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Asymptotically Stabilizing Controller Generating Sparse Input for Nonlinear Systems*

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***Abstract*— In this report, we propose an asymptotically stabilizing controller using a control Lyapunov function (clf). The proposed controller generates a sparse input vector, when no input constraint is active, for energy savings. For the cases with no input constraint, the control law can be described explicitly in a variant of Sontag-type controller. When some input constraints are subjected to the system, the control input can be obtained by solving an LP problem. The controller makes the time derivative of the clf negative, if possible. Otherwise, an input minimizing the time derivative of the clf is chosen, where all inputs are saturated. We have also proposed a chattering-suppression mechanism. The effectiveness of the proposed method is confirmed by computer simulations.**

I. Introduction

A vector whose most elements are zero is called ‘sparse.’ A sparse input vector implies that most actuators are not working, and these actuators can be deactivated. Deactivated actuators consume no standby power, and therefore the sparse input is effective for energy savings. Examples of the actuator deactivation are the cylinder deactivation system of automobile engines, and the on-off control of multiple pumps in chemical plants. Sparse property of input vector is useful in redundant-input systems. A redundant-input system includes multiple actuators having same effect to the system. Some of redundant actuators do not have to work always, and the sparse property of the input is preferred in redundant-input systems. For example, consider a system with two actuators, where one has a good energy efficiency, but its maximum power is limited, and the other is more powerful but it wastes more energy. Around the origin, the second actuator can be shutdown, while when large power is required, both the actuators should work. Sparse input is also preferred in systems with emergency actuators. For example, yaw-rate control systems of automobiles use several kinds of actuators — drive-torque split mechanism, breaking force vectoring, active aero vectoring, active camber-angle control, steering angle control, etc. However, nonzero camber-angle increases the side-slip angle of the vehicle, and an override of the steering angle harms drivers’ feeling. Hence, some of these actuators are emergency, and should be deactivated in usual situations.

Control using sparse input is often called ‘maximum hands-off control’ [1], [2], [3], [4], where a weighted one-norm or a CLOT (combined L-one and two) norm is adopted for the input-cost term in optimal control or model predictive control. However, in nonlinear system cases, optimal control scheme requires a solution of Hamilton-Jacobi partial differential equation, and nonlinear model predictive control must solve nonlinear programming online. On the other hand, control allocation techniques for redundant-input systems often use one-norm cost function, which may generate sparse input [5], [6], [7], [8]. In usual control allocations, when a desired value of the time derivative of the state has already designed, the redundant inputs are determined by solving an optimization problem. The computational complexity of the control allocation for input-affine nonlinear systems is same as one for linear system cases. However, in many cases the constraints for the time derivative of the state are restrictive. Due to the restriction, a dense (non-sparse) input vector often emerges as the result of the optimization.

In this study, an asymptotic stabilization controller, which generates sparse input vectors, for nonlinear systems. We adopt an intermediate approach between the maximum hands-off control and the control allocation technique. We assume that a control Lyapunov function (clf) is designed a priori, instead of the desired time derivative of the state in control-allocation methods. The time derivative of the state, which consists of multiple equations, is replaced by a constraint on the decreasing rate of the clf, which can be expressed by one inequality. The obtained control is given as a solution of a linear programming (LP) problem, which can be calculated efficiently online. If there is no input constraint, the control law is described explicitly as a variant of Sontag-type controller. When all the inputs are saturated, the the upper bound of the decreasing rate of the clf is weaken by multiplying a coefficient γ (≤ 1). Hence, the decision variables of the LP are the input variables and γ. We also obtain a condition on the penalty term for decreasing the value γ. Moreover, we propose a mechanism to suppress the chattering phenomenon, which is caused by the discontinuity of the optimized solution with respect to the state.

This report is organized as follows. The properties of the one-norm optimization problem are described in Section II. In Section III, the problem setting is stated, the Sontag-type control laws for the cases without input constraint, which generate sparse inputs, are proposed, and it is shown that the control laws can be expressed by LP problems. Section IV shows the controller under input constraints. A chattering-suppression mechanism is proposed in Section V. Simulation results for an example are given in Section VI to show the effectiveness of the proposed method. Section VII summarizes the obtained results.

