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Commuting operators over Pontryagin spaces with applications to system theory*

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***Abstract*— In this talk we extend the vessel theory, or equivalently, the theory of overdetermined $2D$ systems to the Pontryagin space setting. The associated transfer function becomes a mapping with a finite number of negative squares between certain vector bundles defined on a compact Riemann Surface. Furthermore, we present a realization theorem of these mappings. In particular, we develop an indefinite version of the de Branges Rovnyak spaces over real compact Riemann surfaces, i.e. reproducing kernel Pontryagin spaces of analytic sections defined on real compact Riemann surfaces.**

I. $2D$ SYSTEM OVER PONTRYAGIN SPACES

A $2D$ overdetermined linear time–invariant system over a Pontryagin space is the set of equations

$$i\frac{\partial f}{\partial t_1}+A_1 f=\Phi^{[*]}\sigma_1 u, \tag{1}$$

$$i\frac{\partial f}{\partial t_2}+A_2 f=\Phi^{[*]}\sigma_2 u, \tag{2}$$

$$v=u-i\Phi f, \tag{3}$$

where A_1 and A_2 are commuting bounded operators on Pontryagin space $\mathscr{P}$ with κ negative index such that $A_k-A_k^{[*]}=i\Phi^{[*]}\sigma_k\Phi$ for $k=1,2$. The functions u, f and v are two-variable functions of t_1 and t_2 denoted by the input, system and output functions of the system, respectively. E is a finite dimensional Pontryagin space, σ_k are selfadjoint operators on E and $\Phi:\mathscr{P}\to E$.

Equivalently, the collection

$$\mathscr{V}=(A_1\,,A_2\,;\mathscr{P}\,,\Phi\,,E\,;\sigma_1\,,\sigma_2\,,\gamma\,,\widetilde{\gamma})$$

equipped with the equations

$$\gamma\Phi=\sigma_1\Phi A_2^{[*]}-\sigma_2\Phi A_1^{[*]} \tag{4}$$

$$\widetilde{\gamma}\Phi=\sigma_1\Phi A_2-\sigma_2\Phi A_1 \tag{5}$$

$$\widetilde{\gamma}-\gamma=i\left(\sigma_1\Phi\Phi^{[*]}\sigma_2-\sigma_2\Phi\Phi^{[*]}\sigma_1\right) \tag{6}$$

where γ and $\widetilde{\gamma}$ are selfadjoint operators on E, is called a commutative two-operator vessel over Pontryagin space.

One can notice that Equations (1-3) together with the compatability equations

$$\left(\sigma_2\frac{\partial}{\partial t_1}-\sigma_1\frac{\partial}{\partial t_2}+i\gamma\right)u=0 \tag{7}$$

$$\left(\sigma_2\frac{\partial}{\partial t_1}-\sigma_1\frac{\partial}{\partial t_2}+i\widetilde{\gamma}\right)v=0. \tag{8}$$

are exactly commutative two-operator vessel over Pontryagin space.

When assuming that the principal subspace of a vessel

$$\widehat{\mathscr{P}}\overset{\text{def}}{=}\bigvee_{n=0}^{\infty}A_1^{k_1}A_2^{k_2}\Phi^{[*]}(E)$$

is non–degenerate (and hence a Pontryagin space) then the input and the output discriminant polynomials coincide

$$p(z_1,z_2)=\det(z_1\sigma_2-z_2\sigma_1+\gamma)=\det(z_1\sigma_2-z_2\sigma_1+\widetilde{\gamma})$$

and furthermore the generalized Cayley-Hamilton theorem, i.e. $p(A_1,A_2)=0$, holds in $\widehat{\mathscr{P}}$. These two results are first introduced in the Hilbert space setting by Livšic [6], [7].

