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Constraints on Non-Uniqueness in Two-Point Boundary Value Problems of Conservative Dynamical Systems

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***Abstract*— Stationary-action problems for finite-dimensional conservative systems are considered. On sufficiently short time intervals, stationary-action trajectories typically correspond to least-action trajectories, and existence and uniqueness of these trajectories is easily demonstrated. However, for arbitrary duration problems, the stationary trajectory is not an optimizer, and in particular, there exist problem data where uniqueness does not hold. It is shown that, under certain conditions, points of non-uniqueness are isolated as a function of initial position. The conditions are verified in a classic two-body problem example. An extension of the implicit function theorem to certain cases where the usual first-order conditions are not satisfied is demonstrated and applied.**

Key words. two-point boundary value problems, stationary action, staticization, uniqueness.

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I. Introduction

Stationary-action problems for finite-dimensional conservative systems are considered. With this approach, certain questions regarding dynamical systems may be very effectively addressed through the use of tools from optimal control theory. We specifically consider problems where the initial position component of the system is known, and where the terminal data is a function of the terminal position component. This class of problems underlies recent results regarding fundamental solutions of certain two-point boundary value problems (TPBVPs) [10], [11], [12], [13], [21], [24], [25]. On sufficiently short time intervals, stationary-action trajectories typically correspond to least-action trajectories, and existence and uniqueness of these trajectories is easily demonstrated [24], [25]. However, for arbitrary duration problems, the stationary trajectory is not an optimizer, and in particular, there exist problem data where uniqueness does not hold. Points where uniqueness fails are a longstanding issue in stationary-action analyses, cf. [3], [4], [5], [17]. It will be shown that, under certain reasonable conditions, points of non-uniqueness are isolated as a function of initial position. The case of gravitational systems is indicated, where for simplicity, the conditions are verified only in the classic two-body problem example.

The result relies on an extension of the implicit function theorem to certain cases where the usual first-order conditions are not satisfied. That is, the implicit function theorem is generally only demonstrated in the case where the operator defined by a certain derivative is invertible, although one may see for example [16] and the references therein. Here, the operator generated by the derivative is a resolvent that might be evaluated at an eigenvalue, and we have a perturbation that is compact and bijective onto a subspace. Solvability of a linear equation with such a perturbed operator is demonstrated. In the context of the implicit function theorem usage, the perturbation is generated by the next higher-order term in the Taylor series expansion.

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II. Problem Definition and Staticization

Let $t \in (0,\infty)$ and $n \in \mathbb{N}$, $\mathcal{D} \doteq (0,t) \times \mathbb{R}^n$ and $\overline{\mathcal{D}} \doteq (0,t] \times \mathbb{R}^n$. For $s \in (0,t]$, let $\mathcal{U}_s \doteq L_2((s,t);\mathbb{R}^n)$. We suppose potential, V, and terminal payoff ϕ satisfy

$$V, \phi \in C^4(\mathbb{R}^n;\mathbb{R}). \qquad (A.1)$$

For $s \in (0,t]$, consider the action functional, $J(s,\cdot,\cdot) : \mathbb{R}^n \times \mathcal{U}_s \to \mathbb{R}$ be given by

$$J(s,x,u) \doteq \int_s^t \tfrac{1}{2}|u_r|^2 - \tfrac{1}{m} V(\xi_r)\,dr + \phi(\xi_t), \quad (1)$$

$$\text{where} \quad \xi_r \doteq x + \int_s^r u_\rho\,d\rho \quad \forall\, r \in [s,t]. \quad (2)$$

We remark that some particularly useful terminal payoffs are of the form $\phi(x) = \phi(x;\bar{v}) \doteq \bar{v}^T x$, $\phi(x) = \phi(x;z) \doteq (c/2)|x-z|^2$ with $c \in (0,\infty)$, and

$$\phi(x) = \phi(x;z) \doteq \begin{cases} 0 & \text{if } x = z, \\ +\infty & \text{otherwise,} \end{cases} \quad (3)$$

cf. [24].

Before continuing with the definition of the stationary value, it is useful to make some clear definitions regarding stationarity

A. Stationarity

In analogy with the language for minimization and maximization, we will refer to the search for stationary points as *staticization,* with these points being *statica* (in analogy with minima/maxima), cf. [23]. Prior to the development, we make the following definitions. Suppose $(\mathcal{U}, |\cdot|)$ is a generic normed vector space with $\mathcal{G} \subseteq \mathcal{U}$, and suppose $F : \mathcal{G} \to \mathbb{R}$. We say $\bar{v} \in \operatorname{argstat}\{F(v)\,|\, v \in \mathcal{G}\}$ if $\bar{v} \in \mathcal{G}$ and either

$$\limsup_{v \to \bar{v}, v \in \mathcal{G}\setminus\{\bar{v}\}} |F(v) - F(\bar{v})|/|v - \bar{v}| = 0, \quad (4)$$

or there exists $\delta > 0$ such that $\mathcal{G} \cap B_\delta(\bar{v}) = \{\bar{v}\}$ (where $B_\delta(\bar{v})$ denotes the ball of radius δ around $\bar{v}$). If $\operatorname{argstat}\{F(v)\,|\, v \in$