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II. One-Norm Optimization

To obtain sparse solutions of an optimization problem, a weighted one-norm is often used as an evaluated function to be minimized. For example, consider an optimization problem finding $u = (u_1, \ldots, u_m)^\top$ ($\in \mathbb{R}^m$) that minimizes

$$J_1(u) = \tilde{k}_1|u_1| + \cdots + \tilde{k}_m|u_m| = \tilde{k}(|u_1|, \ldots, |u_m|)^\top$$

under a constraint $\ell u + a \le 0$, where $\tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_m)$, $\tilde{k}_1 > 0, \ldots, \tilde{k}_m > 0$, $\ell = (\ell_1, \ldots, \ell_m)$, and $a \in \mathbb{R}$. This problem can be converted into a linear programming (LP) problem by using slack variables $u_{\mathrm{abs},i}$ with additional constraints

$$u_i \le u_{\mathrm{abs},i}, \quad -u_i \le u_{\mathrm{abs},i} \quad (i = 1, \ldots, m). \tag{1}$$

One of the optimal solution can be obtained as

$$u = \begin{cases} -\dfrac{a}{\|\tilde{\ell}\|_\infty} \mathrm{diag}(\tilde{k})^{-1} b_s(\tilde{\ell})^\top & (a > 0 \text{ and } \ell \ne 0) \\ 0 & (a \le 0) \\ \text{no solution} & (a > 0 \text{ and } \ell = 0), \end{cases} \tag{2}$$

where $\mathrm{diag}(\cdot)$ generates a diagonal matrix,

$$\begin{aligned} \tilde{\ell} &= \ell \,\mathrm{diag}(\tilde{k})^{-1} = (\ell_1/\tilde{k}_1, \ldots, \ell_m/\tilde{k}_m) \\ b_s(\tilde{\ell}) &= \lim_{p \to +\infty} \frac{(\mathrm{sgn}(\tilde{\ell}_1)|\tilde{\ell}_1|^p, \ldots, \mathrm{sgn}(\tilde{\ell}_m)|\tilde{\ell}_m|^p)}{|\tilde{\ell}_1|^p + \cdots + |\tilde{\ell}_m|^p} \\ \mathrm{sgn}(q) &= \begin{cases} 1 & (q > 0) \\ 0 & (q = 0) \\ -1 & (p < 0), \end{cases} \end{aligned}$$

and $\|\cdot\|_\infty$ denotes the infinity norm, i.e. $\|(\ell_1, \ldots, \ell_m)\|_\infty = \max_i |\ell_i|$. If $i^* = \mathrm{argmax}_i |\tilde{\ell}_i|$ is determined uniquely, then $b_s(\tilde{\ell}) = (0, \ldots, 0, \mathrm{sgn}(\tilde{\ell}_{i^*}), 0, \ldots, 0)$. Note that

$$\tilde{\ell} b_s(\tilde{\ell})^\top = \|\tilde{\ell}\|_\infty, \tag{3}$$

and therefore (2) satisfies the constraint $\ell u + a \le 0$ when $a > 0$ and $\ell \ne 0$. If a solution exists uniquely and $a > 0$, then the optimal solution has one nonzero element, and the other elements are zero. Therefore, the result of the one-norm optimization problem has sparse property. When most elements of a vector or a matrix are zero, the vector (or the matrix) is called 'sparse.'

When $i^* = \mathrm{argmax}_i |\tilde{\ell}_i|$ is fixed, the optimal solution is not affected by changes of the values of $\tilde{k}$, because $1/\tilde{k}_{i^*}$ that is included in $\mathrm{diag}(\tilde{k})^{-1}$ of (2) is canceled by the denominator $\|\tilde{\ell}\|_\infty = \ell_{i^*}/\tilde{k}_{i^*}$. Therefore, when $\ell_i \ne 0$ $(i = 1, \ldots, m)$, the solution for

$$\tilde{k}_i = k_i(1 - \exp(-|\ell_i|/k_i)) \quad (k_i > 0) \tag{4}$$

coincides with that for $\tilde{k}_i = k_i$, because

$$\tilde{\ell} = (\mathrm{sgn}(\ell_1) s(|\ell_i|/k_i), \ldots, \mathrm{sgn}(\ell_1) s(|\ell_m|/k_m))$$

holds for (4), and

$$\underset{i}{\mathrm{argmax}} |\tilde{\ell}_i| = \underset{i}{\mathrm{argmax}}\, s(|\ell_i|/k_i) = \underset{i}{\mathrm{argmax}} |\ell_i|/k_i,$$

where

$$s(p) = \frac{p}{1 - \exp(-p)} \tag{5}$$

is an increasing function.

III. Sontag-Type Control Law Generating Sparse Input

A. Controlled system and clf

We consider a nonlinear system

$$\dot{x} = f(x) + g(x)u = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m, \tag{6}$$

where $x \in \mathbb{R}^n$ denotes the state and $u = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$ is the input vector. We assume that $f(x)$ and $g_1(x), \ldots, g_m(x)$ are smooth vector fields, and $f(0) = 0$.

Each element of u represents the effect of a real actuator, and the sparse property of u causes deactivation of some actuators. The inactive actuator does not consume standby power, and therefore the sparse input contributes energy efficiency. Our aim is the asymptotical stabilization of the system (6) with some input constraints by a sparse input. In this section, we consider the stabilizing problem without input restrictions, and in the next section we extend it for cases with input constraints.

