[1,2], D Napp[3], R Pereira[3], R Pinto[3], and P Rocha[1]

***Abstract*— In this paper we consider 2D behaviors with periodic image representations and provide conditions under which a simple method for obtaining state space realizations by means of 2D periodic (separable) Roesser models can be applied. For the sake of simplicity we restrict our attention to the $(2,2)$-periodic case.**

***Index Terms*— Periodic 2D systems, image behaviors, realizations.**

## I. Introduction

The state space realization problem for periodic 1D systems has been studied by several authors, both in the classical transfer function or input/output framework [3], [4], [5], [8], [10] and within the behavioral approach [2]. However, there are few results available for multidimensional (nD) systems, [1]. The aim of this paper is to further contribute to the study of this problem for the particular case of 2D behavioral systems.

We focus our attention on discrete 2D systems whose behaviors can be described as the image of special polynomial operators in the (inverse) 2D shifts with periodically varying coefficients. Our aim is to obtain an equivalent description of such behaviors by means of periodic 2D Roesser state space models of the separable type, i.e., where one of the states (in our case the horizontal state) has an evolution which is independent from the other one (here, the vertical state).

A naive approach would be try to construct the periodic 2D state space realization by combining the invariant state space realizations of each of the different invariant operators obtained by "freezing" the coefficients of the original periodic operator. Unfortunately, as we here show, this procedure does not work for all 2D periodic polynomial operators. An alternative approach is to construct a lifted invariant version of the original periodic behavior, obtain an invariant state space realization by standard procedures, and then try to obtain a periodic state space realization from the lifted invariant one. However, this is a very difficult task [1]. Given this situation, the search for conditions on the periodically varying polynomial operators that ensure that the aforementioned naive approach actually works becomes an interesting question. In this paper we concentrate on the (2,2)-periodic case, and give an answer to this question for a very particular type of 2D periodic polynomial operators that can be factored as the product of two periodic 1D polynomial operators associated to full column rank polynomial matrices. However, we conjecture that our main result may be extended to other cases.

[1]SYSTEC, Faculty of Engineering, University of Porto, Portugal, mprocha@fe.up.pt
[2] Department of Mathematics, University of Beira Interior, Portugal, jcaleixo@ubi.pt
[3]CIDMA, Department of Mathematics, University of Aveiro, Portugal, diego@ua.pt, ricardopereira@ua.pt, raquel@ua.pt

## II. 2D Periodic Image Behaviors

2D periodic image behaviors are sets of signals that can be described as the image of a periodically varying polynomial operator in the inverse 2D shifts, $M_{(Pk+i,Ql+j)}(\sigma_1^{-1},\sigma_2^{-1})$, where $P$ and $Q$ stand for the horizontal and for the vertical period, respectively. More concretely,

$$M_{(Pk+i,Ql+j)}(\sigma_1^{-1},\sigma_2^{-1}) = M_{(i,j)}(\sigma_1^{-1},\sigma_2^{-1}),$$

for $i=0,\dots,P-1$, $j=0,\dots,Q-1$, $l,k\in\mathbb{N}_0$, and where, for a function $w$ defined over $\mathbb{Z}^2$, the action of the inverse 2D shifts is given by $(\sigma_1^{-1}w)(i,j)=w(i-1,j)$ (for the horizontal direction) and $(\sigma_2^{-1}w)(i,j)=w(i,j-1)$ (for the vertical direction).
We consider a special class of periodically varying 2D polynomial operators $M_{(Pk+i,Ql+j)}(\sigma_1^{-1},\sigma_2^{-1})$ that can be factored as:

$$M_{(Pk+i,Ql+j)}(\sigma_1^{-1},\sigma_2^{-1}) = V_{Ql+j}(\sigma_2^{-1})H_{Pk+i}(\sigma_1^{-1}),$$

where $H_{Pk+i}(\sigma_1^{-1})$ and $V_{Ql+j}(\sigma_2^{-1})$ are periodically varying 1D polynomial operators in the horizontal and in the vertical directions, with period $P$ and $Q$, respectively, i.e., for $k\in\mathbb{N}_0$

$$H_{Pk+i}(\sigma_1^{-1}) = H_i(\sigma_1^{-1}),\ i=0,\dots,P-1,$$

and, for $l\in\mathbb{N}_0$,

$$V_{Ql+j}(\sigma_2^{-1}) = V_j(\sigma_2^{-1}),\ j=0,\dots,Q-1.$$

Moreover, for the sake of simplicity we take $P=Q=2$.