$\mathcal{G}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\begin{aligned}\operatorname*{stat}^s_{v\in\mathcal{G}} F(v) &\doteq \mathrm{stat}^s\{F(v)\,|\,v\in\mathcal{G}\}\\ &\doteq \big\{F(\bar v)\,\big|\,\bar v\in \mathrm{argstat}\{F(v)\,|\,v\in\mathcal{G}\}\big\}. \quad (5)\end{aligned}$$

If $\mathrm{argstat}\{F(v)\,|\,v\in\mathcal{G}\}=\emptyset$, $\mathrm{stat}^s_{v\in\mathcal{G}} F(v)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript s). In particular, if there exists $a\in\mathbb{R}$ such that $\mathrm{stat}^s_{v\in\mathcal{G}} F(v)=\{a\}$, then $\mathrm{stat}_{v\in\mathcal{G}} F(v)\doteq a$; otherwise, $\mathrm{stat}_{v\in\mathcal{G}} F(v)$ is undefined. At times, we may abuse notation by writing $\bar v=\mathrm{argstat}\{F(v)\,|\,v\in\mathcal{G}\}$ in the event that the argstat is the single point $\{\bar v\}$.

We are often interested in the case where $\mathcal{U}$ is a Hilbert space, $\mathcal{G}\subseteq\mathcal{U}$ is an open set, and $F:\mathcal{G}\to\mathbb{R}$ is Fréchet differentiable at $\bar v\in\mathcal{G}$ with Riesz representation $F_v(\bar v)\in\mathcal{U}$. We note that throughout, we will denote such Fréchet derivatives with subscript notation, $F_v(\bar v)$, and where more convenient, with the notation $\frac{d}{du}$ and $\frac{\partial}{d\partial u}$. . The following is immediate from the above definitions [23].

Lemma 1: Suppose $\mathcal{U}$ is a Hilbert space, with open set $\mathcal{G}\subseteq\mathcal{U}$ and $\bar v\in\mathcal{G}$. Then, $\bar v\in\mathrm{argstat}\{F(y)\,|\,y\in\mathcal{G}\}$ if and only if $F_v(\bar v)=0$.

B. The Stationary-Action Value Function

Fix $(\bar s,\bar x)\in(0,t)\times\mathbb{R}^n$ and $\mathcal{G}\subseteq\mathcal{U}_{\bar s}$. We assume there exists $d>0$ such that

$$W(\bar s,x)\doteq\operatorname*{stat}_{u\in\mathcal{G}} J(\bar s,x,u)\ \text{ exists } \forall x\in B_d(\bar x), \qquad (A.2)$$

which obviously also implies that there exists $\bar u\in\mathrm{argstat}_{u\in\mathcal{G}} J(\bar s,\bar x,u)$ and $W(\bar s,\bar x)=J(\bar s,\bar x,\bar u)$. A short comment regarding this assumption will appear in Remark 4. We also remark (cf. [24]) that with the terminal payoff of (3), $\bar u$ corresponds to a solution of the TPBVP with $\xi_0=x$ and $\xi_t=z$, while $\phi(x)=-\bar v^Tx$ corresponds to the TPBVP with $\xi_0=x$ and $\dot\xi_t=\bar v$. Also, and more generally, the form of terminal payoff in (3) can be used to construct a fundamental solution for a class of TPBVPs, cf. [24]. Lastly, we note that the HJ PDE associated to this staticization problem is, at least formally, the small-noise limit (also known as the semiclassical limit) HJ PDE corresponding to a Schrödinger initial value problem, where Planck's constant plays the role of the small parameter, cf. [7], [9], [8], [14], [19], [20], [22] among many others.

III. Some Derivatives and the Implicit Function Theorem

Recalling the characterization of Lemma 1, we examine derivatives of the payoff J. (See also [21], and for extensions to infinite-dimensional cases, [12].) One may write J of (1) as a sum of three terms, with

$$J(s,x,u)=F^0(s,u)+\tfrac{1}{m}F^1(s,x,u)+\tfrac{1}{m}F^2(s,x,u), \quad (6)$$

$$F^0(s,u)\doteq\int_s^t \tfrac12|u_r|^2\,dr,\quad F^1(s,x,u)\doteq\int_s^t -V(\xi_r)\,dr, \quad (7)$$

$$F^2(s,x,u)\doteq\phi(\xi_t), \quad (8)$$

where ξ satisfies (2). Obtaining the Fréchet derivatives of F^0 and F^2 is trivial, and the computations are not included. As it is slightly more complicated, we provide detail on the derivative of F^1. Fix $(s,x)\in\mathcal{D}$ and $u,\hat u\in\mathcal{U}_{s,t}$. By Assumption $(A.1)$, for each $r\in(s,t)$, there exists $\lambda_r\in[0,1]$ such that

$$\begin{aligned}&V(\xi_r+\textstyle\int_s^r\hat u_\rho\,d\rho)-V(\xi_r)-V_x(\xi_r)\textstyle\int_s^r\hat u_\rho\,d\rho\\ &=\tfrac12(\textstyle\int_s^r\hat u_\rho\,d\rho)^TV_{xx}(\tilde x_r)(\textstyle\int_s^r\hat u_\rho\,d\rho) \qquad (9)\end{aligned}$$