We suppose that there exists a smooth control Lyapunov function (clf) $V(x)$ [9], [10] for the system (6), i.e.

$$\begin{aligned} & V(0) = 0, \quad V(x) > 0 \ (x \ne 0) \\ & \{x \in \mathbb{R}^n \mid V(x) \le a\} \text{ is compact for all } a > 0 \\ & L_f V(x) < 0 \text{ for } x \text{ such that } L_g V(x) = 0,\ x \ne 0, \end{aligned} \tag{7}$$

where $L_f V$ and $L_g V = (L_{g_1} V, \ldots, L_{g_m} V)$ are Lie derivatives of V along f and g, respectively. Moreover, we make the following assumption, which is a sufficient condition of the small control property (scp) of $V(x)$ for (6).

Assumption 1: The quadratic approximation of $V(x)$

$$V_{\mathrm{lin}}(x) = \frac{1}{2} x^\top P x, \quad P = \frac{\partial^2 V}{\partial x^2}(0)$$

is a clf of the linearly approximated system of (6)

$$\dot{x} = Ax + Bu, \quad A = \frac{\partial f}{\partial x}(0),\ B = g(0).$$

B. Variant of Sontag-type controller

For the asymptotic stabilization with a sparse input, we can establish the following theorem.

Theorem 1: The system (6) is globally asymptotically stabilized by the following control law:

$$u = \alpha_s(x) = -\beta(x) K_{\mathrm{iv}} b_s(L_g V \cdot K_{\mathrm{iv}})^\top, \tag{8}$$

which is a variant of Sontag-type controller, where

$$\begin{aligned} & K_{\mathrm{iv}} = \mathrm{diag}(1/k_1, \ldots, 1/k_m) \\ & k_i > 0 \ (i = 1, \ldots, m), \quad \rho > 0 \\ & \beta(x) = \begin{cases} \dfrac{L_f V + \sqrt{L_f V^2 + \rho \|L_g V \cdot K_{\mathrm{iv}}\|_\infty^4}}{\|L_g V \cdot K_{\mathrm{iv}}\|_\infty} & (L_g V \ne 0) \\ 0 & (L_g V = 0). \end{cases} \end{aligned}$$

Proof: Under the control law (8), we obtain

$$\begin{aligned} & \dot{V} = L_f V(x) + L_g V(x) \alpha_s(x) = -W_s(x) < 0 \ (x \ne 0) \\ & W_s(x) = \sqrt{L_f V^2 + \rho \|L_g V \cdot K_{\mathrm{iv}}\|_\infty^4} \end{aligned}$$

because of (3). Hence, the closed-loop system is globally asymptotically stable. ■

The Sontag-type control law (8) has a sparse property because most elements of $b_s(\cdot)$ are zero. As with the normal Sontag's universal formula, when the denominator in the control law tends to zero, the input does not go to infinity.

Theorem 2: The control law (8) is locally bounded.

Proof: A function

$$\phi(p,q) = \begin{cases} 0 & (q=0 \text{ and } p<0) \\ \dfrac{p+\sqrt{p^2+q^2}}{q} & (\text{elsewhere}) \end{cases}$$

is real analytic on $S = \{(p,q) \in \mathbb{R}^2 \mid q>0 \text{ or } p<0\}$ [9]. Since (7) holds, the point $(p,q) = (L_fV(x), \sqrt{\rho}\|L_gV(x)\cdot K_{\mathrm{iv}}\|_\infty^2)$ $(x \neq 0)$ is in the interior of S. Hence,

$$\beta(x) = \sqrt{\rho}\|L_gV(x)\cdot K_{\mathrm{iv}}\|_\infty \phi(L_fV(x), \sqrt{\rho}\|L_gV(x)\cdot K_{\mathrm{iv}}\|_\infty^2)$$

is locally bounded around any $x \neq 0$. To check the case around the origin, we set $x = rx_0$ $(x_0 \neq 0,\ r \geq 0)$. The value of $\beta(x)$ around the origin is evaluated as

$$\begin{aligned}
\lim_{x\to 0}\beta(x) &= \lim_{r\to 0}\sqrt{\rho}\|rx_0^\top PBK_{\mathrm{iv}} + O(r^2)\|_\infty \\
&\quad \cdot \phi((r^2/2)x_0^\top(PA+A^\top P)x_0 + O(r^3), \\
&\qquad\qquad r^2\sqrt{\rho}\|x_0^\top PBK_{\mathrm{iv}}\|_\infty^2 + O(r^3)) \\
&= \lim_{r\to 0} r\sqrt{\rho}\|x_0^\top PBK_{\mathrm{iv}} + O(r)\|_\infty \\
&\quad \cdot \phi(x_0^\top(PA+A^\top P)x_0/2 + O(r), \sqrt{\rho}\|x_0^\top PBK_{\mathrm{iv}}\|_\infty^2 + O(r))
\end{aligned}$$

Because Assumption 1 holds, $\phi(x_0^\top(PA+A^\top P)x_0/2, \sqrt{\rho}\ \|x_0^\top PBK_{\mathrm{iv}}\|_\infty^2)$ is bounded. Therefore, we can conclude that $\beta(x) \to 0$ $(x \to 0)$. Consequently, the local boundedness of $\alpha_s(x)$ is shown. ■

C. LP formulation of the controller

In this subsection, we reveal that the sparse-input controller (8) can be expressed by the solution of an LP problem.