Thus our object of study are behaviors $\mathcal{B}$ such that:

$$\begin{aligned}&\mathcal{B}=\{w\in\mathcal{W}:\ \exists v\in\mathcal{V}\ s.t.\\ &w(2k+i,2l+j)=\big(V_j(\sigma_2^{-1})H_i(\sigma_1^{-1})v\big)(2k+i,2l+j),\\ &k,l\in\mathbb{N}_0, i,j=0,1\},\end{aligned} \tag{1}$$

where $\mathcal{W}$ and $\mathcal{V}$ are the sets of signals $\big(\mathbb{R}^q\big)^{\mathbb{Z}^2}$ and $\big(\mathbb{R}^p\big)^{\mathbb{Z}^2}$, respectively, with support in $\mathbb{N}_0^2$, and $H_i(z_1^{-1})\in\mathbb{R}^{r\times p}[z_1^{-1}]$, $V_j(z_2^{-1})\in\mathbb{R}^{q\times r}[z_2^{-1}]$, $i,j=0,1$, are polynomial matrices in $z_1^{-1}$ and $z_2^{-1}$ of sizes $r\times p$ and $q\times r$, respectively. In this case

$$\big(M_{(0,0)},M_{(1,0)},M_{(0,1)},M_{(1,1)}\big) = (V_0H_0,V_0H_1,V_1H_0,V_1H_1)$$

is said to be a 2D (2,2)-periodic image representation of $\mathcal{B}$. We assume that the polynomial matrices $H_i$ and $V_j$ have full column rank (over the corresponding polynomial rings).

## III. STATE SPACE REALIZATIONS

Here we focus on the state space realizations of the special class of 2D periodic image behaviors introduced in the previous section by means of 2D periodic Roesser models. In general, this is a nontrivial matter, mainly due to the fact that a 2D periodic state space realization cannot be obtained by independently realizing each of the invariant polynomial operators $M_{(i,j)} = V_j H_i$, [1]. However, in this paper we show that under certain conditions this problem does not arise, i.e., combining independent realizations of the invariant operators $M_{(i,j)}$ does yield a 2D periodic realization of the corresponding 2D periodic image behavior. Before presenting our result, we first consider the invariant 2D case as well as the periodic 1D case.

### A. The invariant 2D case

As is well-known, in the 2D invariant case, a separable Roesser model realization for a behavior which is the image of a 2D polynomial operator

$$M(\sigma_1^{-1}, \sigma_2^{-1}) = V(\sigma_2^{-1})H(\sigma_1^{-1})$$

can be obtained as the series connection of the 1D state space realizations of $H$ and $V$. Indeed, if $(A^H, B^H, \tilde{C}^H, D^H)$ and $(A^V, \tilde{B}^V, C^V, D^V)$ are respectively state space realizations of $H\left(z_1^{-1}\right)$ and $V\left(z_2^{-1}\right)$, now regarded as finite impulse response transfer functions, then the separable Roesser model $\Sigma = (A^H, A^V, A^{VH}, B^H, B^V, C^H, C^V, D)$:

$$\begin{cases} \sigma_1 x^H(t_1,t_2) &= A^H x^H(t_1,t_2) + B^H v(t_1,t_2) \\ \sigma_2 x^V(t_1,t_2) &= A^V x^V(t_1,t_2) + A^{VH} x^H(t_1,t_2) \\ & \quad +B^V v(t_1,t_2) \\ w(t_1,t_2) &= C^H x^H(t_1,t_2) + C^V x^V(t_1,t_2) \\ & \quad +Dv(t_1,t_2) \end{cases} \tag{2}$$

with $A^{VH} = \tilde{B}^V \tilde{C}^H$, $B^V = \tilde{B}^V D^H$, $C^H = D^V \tilde{C}^H$, and $D = D^V D^H$, is a realization of $\mathcal{B} = \mathrm{im}(M)$ in the sense that the signals in $\mathcal{B}$ corresponding to $v$ coincide with the outputs of (2) produced by the same input $v$ with zero initial conditions, i.e., $x^H(0,t_2) = 0$ and $x^V(t_1,0) = 0$, $t_1, t_2 \in \mathbb{N}_0$.