where $\tilde x_r\doteq\lambda_r\xi_r+(1-\lambda_r)(\xi_r+\int_s^r\hat u_\rho\,d\rho)$. By the measurable selection theorem, cf. [15, Appendix B], one sees that there exists measurable $\hat x:(s,t)\to\mathbb{R}^n$ such that $\hat x_r=\tilde x_r$ a.e. $r\in(s,t)$, and by (9),

$$\begin{aligned}&V(\xi_r+\textstyle\int_s^r\hat u_\rho\,d\rho)-V(\xi_r)-V_x(\xi_r)\textstyle\int_s^r\hat u_\rho\,d\rho\\ &=\tfrac12(\textstyle\int_s^r\hat u_\rho\,d\rho)^TV_{xx}(\hat x_r)(\textstyle\int_s^r\hat u_\rho\,d\rho) \qquad (10)\end{aligned}$$

for a.e. $r\in(s,t)$. Using (7), (8) and (10), we find that for all $(s,x)\in\mathcal{D}$ and $u,\hat u\in\mathcal{U}_{s,t}$,

$$\begin{aligned}&\Big|F^1(s,x,u+\hat u)-F^1(s,x,u)-\int_s^t[-V_x(\xi_r)]\int_s^r\hat u_\rho\,d\rho\,dr\Big|\\ &\le\tfrac{(t-s)}{2}\int_s^t\big|V_{xx}(\hat x_r)\big|\,dr\,\|\hat u\|^2_{\mathcal{U}_{s,t}}.\end{aligned}$$

Fixing $(s,x)\in\mathcal{D}$ and $u\in\mathcal{U}_s$, and restricting this to the unit ball, i.e., $\hat u\in B_1(0)$, one easily sees that there exists $D_1=D_1(s,x,u)<\infty$ such that

$$\begin{aligned}&\Big|F^1(s,x,u+\hat u)-F^1(s,x,u)+\int_s^tV_x(\xi_r)\int_s^r\hat u_\rho\,d\rho\,dr\Big|\\ &\le D_1\,\|\hat u\|^2_{\mathcal{U}_{s,t}}, \qquad (11)\end{aligned}$$

for all $\hat u\in B_1(0)$. Using integration by parts, one finds that the Fréchet derivative of F^1 with respect to u has Riesz representation given by

$$[F^1_u(s,x,u)]_r=-\int_r^tV_x(\xi_\rho)\,d\rho \qquad \text{a.e. } r\in(s,t). \quad (12)$$

We have obtained the following.

Theorem 2: For all $(s,x)\in\mathcal{D}$, $J(s,x,\cdot)$ is Fréchet differentiable, where for all $u\in\mathcal{U}_s$, the derivative has Riesz representation

$$[J_u(s,x,u)]_r=u_r-\tfrac1m\int_r^tV_x(\xi_\rho)\,d\rho+\tfrac1m\phi_x(\xi_t) \quad (13)$$

for a.e. $r\in(s,t)$.

By Lemma 1 and Theorem 2, we obtain the following.

Theorem 3: Let $(\bar s,\bar x)\in\mathcal{D}$. $\bar u\in\mathrm{argstat}_{u\in\mathcal{G}} J(\bar s,\bar x,u)$ if and only if

$$\begin{aligned}0&=\bar u_r-\tfrac1m\Big[\int_r^tV_x(\bar\xi_\rho)\,d\rho-\phi_x(\bar\xi_t)\Big]\\ &\doteq[\mathcal{I}\bar u-G_u(\bar s,\bar x,\bar u)]_r\quad\forall\, r\in[\bar s,t], \qquad (14)\end{aligned}$$

where $\bar\xi$ satisfies (2) with input $\bar u$, and abusing notation, we identify the equivalence class $\bar u\in\mathcal{U}_{\bar s}$ with single element $\bar u\in C[\bar s,t]\cap C^4(\bar s,t)$. Further, $\bar\xi\in C[\bar s,t]\cap C^5(\bar s,t)$.

Proof: Using Theorem 2, we find $\bar{u} \in \mathcal{U}_{\bar{s}}$, which by (2), implies that $\bar{\xi}$ is absolutely continuous. Then, recalling the smoothness of V and ϕ from Assumption $(A.1)$, we see that $\int_r^t V_x(\bar{\xi}_\rho)\, d\rho \in C^1$ as a function of $r \in (\bar{s}, t)$. Combining this with the expression for $\bar{u}$ implied by Theorem 2 and Lemma 1, we also see that we may take $\bar{u}$ satisfying (14), which yields $\bar{u} \in C[\bar{s}, t] \cap C^1(\bar{s}, t)$. An easy induction argument may then be implied to yield the smoothness assertions. ∎

Remark 4: It may be worth noting that differentiating (14) with respect to r yields Newton's second law. Hence, Assumption $(A.2)$ is equivalent to the existence of an initial velocity such that the system, acting under Newton's second law, is such that, in the cases where the terminal payoff corresponds to either a terminal position or a terminal velocity, achieves that terminal state condition.