By comparing (2) and (8), we can see that (2) and (8) are equivalent to each other by setting $\tilde{k} = k = (k_1,\dots,k_m)$, $\ell = L_gV(x)$, and $a = L_fV(x) + W_s(x)$. Note that $L_fV(x) + W_s(x) \geq 0$ holds, and $L_fV(x)+W_s(x) = 0$ implies $L_gV(x) = 0$. Therefore, the controller (8) solves the one-norm optimization problem

$$J_2(u) = k_1|u_1| + \cdots + k_m|u_m| \to \min \tag{9}$$

with a constraints

$$\dot{V} = L_fV(x) + L_gV(x)u \leq -W_s(x). \tag{10}$$

Remark: The usual Sontag controller

$$u = -\frac{L_fV + \sqrt{L_fV^2 + (L_gVL_gV^T)^2}}{L_gVL_gV^T}L_gV^T$$

coincides with the optimal solution of a quadratic programming problem $u^\top u \to \min$ under the constraints $\dot{V} = L_fV(x) + L_gV(x)u \leq -\sqrt{L_fV^2 + (L_gVL_gV^T)^2}$. ◇

Since $L_fV(x) + W_s(x) \leq 0$ means $L_gV(x) = 0$, the control input of the Sontag-type controller becomes zero only on a zero-measure set. Around the set $\{x \mid L_gV(x) = 0\}$ the input has little influence on $\dot{V}$, and therefore it is effective in energy savings to deactivate all actuators around the set. Recall that the sparse property of the input is preferred by the same reason. To make u zero in a neighborhood of each point in $\{x \mid L_gV(x) = 0, x \neq 0\}$, we relax the constraint (10) as

$$\dot{V} = L_fV(x) + L_gV(x)u \leq -W(x), \tag{11}$$

where $W(x) = \eta W_s(x)$ $(0 < \eta < 1)$.

The solution of (9) with the constraints (11) provides an asymptotically stabilizing control, which has a sparse property, when there is no input restriction. The proposed controller can be expressed in an explicit form as

$$\begin{aligned}
u = \alpha_n(x) &= -\beta_n(x)K_{\mathrm{iv}}b_s(L_gV\cdot K_{\mathrm{iv}}) \\
\beta_n(x) &= \frac{\zeta(L_fV + \eta\sqrt{L_fV^2 + \rho\|L_gV\cdot K_{\mathrm{iv}}\|_\infty^4})}{\|L_gV\cdot K_{\mathrm{iv}}\|_\infty},
\end{aligned} \tag{12}$$

where

$$\zeta(p) = \frac{p+|p|}{2} = \begin{cases} p & (p>0) \\ 0 & (p \leq 0). \end{cases}$$

IV. Controller Generating Sparse Input under Input Constraints

A. Controller under Input Constraints

In this section, we consider input constraints

$$\bar{u}_{i-} \leq u_i \leq \bar{u}_{i+} \quad (i = 1,\dots,m) \tag{13}$$

where $\bar{u}_{i-}(<0)$ $(i=1,\dots,m)$ denote lower bounds and $\bar{u}_{i+}(>0)$ $(i=1,\dots,m)$ are upper bounds of the inputs. We define a constraint set of u as

$$U = \{(u_1,\dots,u_m)^\top \mid \bar{u}_{i-} \leq u_i \leq \bar{u}_{i+}\ (i=1,\dots,m)\}.$$

Since the clf $V(x)$ is designed without the consideration on the input restriction, our purpose in this section is not global stabilization. The stabilizable region D $(\in \mathbb{R}^n)$ is definied as

$$\begin{aligned}
D &= \{x \in \mathbb{R}^n \mid V(x) < a_D\} \\
a_D &= \inf_{x\in\overline{D}_0} V(x) \\
\overline{D}_0 &= \left\{x \in \mathbb{R}^n \,\middle|\, \min_{u\in U}\left(L_fV(x) + L_gV(x)u\right) \geq 0\right\}.
\end{aligned}$$

Our purpose is the asymptotic stabilization in D where the control input is sparse near the origin. Moreover, the controller should minimize $\dot{V}$ even when $x \in \overline{D}_0$.