*Remark 1:* Note that, due to the fact that $H$ and $V$ are polynomial operators in $\sigma_1^{-1}$ and $\sigma_2^{-1}$, respectively, it is always possible to construct 2D separable Roesser models with the same input/output behavior as $M$ where the horizontal state at the point $(t_1,t_2)$, $x^H(t_1,t_2)$, only depends on the values of $v$ at (some) points $(t_1-\tau_1, t_2)$, with $\tau_1 \geq 1$, and the vertical state $x^V(t_1,t_2)$ only depends on the values of $v$ at (some) points $(t_1-\tau_1, t_2-\tau_2)$, with $\tau_1 \geq 0$ and $\tau_2 \geq 1$. As a consequence, the initial states $x^H(0,t_2)$ and $x^V(t_1,0)$ corresponding to a signal $v \in \mathcal{V}$ (which has support in $\mathbb{N}_0^2$) are clearly zero. Here only such models are considered to be realizations of $\mathcal{B} = \mathrm{im}(M)$.

In a similar way, under certain conditions, in the 2D periodic case, a periodic separable Roesser model realization can be obtained as a series connection of two 1D periodic state space realizations of the periodic operators $H_{2k+i}$ and $V_{2l+j}$, $i,j = 0,1$. It is therefore important to first analyse the 1D case.

### B. The periodic 1D case

A periodic 1D image behavior (with period 2) is a set of signals $\mathcal{B}$ that can be described as:

$$\mathcal{B} = \{w \in \mathcal{U} : \; \exists \ell \in \mathcal{L} \; s.t. \; w(2\theta+\tau) = \left(M_\tau(\sigma^{-1})\ell\right)(\tau), \\ \theta \in \mathbb{N}_0, \tau = 0,1\}, \tag{3}$$

where $\sigma$ denotes the 1D shift, $\mathcal{U}$ and $\mathcal{L}$ are the sets of signals $(\mathbb{R}^q)^{\mathbb{Z}}$ and $(\mathbb{R}^s)^{\mathbb{Z}}$, respectively, with support in $\mathbb{N}_0$, and $M_\tau(z^{-1})$, $\tau = 0,1$, are 1D polynomial matrices in $z^{-1}$ of suitable size. In this case $(M_0, M_1)$ is said to be a 2-periodic image representation of $\mathcal{B}$.

On the other hand, given two 1D state space systems $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$, $\theta = 0,1$, with the same state dimension, we define a *2-periodic 1D state space system* $\Sigma_{\text{per}}^{1D}$ as

$$\begin{cases} \sigma x(t) &= A(t)x(t) + B(t)\ell(t) \\ w(t) &= C(t)x(t) + D(t)\ell(t) \end{cases}, \; t \in \mathbb{Z} \tag{4}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are periodic functions with period 2, such that, for each $\theta \in \mathbb{N}_0$,

$$\left(A(2\theta), B(2\theta), C(2\theta), D(2\theta)\right) = (A_0, B_0, C_0, D_0)$$

and

$$\left(A(2\theta+1), B(2\theta+1), C(2\theta+1), D(2\theta+1)\right) = \\ = (A_1, B_1, C_1, D_1).$$

The dimension of $\Sigma_{\text{per}}^{1D}$ is defined as the dimension of the state vector $x$. In this case we say that $\Sigma_{\text{per}}^{1D}$ is obtained from $\Sigma_0$ and $\Sigma_1$, and write $\Sigma_{\text{per}}^{1D} = (\Sigma_0, \Sigma_1)$.

Moreover, $\Sigma_{\text{per}}^{1D}$ is a realization of a 2-periodic image representation $(M_0, M_1)$ and of the associated 2-periodic image behavior, if the output $w$ of $\Sigma_{\text{per}}^{1D}$ that corresponds to an input $\ell$ and zero initial conditions, i.e., $x(0) = 0$, equals the trajectory $w$ corresponding to $\ell$ according to (3).