We will be interested in looking at (14) as an implicit function definition for some single-valued argstat in a neighborhood of $(\bar{s}, \bar{x}, \bar{u})$, i.e., as generating a continuous function $\tilde{u}(x)$ in a neighborhood, $\mathcal{O}$, of $\bar{x}$, with $\tilde{u}(\bar{s}, \bar{x}) = \bar{u}$ and $\tilde{u}(x) = \operatorname{argstat}_{u \in \mathcal{G}} J(\bar{s}, x, u)$ for $x \in \mathcal{O} \setminus \{\bar{x}\}$. (Note the implicit claim that the argstat will be single-valued on $\mathcal{O} \setminus \{\bar{x}\}$.) We begin by attempting to apply the classic implicit function theorem. In this regard, we must compute $J_{uu}(\bar{s}, \bar{x}, \bar{u}) = D_u[\mathcal{I}u - G_u(\bar{s}, \bar{x}, u)]\big|_{u=\bar{u}}$. Proceeding similarly as above, and not including the details, we obtain the following.

Theorem 5: For all $(s, x) \in \mathcal{D}$, $J_u(s, x, \cdot)$ is Fréchet differentiable, where for all $u \in \mathcal{U}_s$, the derivative has Riesz representation evaluated at (s, x, u) and applied to $v \in \mathcal{U}_s$ given by

$$\begin{aligned} &[J_{uu}(s,x,u)v]_r = \big\{[\mathcal{I} - G_{uu}(s,x,u)]v\big\}_r \\ &= v_r - \tfrac{1}{m}\Big[\int_r^t V_{xx}(\xi_\rho)\; w_\rho\, d\rho + \phi_{xx}(\xi_t) w_t\Big] \quad \forall\, r \in (s,t), \end{aligned} \tag{15}$$

where $w_r \doteq \int_s^r v_\rho\, d\rho$ and $\xi_r \doteq x + \int_s^r u_\rho\, d\rho$ for all $r \in [s, t]$.

The classic implicit function theorem applied to the problem at hand takes the following form.

Theorem 6: Suppose there does not exist $v \in \mathcal{U}_{\bar{s}}$, $v \neq 0$ such that

$$[\mathcal{I} - G_{uu}(\bar{s}, \bar{x}, \bar{u})]v = 0. \tag{16}$$

Then, $[\mathcal{I}u - G_u(\bar{s}, x, u)] = 0$ uniquely defines differentiable $\tilde{u} : \mathcal{O} \to \mathcal{U}_s$ on some neighborhood, $\mathcal{O}$, of $\bar{x}$, with $\tilde{u}(\bar{x}) = \bar{u}$.

Proof: Note that $G_{uu}(\bar{s}, \bar{x}, \bar{u})$ is a compact linear operator, and hence, $\mathcal{I} - G_{uu}(\bar{s}, \bar{x}, \bar{u})$ is bijective if and only if (16) is satisfied, cf. [26], Sec. 5.7. The assertion then follows from the implicit function theorem, cf. [2], Th 12.8.1. ∎

The case of interest here is that where condition (16) is *not* satisfied, which correspond to points of nonuniqueness of the stationary action trajectories over the set of TPBVP boundary data. More specifically, we will show that such points are isolated. We begin with an abstract result regarding perturbations of the resolvent at eigenvalues.

IV. Perturbation of the Resolvent at an Eigenvalue

Let $\mathcal{X}$ be a Hilbert space, and let subspace $\mathcal{Y} \subseteq \mathcal{X}$. Let $H \in \mathcal{L}(\mathcal{Y}; \mathcal{Y})$ be compact and self-adjoint. Suppose $\lambda = 1$ is an eigenvalue of H, and let $R_1 \doteq \mathcal{I} - H$ where $\mathcal{I}$ is the identity operator. Let Γ be a subset of a Hilbert space. Suppose that for all $\gamma \in \Gamma$, $S_\gamma \in \mathcal{L}(\mathcal{Y}; \mathcal{Y})$ is compact and bijective, and further, that there exists $\bar{K} < \infty$ such that $\|S_\gamma\| \le \bar{K}$ for all $\gamma \in \Gamma$. We will consider the existence of a nontrivial solution of

$$(R_1 + \epsilon S_\gamma)v = 0. \tag{17}$$

Let $\mathcal{R} = \mathcal{R}(R_1)$ and $\mathcal{N} = \mathcal{N}(R_1)$ denote the range and null space of R_1.

Lemma 7: $\lambda = 1$ is an isolated eigenvalue of H. $\mathcal{R}$ and $\mathcal{N}$ are closed, with $\dim(\mathcal{N}) < \infty$. Also, $\mathcal{X} = \mathcal{R} \oplus \mathcal{N}$ where $\mathcal{R} \perp \mathcal{N}$, and there exist corresponding orthogonal projections, $P_\mathcal{R}$ and $P_\mathcal{N}$ with $P_\mathcal{R}, P_\mathcal{N} \in \mathcal{L}(\mathcal{X}; \mathcal{X})$. Lastly, R_1 has a pseudo-inverse, denoted here as $R_1^\#$, with $R_1^\# \in \mathcal{L}(\mathcal{R}; \mathcal{R})$.