The optimization problem of (9) with (11) and (13) may have no solution for some $x \in D$, because (11) requires $\dot{V} \leq -W(x)$. Hence, we consider an optimization problem with a relaxed constraint $\dot{V} \leq -\gamma W(x)$ $(\gamma \leq 1)$. The value of γ is determined by the optimization problem, and the cost for $(1-\gamma)$ is added to the cost function. Therefore, the control input is obtained by solving the following problem, which can be written in an LP by adding slack variables as (1).

Problem 1: Suppose that $x \in \mathbb{R}^n$ is fixed, and $k_i > 0$ $(i = 1,\dots,m)$ and a sufficiently large $\xi(x) \in \mathbb{R}$ are given. Find

$$u = (u_1,\dots,u_m)^\top \in \mathbb{R}^m \text{ and } \gamma \in \mathbb{R}$$

that minimize

$$J_3(u,\gamma;x)=\tilde{k}_1(x)|u_1|+\cdots+\tilde{k}_m(x)|u_m|+\xi(x)(1-\gamma), \quad (14)$$

subject to

$$\dot{V}=L_fV(x)+L_{g_1}V(x)u_1+\cdots+L_{g_m}V(x)u_m \leq -\gamma W(x) \quad (15)$$

$$\gamma \leq 1 \quad (16)$$

$$\bar{u}_{i-} \leq u_i \leq \bar{u}_{i+} \ (i=1,\ldots,m), \quad (17)$$

where

$$\tilde{k}_i(x)=\begin{cases} k_i\{1-\exp(-|L_{g_i}V(x)|/k_i)\} & (L_{g_i}V(x)\neq 0)\\ k_i & (L_{g_i}V(x)=0)\end{cases}$$

$$W(x)=\eta\sqrt{L_fV^2+\rho\|L_gV\cdot K_{\mathrm{iv}}\|_\infty^4} \quad (0<\eta<1,\ \rho>0).$$

Notice that the coefficients $k_1,\ldots,k_m$ are replaced by $\tilde{k}_i(x)$ like (4). As discussed in Section II, this modification has no effect on the solution insofar as the solution satisfies $\gamma=1$.

B. Selection of $\xi(x)$

Because $\xi(x)(1-\gamma)$ is the penalty for an increase in the upper bound of $\dot{V}$, we suppose that $\xi(x)$ is sufficiently large. In this subsection, we provide a suitable selection of $\xi(x)$.

If $L_{g_i}V(x)=0$ for some i, then the input u_i does not appear in the constraint (15), i.e. $\dot{V}$ is independent of u_i, and the optimal solution satisfies $u_i=0$. We define the set of indices of non-trivial inputs as

$$I(x)=\{i \mid L_{g_i}V(x)\neq 0\}.$$

Theorem 3: Assume that

$$\xi(x)>W(x) \quad (18)$$

holds for any x. For a specified x, if there exists an index $i^*\in I(x)$ such that the solution of Problem 1 satisfies

$$\bar{u}_{i^*-}<u_{i^*}<\bar{u}_{i^*+}, \quad (19)$$

then $\gamma=1$ holds.

This theorem means that if $\gamma<1$, all input elements u_i ($i\in I(x)$) are saturated under (18).

Proof: By introducing some Lagrange multipliers, the cost function (14) is extended as

$$\begin{aligned} J_{\mathrm{ext}}=&\sum_{i=1}^m \tilde{k}_i(x)|u_i|+\xi(x)(1-\gamma)\\ &-\mu_1\{-L_fV(x)-L_gV(x)u-\gamma W(x)\}-\mu_2(1-\gamma)\\ &-\mu_3^\top(|\bar{u}_{1,\widetilde{\mathrm{sgn}}(u_1)}|-|u_1|,\ldots,|\bar{u}_{m,\widetilde{\mathrm{sgn}}(u_m)}|-|u_m|)^\top\end{aligned}$$

where

$$\widetilde{\mathrm{sgn}}(s)=\begin{cases}\text{“}+\text{”} & (s\geq 0)\\ \text{“}-\text{”} & (s<0),\end{cases}$$

and μ_1, μ_2, and $\mu_3=(\mu_{3,1},\ldots,\mu_{3,m})^\top$ are Lagrange multipliers for (15), (16), and (17), respectively. We obtain the necessary conditions for the optimality

$$\frac{\partial J_{\mathrm{ext}}}{\partial u_i}=\mu_1L_{g_i}V(x)+(\tilde{k}_i(x)+\mu_{3,i})\overline{\mathrm{sgn}}(u_i)=0, \quad i=1,\ldots,m \quad (20)$$

$$\frac{\partial J_{\mathrm{ext}}}{\partial \gamma}=-\xi(x)+\mu_1W(x)+\mu_2=0 \quad (21)$$