As already mentioned, given two realizations $\Sigma_0$ and $\Sigma_1$ of $M_0$ and $M_1$, the periodic state space system $\Sigma_{\text{per}}^{1D} = (\Sigma_0, \Sigma_1)$ obtained from $\Sigma_0$ and $\Sigma_1$ is in general not a periodic state space realization of the 2-periodic image representation $(M_0, M_1)$, nor of the associated periodic behavior. However, if the polynomial matrices $M_0$ and $M_1$ have the same column degrees, it is possible to construct invariant 1D state space realizations $\Sigma_0$ and $\Sigma_1$ such that the 2-periodic state space system $\Sigma_{\text{per}}^{1D} = (\Sigma_0, \Sigma_1)$ is indeed a realization of the 1D 2-periodic behavior associated with $(M_0, M_1)$. Such realizations are obtained as stated in the next proposition, [6], [7].

Recall that the degree of a column is defined as the maximum of the degrees of its entries. Note that such degrees are considered for the indeterminate $z^{-1}$. Therefore, for instance, $z^{-2}$ has degree 2.

*Proposition 1:* Let $M\left(z^{-1}\right) \in \mathbb{R}^{q\times s}\left[z^{-1}\right]$ be a polynomial matrix with rank $s$ and column degrees $\nu_1,\ldots,\nu_s$. Consider $\bar{n}=\sum_{i=1}^{s}\nu_i$. Let $M\left(z^{-1}\right)$ have columns $m_i\left(z^{-1}\right)=\sum_{k=0}^{\nu_i} m_{k,i}z^{-k}$, $i=1,\ldots,s$, where $m_{k,i}\in\mathbb{R}^q$. For $i=1,\ldots,s$ define the matrices

$$A_i=\begin{bmatrix}0&\cdots&\cdots&0\\1&&&\vdots\\&\ddots&&\vdots\\&&1&0\end{bmatrix}\in\mathbb{R}^{\nu_i\times\nu_i},\ B_i=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\in\mathbb{R}^{\nu_i},$$

$$C_i=\begin{bmatrix}m_{1,i}&\cdots&m_{\nu_i,i}\end{bmatrix}\in\mathbb{R}^{q\times\nu_i}.$$

Then a state-space realization of $M$ is given by the matrix quadruple $(A,B,C,D)\in\mathbb{R}^{\bar{n}\times\bar{n}}\times\mathbb{R}^{\bar{n}\times s}\times\mathbb{R}^{q\times\bar{n}}\times\mathbb{R}^{q\times s}$ where

$$A=\begin{bmatrix}A_1&&\\&\ddots&\\&&A_s\end{bmatrix},\ B=\begin{bmatrix}B_1&&\\&\ddots&\\&&B_s\end{bmatrix},$$

$$C=\begin{bmatrix}C_1&\cdots&C_s\end{bmatrix},\ D=\begin{bmatrix}m_{0,1}&\cdots&m_{0,s}\end{bmatrix}=M(0).$$

In the case where $\nu_i=0$ the $ith$ block of $A$ and $C$ are void and in $B$ a zero column occurs.

*Theorem 1:* [9] Consider two image representations $M_0\left(z^{-1}\right)\in\mathbb{R}^{q\times s}\left[z^{-1}\right]$ and $M_1\left(z^{-1}\right)\in\mathbb{R}^{q\times s}\left[z^{-1}\right]$ with the same column degrees and let $\Sigma_i$ be the realizations of $M_i\left(z^{-1}\right), i=0,1$, obtained by Proposition 1. Then, the 1D periodic state-space system $\Sigma_{\text{per}}^{1D}$ obtained from $\Sigma_0$ and $\Sigma_1$ is a realization of the periodic image representation $(M_0,M_1)$ and of the corresponding 2-periodic 1D image behavior.

### C. The periodic 2D case

Let

$$\left(M_{(0,0)},M_{(1,0)},M_{(0,1)},M_{(1,1)}\right)=\left(V_0H_0,V_0H_1,V_1H_0,V_1H_1\right)$$

define a (2,2)-periodic 2D polynomial operator, and let further $\Sigma_i^H=(A_i^H,B_i^H,\tilde{C}_i^H,D_i^H)$ and $\Sigma_j^V=(A_j^V,\tilde{B}_j^V,C_j^V,D_j^V)$ be state space realizations of the invariant operators $H_i$ and $V_j$, $i,j=0,1$, respectively. Assume that $\Sigma_0^H$ and $\Sigma_1^H$ have the same state dimensions and that the same happens for $\Sigma_0^V$ and $\Sigma_1^V$. Combining these realizations yields the following (2,2)-periodic 2D separable Roesser state space system $\Sigma_{\text{per}}^{2D}$:

$$\begin{aligned}\begin{bmatrix}\sigma_1x^H(2k{+}i,2l{+}j)\\\sigma_2x^V(2k{+}i,2l{+}j)\end{bmatrix}&=\begin{bmatrix}A_i^H&0\\A_{ij}^{VH}&A_j^V\end{bmatrix}\begin{bmatrix}x^H(2k{+}i,2l{+}j)\\x^V(2k{+}i,2l{+}j)\end{bmatrix}\\&\quad+\begin{bmatrix}B_i^H\\B_{ij}^V\end{bmatrix}v(2k+i,2l+j)\\w(2k+i,2l+j)&=\begin{bmatrix}C_{ij}^H&C_j^V\end{bmatrix}\begin{bmatrix}x^H(2k+i,2l+j)\\x^V(2k+i,2l+j)\end{bmatrix}\\&\quad+D_{ij}v(2k+i,2l+j)\end{aligned}\tag{5}$$

with $A_{ij}^{VH}=\tilde{B}_j^V\tilde{C}_i^H$, $B_{ij}^V=\tilde{B}_j^VD_i^H$, $C_{ij}^H=D_j^V\tilde{C}_i^H$, and $D_{ij}=D_j^VD_i^H$.

Note that for each pair of *fixed* values of $i$ and $j$ this peridodic 2D system is an invariant separable 2D state space system

$$\Sigma_{(i,j)}=\left(A_i^H,A_j^V,A_{ij}^{VH},B_i^H,B_{ij}^V,C_{ij}^H,C_j^V,D_{ij}\right).$$

Similar to what happens in the 1D case, we say that $\Sigma_{\text{per}}^{2D}$ is obtained from $\Sigma_{(0,0)}$, $\Sigma_{(1,0)}$, $\Sigma_{(0,1)}$ and $\Sigma_{(1,1)}$ and write $\Sigma_{\text{per}}^{2D}=\left(\Sigma_{(0,0)},\Sigma_{(1,0)},\Sigma_{(0,1)},\Sigma_{(1,1)}\right)$.

As shown in the following example the 2D (2,2)-periodic Roesser state space system $\Sigma_{\text{per}}^{2D}=\left(\Sigma_{(0,0)},\Sigma_{(1,0)},\Sigma_{(0,1)},\Sigma_{(1,1)}\right)$ is not necessarily a realization of the (2,2)-periodic image representation

$$\left(M_{(0,0)},M_{(1,0)},M_{(0,1)},M_{(1,1)}\right)=\left(V_0H_0,V_0H_1,V_1H_0,V_1H_1\right).$$

*Example 1:* Consider the (2-2)-periodic image representation

$$\left(M_{(0,0)},M_{(1,0)},M_{(0,1)},M_{(1,1)}\right)=\left(V_0H_0,V_0H_1,V_1H_0,V_1H_1\right)$$

with

$$\begin{aligned}H_0\left(z_1^{-1}\right)&=H_0^0+H_0^1z_1^{-1}+H_0^2z_1^{-2}\\&=\begin{bmatrix}1+z_1^{-2}&1&0\\z_1^{-2}&1+z_1^{-1}&1\\1+z_1^{-1}&1&1\\1&1&1+z_1^{-1}\end{bmatrix},\end{aligned}$$

$$\begin{aligned}H_1\left(z_1^{-1}\right)&=H_1^0+H_1^1z_1^{-1}+H_1^2z_1^{-2}\\&=\begin{bmatrix}1+z_1^{-1}&1&0\\1+z_1^{-2}&1+z_1^{-1}&1\\1&1+z_1^{-2}&1\\0&1&1\end{bmatrix},\end{aligned}$$

$$V_0\left(z_2^{-1}\right)=\left(1+z_2^{-1}\right)I_4\ \text{ and }\ V_1\left(z_2^{-1}\right)=\left(1+2z_2^{-1}\right)I_4.$$

Realizing $H_0\left(z_1^{-1}\right)$ as in Proposition 1 we obtain the state-space realization $\Sigma_0^H=(A_0^H,B_0^H,\tilde{C}_0^H,D_0^H)$ with

$$A_0^H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B_0^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{C}_0^H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D_0^H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Proceeding in the same way, we obtain a state-space realization $\Sigma_1^H = (A_1^H, B_1^H, \tilde{C}_1^H, D_1^H)$ for $H_1\left(z_1^{-1}\right)$ with

$$A_1^H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B_1^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{C}_1^H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D_1^H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