Proof: The lemma is a recollection of classical results. In particular, the compactness of H yields that $\lambda = 1$ is isolated, and that $\dim(\mathcal{N}) < \infty$. which also implies that $\mathcal{N}$ is closed. Also by the compactness of H, $\mathcal{R}$ is closed, cf. [26], Th. 5.7.8. As $\mathrm{dom}(R_1) = \mathcal{X}$ and $\mathcal{R}$ is closed, R_1 is closed by definition. By the closedness of R_1 and the self-adjoint assumption on H, $\mathcal{R} \perp \mathcal{N}$, cf. [26], Th. 4.11.2. By the above and [26], Th. 6.3.6, $\mathcal{X} = \mathcal{R} \oplus \mathcal{N}$. Further, $P_\mathcal{R}, P_\mathcal{N} \in \mathcal{L}(\mathcal{X}; \mathcal{X})$ by, for example, [26], Th. 4.12.6. Lastly, $R_1 : \mathcal{R} \subset \mathcal{X} \to \mathcal{R}$ is bounded, and hence $R_1^\# \in \mathcal{L}(\mathcal{R}; \mathcal{R})$, cf., [26] Th. 4.12.9. ∎

For $\gamma \in \Gamma$, let

$$\begin{aligned} \mathcal{R}^\gamma &\doteq S_\gamma^{-1} P_\mathcal{R} \mathcal{Y} \doteq \{v \in \mathcal{X} \mid S_\gamma v \in P_\mathcal{R} \mathcal{Y}\}, \\ \mathcal{N}^\gamma &\doteq S_\gamma^{-1} P_\mathcal{N} \mathcal{Y} \doteq \{v \in \mathcal{X} \mid S_\gamma v \in P_\mathcal{N} \mathcal{Y}\}, \end{aligned}$$

and recalling that S_γ is bijective on $\mathcal{Y} = \mathcal{R}(S_\gamma)$, note that

$$\mathcal{X} = \mathcal{R}^\gamma \oplus \mathcal{N}^\gamma. \tag{18}$$

We note that it is not necessarily true that $\mathcal{R}^\gamma \perp \mathcal{N}^\gamma$.

Theorem 8: There exists $\epsilon_0 > 0$ such that for all $\gamma \in \Gamma$ and $\epsilon \in (0, \epsilon_0]$, there does not exist $\bar{v} \in \mathcal{Y}$, $\bar{v} \neq 0$ satisfying (17).

Proof: Fix $\gamma \in \Gamma$. By the Fredholm alternative (cf. [26], Sec. 5.7.1), the claim is true if and only if for any $y \in \mathcal{Y}$, there exists unique $v \in \mathcal{Y}$ such that

$$(R_1 + \epsilon S_\gamma)v = y. \tag{19}$$

Recalling Lemma 7, note that (19) may be written as $(R_1 + \epsilon S_\gamma)v = y^\mathcal{R} + y^\mathcal{N}$ with $y^\mathcal{R} \doteq P_\mathcal{R} y$ and $y^\mathcal{N} \doteq P_\mathcal{N} y$, or equivalently as the pair of equations, $P_\mathcal{R}(R_1 + \epsilon S_\gamma)v = y^\mathcal{R}$ and $P_\mathcal{N}(R_1 + \epsilon S_\gamma)v = y^\mathcal{N}$. Using (18), one may decompose v as $v = v^{\mathcal{R}^\gamma} + v^{\mathcal{N}^\gamma}$ with $v^{\mathcal{R}^\gamma} \in \mathcal{R}^\gamma$ and $v^{\mathcal{N}^\gamma} \in \mathcal{N}^\gamma$, we may rewrite the above pair of equations as

$$\begin{pmatrix} P_\mathcal{R}(R_1 + \epsilon S_\gamma) & P_\mathcal{R}(R_1 + \epsilon S_\gamma) \\ P_\mathcal{N}(R_1 + \epsilon S_\gamma) & P_\mathcal{N}(R_1 + \epsilon S_\gamma) \end{pmatrix} \begin{pmatrix} v^{\mathcal{R}^\gamma} \\ v^{\mathcal{N}^\gamma} \end{pmatrix} = \begin{pmatrix} y^\mathcal{R} \\ y^\mathcal{N} \end{pmatrix}.$$

Noting that $P_{\mathcal{N}} R_1 : \mathcal{X} \to \{0\}$, this becomes

$$\begin{pmatrix} P_{\mathcal{R}}(R_1+\epsilon S_\gamma) & P_{\mathcal{R}}(R_1+\epsilon S_\gamma) \\ \epsilon P_{\mathcal{N}} S_\gamma & \epsilon P_{\mathcal{N}} S_\gamma \end{pmatrix} \begin{pmatrix} v^{\mathcal{R}^\gamma} \\ v^{\mathcal{N}^\gamma} \end{pmatrix} = \begin{pmatrix} y^{\mathcal{R}} \\ y^{\mathcal{N}} \end{pmatrix}. \tag{20}$$