II. CHARACTERISTIC FUNCTIONS AND REALIZATION THEOREMS

The *complete characteristic function* (CCF) of a vessel is defined by

$$W(\xi_1,\xi_2,z)=I-i\Phi(\xi_1A_1+\xi_2A_2-zI)^{-1}\Phi^{[*]}(\xi_1\sigma_1+\xi_2\sigma_2),$$

and it satisfies

$$W(\xi_1,\xi_2,x)^*(\xi_1\sigma_1+\xi_2\sigma_2)W(\xi_1,\xi_2,x)=\xi_1\sigma_1+\xi_2\sigma_2.$$

for all $x\in\mathbb{R}$ and the kernel

$$\frac{W(\xi_1,\xi_2,w)^*(\xi_1\sigma_1+\xi_2\sigma_2)W(\xi_1,\xi_2,z)-(\xi_1\sigma_1+\xi_2\sigma_2)}{-i(z-\overline{w})},$$

has κ negative squares for $\mathfrak{Im}\ w,\mathfrak{Im}\ z\geq 0$ in any direction (ξ_1,ξ_2).

The joint characteristic function of a vessel over Pontryagin space $\mathscr{P}$ is a mapping from $\ker(\lambda_1\sigma_2-\lambda_2\sigma_1+\gamma)$ to $\ker(\lambda_1\sigma_2-\lambda_2\sigma_1+\widetilde{\gamma})$ and given by

$$S(\lambda_1,\lambda_2)\overset{\text{def}}{=}W(\xi_1,\xi_2,\xi_1\lambda_1+\xi_2\lambda_2)\big|_{\ker(\lambda_1\sigma_2-\lambda_2\sigma_1+\gamma)}$$

where (λ_1,λ_2) belongs to C_0 and $\xi_1\lambda_1+\xi_2\lambda_2$ does not belong to the spectrum of $\xi_1A_1+\xi_2A_2$. The joint characteristic function defines the frequency domain relation between input and the output [8].

Assuming the homogeneous discriminant polynomial is irreducible and the determinantal representations are maximal (see [8]), there exist isomorphisms from the input and output kernel bundles to certain vector bundles on a real compact Riemann surface X. Under these isomorphisms the joint transfer function becomes the normalized joint transfer function $T:L_\zeta\otimes\Delta\to L_{\widetilde{\zeta}}\otimes\Delta$. Here $L_{\widetilde{\zeta}}$ is a unitary flat line bundle corresponding to $\widetilde{\zeta}$ in $J(X)$ and Δ satisfies $\Delta\otimes\Delta=K_X$, that is, the square root of the canonical line

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bundle K_X on X. The Cauchy kernel (in the line bundle case) is given by

$$K_\zeta(u,v) = \frac{\theta[\zeta](\overline{v}-u)}{i\theta[\zeta](0)E(u,\overline{v})}, \tag{9}$$

where $E(\cdot,\cdot)$ is the prime form, see [5] and $\theta[\zeta]$ is the theta function with characteristic ζ. We note that (9) is the counterpart of the kernel $\frac{1}{z-\overline{w}}$ in the case of compact Riemann surface.

A classification theorem of the class of normalized joint characteristic functions of vessels over Pontryagin spaces is stated in the following theorem.

Theorem 1: Let $T(p)$ be a multiplicative function on a real compact Riemann surface X corresponding to $\zeta - \widetilde{\zeta}$. Then $T(p)$ is the normalized joint characteristic function of a vessel $\mathscr{V}$ over Pontryagin space with κ negative index and with discriminant polynomial $p(\lambda_1, \lambda_2)$ with maximal input and output determinantal representations corresponding to $\zeta, \widetilde{\zeta} \in J(X)$ if and only if $T(p)$ is a non-zero holomorphic function in the neighborhood of C at infinity, meromorphic on $X \setminus X_{\mathbb{R}}$, $T(p)T(\overline{T(\overline{p})}) = 1$ and the kernel

$$\frac{\theta[\zeta](\overline{q}-p)}{i\theta[\zeta](0)E(p,\overline{q})} - T(p)\frac{\theta[\widetilde{\zeta}](\overline{q}-p)}{i\theta[\widetilde{\zeta}](0)E(p,\overline{q})}\overline{T(q)}, \tag{10}$$

has κ negative squares.
In [1] we obtain realization theorems also for kernels associated to the complete characteristic function and the joint characteristic function, respectively.