As for $V_0\left(z_2^{-1}\right)$ and $V_1\left(z_2^{-1}\right)$, it is easily seen that they can be realized by $\Sigma_0^V = (A_0^V, \tilde{B}_0^V, C_0^V, D_0^V)$ and $\Sigma_1^V = (A_1^V, \tilde{B}_1^V, C_1^V, D_1^V)$ with

$$A_0^V = 0_4,\ \tilde{B}_0^V = C_0^V = D_0^V = I_4$$

and

$$A_1^V = 0_4,\ \tilde{B}_0^V = D_0^V = I_4,\ C_0^V = 2I_4,$$

where $0_4$ denotes the $4 \times 4$ zero matrix.

Let us consider, for every $t_2 \in \mathbb{N}_0$,

$$v(0,t_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ v(1,t_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ v(t_1,t_2) = 0, t_1 \geq 2.$$

From (1) it follows that, for $l \in \mathbb{N}_0$, $j = 0,1$,

$$\begin{aligned} w(1,2l+j) &= \left(V_j(\sigma_2^{-1})H_1(\sigma_1^{-1})v\right)(1,2l+j) \\ &= \left(V_j(\sigma_2^{-1})u\right)(1,2l+j) \end{aligned}$$

where

$$\begin{aligned} u(1,2l+j) &= H_1^0 v(1,2l+j) + H_1^1 v(0,2l+j) \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

thus

$$w(1,2l+j) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, l \in \mathbb{N}_0, j = 0,1$$

or simply

$$w(1,t_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{for } t_2 \in \mathbb{N}_0.$$

On the other hand, using (5), we have

$$w(1,0) = \begin{bmatrix} C_{10}^H & C_0^V \end{bmatrix} \begin{bmatrix} x^H(1,0) \\ x^V(1,0) \end{bmatrix} + D_{10}v(1,0)$$

Note that, due to the fact that the initial conditions must be zero (according to our definition of realization), $x^V(1,0) = 0$ and $x^H(0,0) = 0$. Moreover,

$$\begin{aligned} x^H(1,0) &= \left(\sigma_1 x^H\right)(0,0) = A_0^H x^H(0,0) + B_0^H v(0,0) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} w(1,0) &= C_{10}^H x^H(1,0) = D_0^V \tilde{C}_1^H x^H(1,0) \\ &= I_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \end{aligned}$$

i.e., the output $w$ of the 2D (2,2)-periodic Roesser state space system $\Sigma_{\text{per}}^{2D}$ corresponding to $v$ is different from the trajectory $w$ corresponding to $v$ according to (1). ☐

However, in the next theorem is shown that it is possible to obtain periodic 2D separable Roesser model realizations for 2D periodic image behaviors (1), by independently realizing the operators $V_0$, $V_1$, and $H_0$, $H_1$, provided that $V_0$ and $V_1$ have the same column degress and the same happens for $H_0$ and $H_1$.

*Theorem 2:* Consider the polynomial operators in $\sigma_1^{-1}$ corresponding to the polynomial matrices $H_0\left(z_1^{-1}\right) \in \mathbb{R}^{q\times r}\left[z_1^{-1}\right]$ and $H_1\left(z_1^{-1}\right) \in \mathbb{R}^{q\times r}\left[z_1^{-1}\right]$, and assume that they have the same column degrees. Let $\Sigma_i^H$ be the realizations of $H_i\left(\sigma_1^{-1}\right), i = 0,1$, obtained by Proposition 1. Consider further the polynomial operators in $\sigma_2^{-1}$ corresponding to the polynomial matrices $V_0\left(z_2^{-1}\right) \in \mathbb{R}^{r\times p}\left[z_2^{-1}\right]$ and $V_1\left(z_2^{-1}\right) \in \mathbb{R}^{r\times p}\left[z_2^{-1}\right]$, and assume that they have the same column degrees. Let $\Sigma_j^V$ be the realizations of $V_j\left(\sigma_2^{-1}\right), j = 0,1$, obtained by Proposition 1. Define the 2D periodic Roesser separable model $\Sigma_{\text{per}}^{2D}$ obtained from $\Sigma_i^H$ and $\Sigma_j^V$ as in (5). Then $\Sigma_{\text{per}}^{2D}$ is a state space realization of the 2D periodic image behavior $\mathcal{B}$ given by (1).