By definition, $S_\gamma v^{\mathcal{R}^\gamma} \in \mathcal{R}$ and $S_\gamma v^{\mathcal{N}^\gamma} \in \mathcal{N}$, and hence, using Lemma 7, $P_{\mathcal{N}} S_\gamma v^{\mathcal{R}^\gamma} = 0 = P_{\mathcal{R}} S_\gamma v^{\mathcal{N}^\gamma}$. Applying these in (20) yields

$$\begin{pmatrix} P_{\mathcal{R}}(R_1+\epsilon S_\gamma) & P_{\mathcal{R}} R_1 \\ 0 & \epsilon P_{\mathcal{N}} S_\gamma \end{pmatrix} \begin{pmatrix} v^{\mathcal{R}^\gamma} \\ v^{\mathcal{N}^\gamma} \end{pmatrix} = \begin{pmatrix} y^{\mathcal{R}} \\ y^{\mathcal{N}} \end{pmatrix}. \tag{21}$$

By the nonsingularity of S_γ and the definition of $\mathcal{N}^\gamma$, there exists unique $\tilde{v}^{\mathcal{N}^\gamma} \in \mathcal{N}^\gamma$ such that $\epsilon P_{\mathcal{N}} S_\gamma \tilde{v}^{\mathcal{N}^\gamma} = y^{\mathcal{N}}$. Using this, we see that (21) reduces to

$$P_{\mathcal{R}}(R_1+\epsilon S_\gamma) v^{\mathcal{R}^\gamma} + P_{\mathcal{R}} R_1 \tilde{v}^{\mathcal{N}^\gamma} = y^{\mathcal{R}},$$

which by the definition of $P_{\mathcal{R}}$, is equivalently written as

$$R_1 v^{\mathcal{R}^\gamma} = \big[y^{\mathcal{R}} - R_1 \tilde{v}^{\mathcal{N}^\gamma}\big] - \epsilon P_{\mathcal{R}} S_\gamma v^{\mathcal{R}^\gamma},$$

or, recalling Lemma 7,

$$v^{\mathcal{R}^\gamma} = \tilde{z} - \epsilon R_1^{\#}\big[P_{\mathcal{R}} S_\gamma v^{\mathcal{R}^\gamma}\big] \doteq \mathcal{F}_{\epsilon,\gamma}\big(v^{\mathcal{R}^\gamma}\big), \tag{22}$$

where $\tilde{z} \doteq R_1^{\#}\big[y^{\mathcal{R}} - R_1 \tilde{v}^{\mathcal{N}^\gamma}\big]$. Recall that $\|S_\gamma\| \le \bar{K}$ for all $\gamma \in \Gamma$, and that $R_1^{\#}$ and $P_{\mathcal{R}}$ are bounded. This implies that there exists $\hat{K} < \infty$ such that $\|R_1^{\#} P_{\mathcal{R}} S_\gamma\| \le \hat{K}$ for all $\gamma \in \Gamma$. Consequently, there exists $\epsilon_0 > 0$ such that for all $\gamma \in \Gamma$ and $\epsilon \in (0, \epsilon_0]$, $\mathcal{F}_{\epsilon,\gamma}$ is a contraction on complete space $\mathcal{R}$. Hence, for all $\gamma \in \Gamma$ there exists a unique solution to (22). ■

V. Isolation of the Points of Nonuniqueness

As indicated above, we consider the case where $(\bar{s}, \bar{x})$ is such that we do not have uniqueness of the stationary-action trajectory. That is, we consider the case where there exists $v = \bar{v} \in \mathcal{U}_{\bar{s}}$, $\bar{v} \neq 0$ satisfying (16). We will show that there is a neighborhood around $\bar{x}$ on which this degeneracy does not occur, i.e., that the degeneracy occurs only at isolated points in $\mathbb{R}^n$. We begin by obtaining some derivatives. Throughout the remainder, we assume that ϕ is a quadratic form, i.e., $\phi(x) = \phi(x; z) = (c/2)|x - z|^2$ for some $c \in (0, \infty)$ and $z \in \mathbb{R}^n$.

A. Derivatives and Usage of the Mean Value Theorem

Lemma 9: The Fréchet derivative $\frac{d}{du}\big\{\big[G_{uu}(s,x,u)\big]\bar{v}\big\} = G_{uuu}(s,x,u)\bar{v}$ applied to $\hat{v} \in \mathcal{U}_{\bar{s}}$ has Riesz representation given component-wise as

$$\Big[\big(\tfrac{d}{du}\{[G_{uu}(s,x,u)]\bar{v}\}\hat{v}\big)_r\Big]_k = \tfrac{1}{m}\sum_{i,j}\Big\{\int_r^t V_{x_i x_j x_k}(\xi_\sigma)[\bar{w}_\sigma]_i[\hat{w}_\sigma]_j\,d\sigma + \phi_{x_i x_j x_k}(\xi_t)[\bar{w}_t]_i[\hat{w}_t]_j\Big\}$$