with Karush–Kuhn–Tucker conditions

$$\begin{cases}-L_fV(x)-L_gV(x)u-\gamma W(x)\geq 0\\ \mu_1\geq 0\\ \mu_1\{-L_fV(x)-L_gV(x)u-\gamma W(x)\}=0\end{cases} \quad (22)$$

$$\begin{cases}1-\gamma\geq 0\\ \mu_2\geq 0\\ \mu_2(1-\gamma)=0\end{cases} \quad (23)$$

$$\begin{cases}\bar{u}_{i-}\leq u_i\leq \bar{u}_{i+}\\ \mu_{3,i}\geq 0 & (i=1,\ldots,m),\\ \mu_{3,i}(|\bar{u}_{i,\widetilde{\mathrm{sgn}}(u_i)}|-|u_i|)=0\end{cases} \quad (24)$$

where

$$\overline{\mathrm{sgn}}(s)\begin{cases}=1 & (s>0)\\ \in[0,1] & (s=0)\\ =-1 & (s<0).\end{cases}$$

From (19) and (24), it is obvious that $\mu_{3,i^*}=0$. Therefore, from (20), we obtain

$$\left|\frac{\mu_1L_{g_{i^*}}V(x)}{\tilde{k}_{i^*}(x)}\right|\leq 1,$$

which derives

$$\mu_1\leq\frac{\tilde{k}_{i^*}(x)}{|L_{g_{i^*}}V(x)|}=\frac{1}{s(|L_{g_{i^*}}V(x)|/k_{i^*})}\leq 1, \quad (25)$$

where $s(\cdot)$ is defined by (5). From (18), (21), and (25), the value of μ_2 is evaluated as

$$\mu_2=\xi(x)-\mu_1W(x)=(\xi(x)-W(x))+(1-\mu_1)W(x)>0.$$

From (23), the above inequality implies $\gamma=1$, and the proof is completed. ■

Theorem 3 derives the following two corollaries.

Corollary 1: Under the solution of Problem 1 with (18),

$$\dot{V}=\max\Big(\min(L_fV(x),-W(x)),\ L_fV(x)+\min_{u\in U}L_gV(x)u\Big) \quad (26)$$

holds.

Proof: From Theorem 3, we can see that the inputs u_i ($i\in I(x)$) are saturated, if $\gamma<1$. Obviously, the solution satisfies $L_{g_i}V(x)u_i\leq 0$, and therefore if $\gamma<1$, then the optimal input minimizes $L_gV(x)u$, and $\dot{V}=-\gamma W(x)>-W(x)$ ($x\neq 0$) holds. When $\gamma=1$, the constraint (15) becomes $\dot{V}\leq -W(x)\leq \min_u(L_fV(x)+L_gV(x)u)$. When $\mu_1>0$, then the constraint (15) is strict. On the other hand, if and only if $L_fV(x)\leq -W(x)$ holds, $u=0$ is optimal, μ_1 becomes zero, and $\dot{V}=L_fV(x)$. From these results, we can conclude that (26) holds. ■

Corollary 2: Under the optimal input of Problem 1 with (18), the origin of the controlled system is asymptotically stable, and the domain of attraction contains the set D.

Proof: From Corollary 1, it is obvious that $\dot{V} < 0$ ($x \in D$, $x \neq 0$). Although the control input is discontinuous with respect to x, from the smoothness of $V(x)$, we can see that $V(x(t))$ is a decreasing function when $x(0) \in D$. Therefore, the closed-loop system is asymptotically stable with a domain of attraction that includes D. ∎

V. Suppression of Chattering

Since the control law proposed in the previous section is discontinuous with respect to the state, chattering phenomenon may occur. For example, in the case with two inputs, if no input constraint is active, a switch of the active input occurs on the curve $|L_{g_1}V(x)|/k_1 = |L_{g_2}V(x)|/k_2$. If on the curve

$$\begin{aligned}
&\frac{1}{k_1}\frac{d|L_{g_1}V|}{dt}\bigg|_{u=(u_{10},0)^\top} < \frac{1}{k_2}\frac{d|L_{g_2}V|}{dt}\bigg|_{u=(u_{10},0)^\top}\\
&\frac{1}{k_1}\frac{d|L_{g_1}V|}{dt}\bigg|_{u=(0,u_{20})^\top} > \frac{1}{k_2}\frac{d|L_{g_2}V|}{dt}\bigg|_{u=(0,u_{20})^\top}\\
&u_{10} = -(W(x)+L_fV(x))/L_{g_1}V(x)\\
&u_{20} = -(W(x)+L_fV(x))/L_{g_2}V(x)
\end{aligned}$$

hold, a chattering phenomenon is caused, and the solution slides on the curve.