*Proof:*
Note that since $H_0(z_1^{-1})$ and $H_1(z_1^{-1})$ have the same column degrees it follows that the corresponding realizations $\Sigma_i^H = (A_i^H, B_i^H, \tilde{C}_i^H, D_i^H)$, $i = 0,1$, are such that $A_0^H = A_1^H$ and $B_0^H = B_1^H$. Let us consider $A^H := A_0^H = A_1^H$ and $B^H := B_0^H = B_1^H$. By the same reason the realizations $\Sigma_j^V = (A_j^V, \tilde{B}_j^V, C_j^V, D_j^V)$, $j = 0,1$, are such that $A_0^V = A_1^V$ and $\tilde{B}_0^V = \tilde{B}_1^V$. Let $A^V := A_0^V = A_1^V$ and $\tilde{B}^V := \tilde{B}_0^V = \tilde{B}_1^V$.

Then, after simple, but cumbersome computations, one concludes that the output $w$ of $\Sigma_{per}^{2D} = (\Sigma_{(0,0)}, \Sigma_{(1,0)}, \Sigma_{(0,1)}, \Sigma_{(1,1)})$, with

$$\Sigma_{(i,j)} = \Big(A^H, A^V, A_{ij}^{VH} = \tilde{B}^V \tilde{C}_i^H, B^H, B_{ij}^V = \tilde{B}^V D_i^H, C_{ij}^H = D_j^V \tilde{C}_i^H, C_j^V, D_{ij}\Big)$$

that corresponds to the input $v$ and zero initial conditions $\big(x^H(0,t_2) = 0, x^V(t_1,0) = 0\big)$ is such that, for $l,k \in \mathbb{N}_0$, $i,j = 0,1$,

$$\begin{aligned} w(2k+i, 2l+j) &= D_j^V D_i^H v(2k+i, 2l+j) \\ &+ \sum_{t_1 \geq 1} C_{ij}^H (A^H)^{t_1-1} B^H v(2k+i-t_1, 2l+j) \\ &+ \sum_{t_2 \geq 1} C_j^V (A^V)^{t_2-1} B_{ij}^V v(2k+i, 2l+j-t_2) \\ &+ \sum_{\substack{t_1 \geq 1 \\ t_2 \geq 1}} C_j^V (A^V)^{t_2-1} A_{ij}^{VH} (A^H)^{t_1-1} B^H v(2k+i-t_1, 2l+j-t_2). \end{aligned}$$

Let us now show that the trajectory

$$\tilde{w} = \big(V_j(\sigma_2^{-1}) H_i(\sigma_1^{-1})\big) v$$

equals $w$. For that, note that since $\Sigma_0^H = (A^H, B^H, \tilde{C}_0^H, D_0^H)$ is a realization of $H_0$ we have that

$$H_0(z_1^{-1}) = D_0^H + \sum_{t_1 \geq 1} \tilde{C}_0^H (A^H)^{t_1-1} B^H z_1^{-t_1}.$$

In the same way

$$H_1(z_1^{-1}) = D_1^H + \sum_{t_1 \geq 1} \tilde{C}_1^H (A^H)^{t_1-1} B^H z_1^{-t_1},$$

$$V_0(z_2^{-1}) = D_0^V + \sum_{t_2 \geq 1} C_0^V (A^V)^{t_2-1} \tilde{B}^V z_2^{-t_2}.$$

and

$$V_1(z_2^{-1}) = D_1^V + \sum_{t_2 \geq 1} C_1^V (A^V)^{t_2-1} \tilde{B}^V z_2^{-t_2}.$$