for all $k \in]1, n[$ and a.e. $r \in [\bar{s}, t]$, where $\hat{w}_\sigma \doteq \int_{\bar{s}}^\sigma \hat{v}_\rho\,d\rho$ and $\xi_\sigma \doteq x + \int_{\bar{s}}^\sigma u_\rho\,d\rho$ for all $\sigma \in [\bar{s}, t]$. Similarly, The derivative $\frac{d}{dx}\big\{[G_{uu}(s,x,u)]\bar{v}\big\} = G_{uux}(s,x,u)\bar{v}$ applied to $y \in \mathbb{R}^n$ has Riesz representation given component-wise as

$$\Big[\big(\tfrac{d}{dx}\{[G_{uu}(s,x,u)]\bar{v}\}y\big)_r\Big]_k = \tfrac{1}{m}\sum_{i,j}\Big\{\int_r^t V_{x_i x_j x_k}(\xi_\sigma)[\bar{w}_\sigma]_i\,d\sigma\,y_j + \phi_{x_i x_j x_k}(\xi_t)[\bar{w}_t]_i\,y_j\Big\}$$

for all $k \in]1, n[$ and a.e. $r \in [\bar{s}, t]$.

The proof is technical, and the details are not included. Throughout the remainder, we assume the following.

Either $\phi(x) = -z^T x$ for some $z \in \mathbb{R}^n$, or $\phi(x) = (c/2)|x - z|^2$ for some $c \in (0, \infty)$ and $z \in \mathbb{R}^n$. $(A.3)$

There exists $D_{v,1} < \infty$ such that $|V_x(x)| \le D_{v,1}$ for all $x \in \mathbb{R}^n$. $(A.4)$

Remark 10: We make two remarks regarding these assumptions. Assumption $(A.3)$ corresponds to TPBVPs where either the terminal velocity or the terminal position is specified, cf. [24]. Assumption $(A.4)$ includes the case of gravitation for spherically symmetric bodies, which will be further indicated in Section VI.

The next lemma follows immediately from Lemma 9 and Assumption $(A.3)$.

Lemma 11: The Fréchet derivative $G_{uuu}(s,x,u)\bar{v} \doteq \frac{d}{du}\big\{\big[G_{uu}(s,x,u)\big]\bar{v}\big\}$ applied to $\hat{v} \in \mathcal{U}_{\bar{s}}$ has Riesz representation given component-wise as

$$\Big[\big(\tfrac{d}{du}\{[G_{uu}(s,x,u)]\bar{v}\}\hat{v}\big)_r\Big]_k = \tfrac{1}{m}\sum_{i,j}\int_r^t V_{x_i x_j x_k}(\xi_\sigma)[\bar{w}_\sigma]_i[\hat{w}_\sigma]_j\,d\sigma$$

for all $k \in]1, n[$ and a.e. $r \in [\bar{s}, t]$, where $\hat{w}_\sigma \doteq \int_{\bar{s}}^\sigma \hat{v}_\rho\,d\rho$ and $\xi_\sigma \doteq x + \int_{\bar{s}}^\sigma u_\rho\,d\rho$ for all $\sigma \in [\bar{s}, t]$. Similarly, The derivative $G_{uux}(s,x,u)\bar{v} \doteq \frac{d}{dx}\big\{[G_{uu}(s,x,u)]\bar{v}\big\}$ applied to $y \in \mathbb{R}^n$ has Riesz representation given component-wise as

$$\Big[\big(\tfrac{d}{dx}\{[G_{uu}(s,x,u)]\bar{v}\}y\big)_r\Big]_k = \tfrac{1}{m}\sum_{i,j}\int_r^t V_{x_i x_j x_k}(\xi_\sigma)[\bar{w}_\sigma]_i\,d\sigma\,y_j$$

for all $k \in]1, n[$ and a.e. $r \in [\bar{s}, t]$.

Let $\bar{z} \in \mathbb{R}^n$, $|\bar{z}| = 1$, $\bar{\beta} \in (0, d)$ (where d was indicated in Assumption $(A.2)$), and let $u^+ \in \operatorname{argstat}_{u \in \mathcal{G}} J(\bar{s}, \bar{x} + \bar{\beta}\bar{z}, u)$ (or equivalently, that (14) holds at $(\bar{s}, \bar{x}+\bar{\beta}\bar{z}, u^+)$). We consider the line segment in $\mathbb{R}^n \times \mathcal{U}_{\bar{s}}$ given by $(\breve{x}(\beta), \breve{u}(\beta)) \doteq (\bar{x}+\beta\bar{z}, \bar{u}+(\beta/\bar{\beta})(u^+ - \bar{u}))$ for $\beta \in [0, \bar{\beta}]$. Given $(\bar{s}, \bar{x}) \in \mathcal{D}$, $\bar{\beta}, \bar{z}$ as above and $\bar{u}, u^+, \bar{v}, \hat{v} \in \mathcal{U}_{\bar{s}}$, let $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ be given by

$$\mathcal{F}(\beta; \bar{s}, \bar{x}, \bar{z}, \bar{\beta}, \bar{u}, u^+, \bar{v}, \hat{v}) \doteq \big\langle \hat{v}, \big[\mathcal{I} - G_{uu}\big(\bar{s}, \bar{x}+\beta\bar{z}, \bar{u}+(\beta/\bar{\beta})(u^+ - \bar{u})\big)\big]\,\bar{v}\big\rangle.$$