A frequently switching input may increase energy consumption, because the activation of an actuator often requires extra power. In this paper, the sparse property has been introduced for the purpose of energy savings, but the chattering phenomenon ruins it. Hence, in this section, we propose a mechanism to suppress the chattering, where the values of $k_1, \ldots, k_m$ vary with respect to time.

Notice that $\tilde{k}_i$ is an increasing function with respect to k_i (> 0) for a fixed $L_{g_i}V(x) \neq 0$. When k_i is large for an index i, the penalty for the activation of the input u_i is also large. Therefore, making the values of k_i for currently inactive inputs large prevents the activation of these inputs. Such a mechanism suppresses the switching of the input. The mechanism is described as

$$k_i(t) = \begin{cases} \sigma k_{i,\text{default}} & \text{if } u_i(t-0) = 0 \\ k_{i,\text{default}} & \text{others} \end{cases} \qquad (i = 1, \ldots, m), \tag{27}$$

where $k_{i,\text{default}}$ ($i = 1, \ldots, m$) are default values of $k_1, \ldots, k_m$, $u_i(t-0)$ is the input just before the current time, and $\sigma > 1$ is a constant. This mechanism can suppress the frequent switching when the input constraints are not active.

Because the constraint set is not strictly convex, chattering due to the constraint set may also occur. To prevent this kind of chattering is our future work.

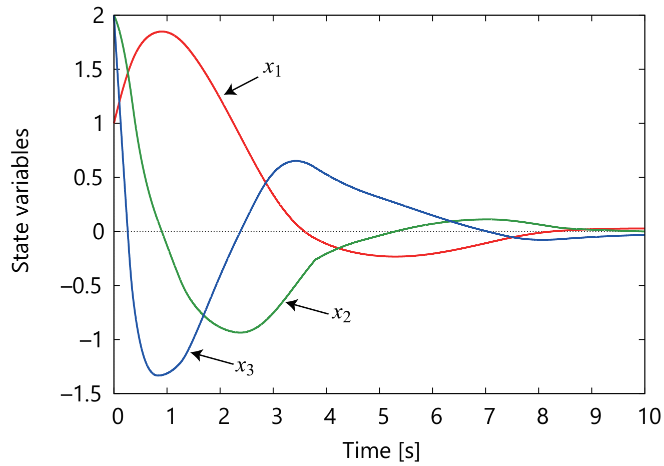

Fig. 1. Responses of state variables for the sparse input controller without chattering suppression.

VI. Simulation

In this section, we show simulation results for an example. We consider a nonlinear system

$$\dot{x} = \begin{pmatrix} x_2 \\ x_3 \\ -x_2 + 0.1\sin x_1 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} u,$$

where $x = (x_1, x_2, x_3)^\top$ is the state and $u = (u_1, u_2)^T$ is the input. We assume that input constraints $-3 \le u_1 \le 3$, $-3 \le u_2 \le 3$ exist. For this system, a clf

$$V(x) = \frac{1}{2}x^T \begin{bmatrix} 1 & 1/2 & 3/4 \\ 1/2 & 1 & 1/2 \\ 3/4 & 1/2 & 1 \end{bmatrix} x$$

is considered here.

We construct two asymptotically stabilizing controllers for the system. One is a controller generated by Problem 1 with (18) for fixed $k_1 = k_2 = 1$. Other parameters are chosen as $\rho = 2$, $\eta = 0.8$, and $\xi(x) = 3W(x) + 0.01$. The other is a controller generated by Problem 1 with (18) and the chattering-suppression mechanism. The default values of k_i are $k_{1,\text{default}} = k_{2,\text{default}} = 1$ and $\sigma = 1.8$. The parameters ρ, η, and $\xi(x)$ are same as the first controller.

Simulations for both the controllers are performed, where the initial state is $x(0) = (1, 2, 2)^\top$. As an LP solver, IBM ILOG CPLEX 12.8 is adopted, which is called from a C++ program. Figures 1 and 2 show the simulation result for the first controller. The time responses of the state variables are plotted in Fig. 1, and the time responses of the input variables and γ are shown in Fig. 2. All state variables converge to zero, and the input vector is sparse when the inputs are free for the input constraints. However, in some time periods the choice of the active input is frequently switched, i.e. a chattering phenomenon occurs. Figures 3 and 4 show the simulation result for the controller with the chattering-suppression mechanism. Figure 3 shows the time responses of the state variables, while the time responses of the input variables and γ are plotted in Fig. 4. All state variables