Thus

$$\tilde{w}(2k+i, 2l+j) = \sum_{\substack{0 \leq t_1 \leq 2k+i \\ 0 \leq t_2 \leq 2l+j}} M(i,j) v(2k+i-t_1, 2l+j-t_2)$$

where $M(i,j)$ is the coefficient of $z_1^{-i} z_2^{-j}$ of the polynomial matrix in $z_1^{-1}$ and $z_2^{-1}$, $V_j(z_2^{-1}) H_i(z_1^{-1})$. It is not difficult to check that

$$\begin{aligned} V_j(z_2^{-1}) H_i(z_1^{-1}) &= D_j^V D_i^H + \sum_{t_1 \geq 1} D_j^V \tilde{C}_i^H (A^H)^{t_1-1} B^H z_1^{-t_1} \\ &+ \sum_{t_2 \geq 1} C_j^V (A^V)^{t_2-1} \tilde{B}^V D_i^H z_2^{-t_2} \\ &+ \sum_{\substack{t_1 \geq 1 \\ t_2 \geq 1}} C_j^V (A^V)^{t_2-1} \tilde{B}^V \tilde{C}_i^H (A^H)^{t_1-1} B^H z_1^{-t_1} z_2^{-t_2}. \end{aligned}$$

Taking into account that $C_{ij}^H = D_j^V \tilde{C}_i^H$, $B_{ij}^V = \tilde{B}^V D_i^H$ and $A_{ij}^{VH} = \tilde{B}^V \tilde{C}_i^H$, this allows to conclude that $\tilde{w} = w$. ■

## IV. Conclusions

In this paper we have studied the state space realization problem for periodic 2D behavioral systems. Conditions were provided under which a simple method for obtaining state space realizations by means of 2D periodic (separable) Roesser models can be implemented. For that purpose we have assumed some relevant polynomial operators to have the same column degrees. Although this requirement is very restrictive, our conviction is that it can be relaxed. Thus, the case where the relevant polynomial operators have different column degrees will be investigated next. Future work also includes the investigation of minimality issues for the obtained realizations.

## Acknowledgement

This work was supported in part by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013. Moreover, it was also partially supported by: Project POCI-01-0145-FEDER-006933 - SYSTEC - Research Center for Systems and Technologies - funded by FEDER funds through COMPETE2020 - Programa Operacional Competitividade e Internacionalização (POCI) - and by national funds through FCT - Fundação para a Ciência e a Tecnologia.

## References

[1] J. C. Aleixo and P. Rocha, "Roesser model representation of 2d periodic behaviors: the (2,2)-periodic siso case," in *2017 10th International Workshop on Multidimensional (nD) Systems (nDS)*, Sept 2017, pp. 1–6.

[2] J. C. Aleixo, P. Rocha, and J. C. Willems, "State space representation of siso periodic behaviors," in *2011 50th IEEE Conference on Decision and Control and European Control Conference*, Dec 2011, pp. 1545–1550.

[3] S. Bittanti, P. Bolzern, L. Piroddi, and G. D. Nicolao, *Representation, prediction, and identification of cyclostationary processes. A state space approach.* New York, NY: IEEE, 1994, pp. 267 – 294.

[4] P. Colaneri and S. Longhi, "The realization problem for linear periodic systems," *Automatica*, vol. 31, no. 5, pp. 775 – 779, 1995. [Online]. Available: http://www.sciencedirect.com/science/article/pii/000510989400155C

[5] C. Coll, R. Bru, E. SÃ¡nchez, and V. HernÃ¡ndez, "Discrete-time linear periodic realization in the frequency domain," *Linear Algebra and its Applications*, vol. 203-204, pp. 301 – 326, 1994. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0024379594902070

[6] E. Fornasini and R. Pinto, "Matrix fraction descriptions in convolutional coding," *Linear Algebra and its Applications*, vol. 392, no. Supplement C, pp. 119 – 158, 2004. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0024379504002836

[7] H. Gluesing-Luerssen and G. Schneider, "State space realizations and monomial equivalence for convolutional codes," *Linear Algebra and its Applications*, vol. 425, no. 2, pp. 518 – 533, 2007, special Issue in honor of Paul Fuhrmann. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S002437950700122X

[8] C. A. Lin and C. W. King, "Minimal periodic realizations of transfer matrices," in *1992 American Control Conference*, June 1992, pp. 353–354.

[9] D. Napp, R. Pereira, R. Pinto, and P. Rocha, "Periodic state-space representations of periodic convolutional codes," *Cryptography and Communications*, submitted.

[10] E. SÃ¡nchez, V. HernÃ¡ndez, and R. Bru, "Minimal realizations for discrete-time linear periodic systems," *Linear Algebra and its Applications*, vol. 162-164, pp. 685 – 708, 1992. [Online]. Available: http://www.sciencedirect.com/science/article/pii/002437959290402V