III. de Branges Rovnyak spaces

de Branges Rovnyak spaces in our setting are reproducing kernel Pontryagin spaces with reproducing kernels of the form (10) have finite number of negative squares.

Then for a meromorphic function on X, denoted by y, with n distinct poles $p_1, ..., p_n$ and where c_m are the residue of $y(\cdot)$ at the pole p_m, the resolvent operator R^y_α is given by (see [8, Equation 3-4] and [3])

$$R^y_\alpha f(u) = \frac{f(u)}{y(u)-\alpha} - \sum_{j=1}^{n} \frac{f(u^{(j)})}{dy(u^{(j)})} \frac{\theta[\widetilde{\zeta}](u^{(j)}-u)}{\theta[\widetilde{\zeta}](0)E(u^{(j)},u)}. \tag{11}$$

Here the points $u^{(j)}$ points on X such that $y(u^{(j)}) = \alpha$. The model operator M^y, satisfying (in the neighborhood of the poles of y) $R^y_\alpha = (M^y - \alpha)^{-1}$, is defined by

$$M^y f(u) = y(u)f(u) + \sum_{m=1}^{n} c_m f(p^{(m)}) \frac{\theta[\widetilde{\zeta}](p^{(m)}-u)}{\theta[\widetilde{\zeta}](0)E(p^{(m)},u)}. \tag{12}$$

de Branges theory is closely related to a certain identity, also known as the structure identity. The structure identity in Compact Riemann surfaces setting is given by

$$\left[R^{y_k}_\alpha f, g\right] - \left[f, R^{y_k}_\beta g\right] - (\alpha - \overline{\beta})\left[R^{y_k}_\alpha f, R^{y_k}_\beta g\right] = \\ -i(\alpha - \overline{\beta}) \sum_{l,t=1}^{n} \frac{f(v^{(l)})\overline{g(\omega^{(t)})}}{dy_k(v^{(l)})\overline{dy_k(\omega^{(t)})}} K(\widetilde{\zeta}; v^{(l)}, \overline{\omega^{(t)}}). \tag{13}$$

Then the analogue of de Branges structure theorem (see [4]) is given in the following theorems.

Theorem 2: Let X be a real compact Riemann surface. Let $\mathscr{X}$ be a reproducing kernel Pontryagin space, with negative index κ, of sections of $L_{\widetilde{\zeta}} \otimes \Delta$ analytic in open and connected set Ω. We choose real meromorphic functions y_1 and y_2 generating $\mathscr{M}(X)$, such that Ω contains all the points above the singular points of C and contains the poles of y_1 and y_2 and all the elements of $\mathscr{X}$ are regular at these points. Furthermore, assume that for every $\alpha, \beta \in \mathbb{C}$ such that their n pre-images lies within Ω and the following two conditions hold:

1) $\mathscr{X}$ is invariant under $R^{y_1}_\alpha$ and $R^{y_2}_\beta$.
2) For every choice of $f, g \in \mathscr{X}$ such that f and g are analytic at the poles of y_1 and y_2, (13) holds.

Then the reproducing kernel of $\mathscr{X}$ is of the form (10) for some $\zeta \in J(X)$ and for some line bundle mapping $T(\cdot)$ with κ negative squares.

The converse statement is given below and its proof is based on the above realization result, i.e. Theorem 1, together with the strategy we used in [2].

Theorem 3: Let X be a real compact Riemann surface. let T be a line bundle mapping corresponds to $(\zeta, \widetilde{\zeta})$ with κ negative squares and let $\mathscr{X}$ be the corresponding reproducing kernel Pontryagin space with reproducing kernel of the form (10). Furthermore, let y be a real meromorphic function on X such that T is regular at the poles of y. Then, for any $\alpha \in \mathbb{C}$ such all its pre-images under $y(\cdot)$ belongs to Ω, $\mathscr{X}$ is R^y_α-invariant and the structure identity (13) holds.

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