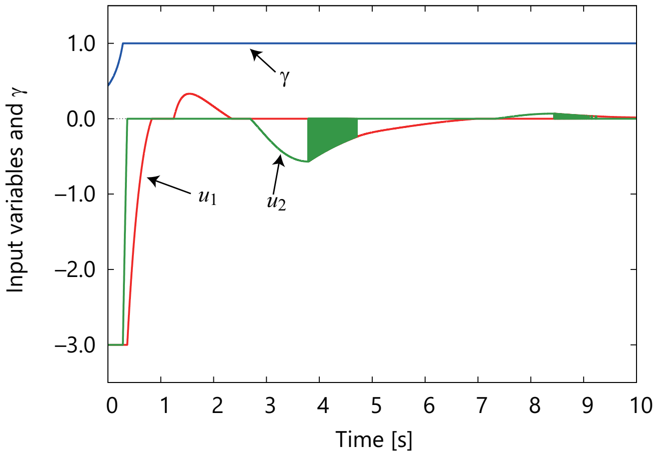

Fig. 2. Responses of input variables and γ for the sparse input controller without chattering suppression.

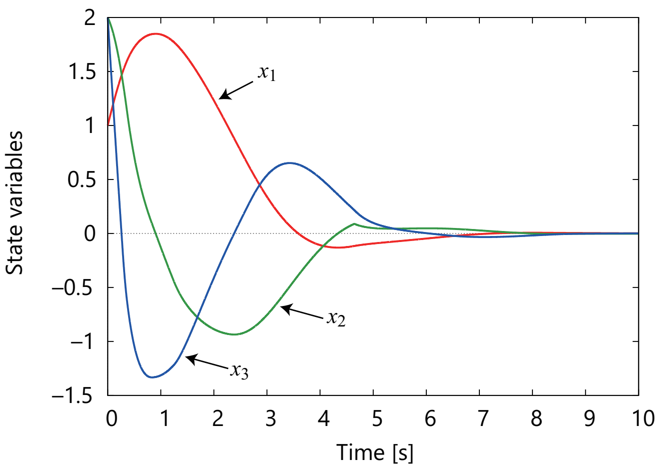

Fig. 3. Responses of state variables for the sparse input controller with chattering suppression.

converge to zero as in the first-controller case. Input vector is sparse when it is inside the input-constraint boundary. The chattering phenomenon is removed by the chattering-suppression mechanism.

As Theorem 3 claims, in both simulations, only for the time period where all inputs are saturated, the value of γ is less than one, and the constraint $\dot{V} \leq -W(x)$ is relaxed as $\dot{V} \leq -\gamma W(x)$. Moreover, around the points where $L_{g_1}V(x) = L_{g_2}V(x) = 0$, both inputs u_1 and u_2 are zero, because $L_fV(x)+W(x)$ becomes $L_fV(x)+\eta|L_fV(x)|$ (<0) at the points such that $L_gV(x) = 0$. This behavior often contributes energy savings.

In these simulations, the switching of the active input occurs even around the origin. The system is controllable only by the input u_2, and hence we can improve the proposed method where only u_2 is active around the origin by changing the definition of $\tilde{k}_i(x)$, but this issue is our future work.

VII. Conclusions

An asymptotically stabilizing controller using a control Lyapunov function $V(x)$ has been proposed. The proposed

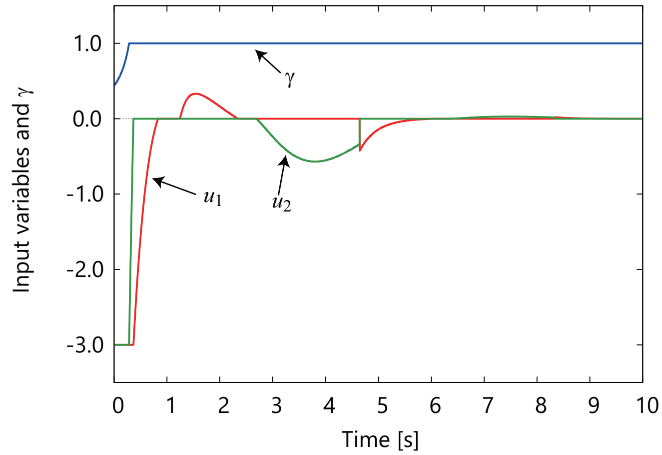

Fig. 4. Responses of input variables and γ for the sparse input controller with chattering suppression.

controller generates a sparse input vector when no input constraint is active, and the sparse property is effective for energy savings. For the cases with no input constraint, the control law can be described in an explicit form, and it is similar to Sontag-type controllers. When some input constraints are subjected to the system, the control input can be obtained by solving an LP problem, which can be easily calculated online. The controller makes $\dot{V}$ negative, if possible. When negative $\dot{V}$ is impossible owing to the input constraints, an input minimizing $\dot{V}$ is chosen. We have also proposed a chattering-suppression mechanism. The effectiveness of the proposed method has been confirmed by computer simulations. To develop a mechanism that changes the input-selection behavior around the origin is our future work.

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