By the mean value theorem, there exists $\hat{\beta} = \hat{\beta}(\bar{v}, \hat{v}) \in (0, \bar{\beta})$ such that

$$\mathcal{F}(\bar{\beta}; \bar{s}, \bar{x}, \bar{z}, \bar{\beta}, \bar{u}, u^+, \bar{v}, \hat{v})$$

$$= \mathcal{F}(0; \bar{s}, \bar{x}, \bar{z}, \bar{u}, \bar{\beta}, u^+, \bar{v}, \hat{v}) + \mathcal{F}_\beta(\hat{\beta}; \bar{s}, \bar{x}, \bar{z}, \bar{\beta}, \bar{u}, u^+, \bar{v}, \hat{v})\bar{\beta}. \tag{23}$$

Equivalently,

$$\left\langle \hat{v}, \left[\mathcal{I} - G_{uu}\left(\bar{s}, \bar{x} + \bar{\beta}\bar{z}, u^+\right)\right] \bar{v} \right\rangle = \left\langle \hat{v}, (R_1 + \bar{\beta} S(\hat{\beta}, \bar{z}))\bar{v} \right\rangle, \tag{24}$$

where

$$R_1 = \left[\mathcal{I} - G_{uu}\left(\bar{s}, \bar{x}, \bar{u}\right)\right], \tag{25}$$

$$S(\hat{\beta}, \bar{z}) = -G_{uux}(\bar{s}, \bar{x} + \hat{\beta}\bar{z}, \bar{u} + (\hat{\beta}/\bar{\beta})(u^+ - \bar{u}))\bar{z} - G_{uuu}(\bar{s}, \bar{x} + \hat{\beta}\bar{z}, \bar{u} + (\hat{\beta}/\bar{\beta})(u^+ - \bar{u}))(1/\bar{\beta})(u^+ - \bar{u}). \tag{26}$$

Using Lemma 11 and the chain rule, we find

$$\bar{\beta}\langle \hat{v}, S(\hat{\beta}, \bar{z})\bar{v}\rangle = \frac{\bar{\beta}}{m} \sum_{i,j,k} \int_{\bar{s}}^t \int_r^t V_{x_i x_j x_k}(\tilde{\xi}_\rho)(\bar{z}_k + [w_\rho^\Delta]_i) [\bar{w}_\rho]_j \, d\rho \, [\hat{v}_r]_k \, dr, \tag{27}$$

where $\tilde{\xi}_r = (\bar{x} + \hat{\beta}\bar{z}) + \int_{\bar{s}}^r \bar{u}_\rho + (\hat{\beta}/\bar{\beta})(u^+_\rho - \bar{u}_\rho)\, d\rho$, $w_r^\Delta \doteq (1/\bar{\beta}) \int_{\bar{s}}^r (u^+_\rho - \bar{u}_\rho)\, d\rho$, $\bar{w}_r \doteq \int_{\bar{s}}^r \bar{v}_\rho \, d\rho$ and $\hat{w}_r \doteq \int_{\bar{s}}^r \hat{v}_\rho \, d\rho$. Also, defining $[\mathcal{V}^k(\tilde{\xi}_r)]_{i,j} \doteq V_{x_i x_j x_k}(\tilde{\xi}_r)$, it is helpful to note the component-wise Riesz representation form

$$[(S(\hat{\beta}, \bar{z})\bar{v})_r]_k = \frac{1}{m} \sum_{i,j} \int_r^t [\mathcal{V}^k(\tilde{\xi}_\rho)]_{i,j} (\bar{z}_i + [w_\rho^\Delta]_i)[\bar{w}_\rho]_j \, d\rho = \frac{1}{m} \left[\int_r^t (\bar{z} + w_\rho^\Delta)^T \mathcal{V}^k(\tilde{\xi}_\rho)\bar{w}_\rho \, d\rho \right]_k. \tag{28}$$

It will be helpful to consider the structure of (28) a bit further. Note that

$$\begin{aligned} \bar{z} + w_r^\Delta &= \bar{z} + (1/\bar{\beta}) \int_{\bar{s}}^r (u^+_\rho - \bar{u}_\rho)\, d\rho \\ &= \tfrac{1}{\bar{\beta}} \left[\bar{x} + \bar{\beta}\bar{z} + \int_{\bar{s}}^r u^+_\rho \, d\rho - (\bar{x} + \int_{\bar{s}}^r \bar{u}_\rho \, d\rho)\right] \\ &= \xi^+_r - \bar{\xi}_r. \end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned} \tilde{\xi}_r &= (\bar{x} + \hat{\beta}\bar{z}) + \int_{\bar{s}}^r \bar{u}_\rho + (\hat{\beta}/\bar{\beta})(u^+_\rho - \bar{u}_\rho)\, d\rho \\ &= \left(1 - \tfrac{\hat{\beta}}{\bar{\beta}}\right)\left[\bar{x} + \int_{\bar{s}}^r \bar{u}_\rho \, d\rho\right] + \left(\tfrac{\hat{\beta}}{\bar{\beta}}\right)\left[\bar{x} + \bar{\beta}\bar{z} + \int_{\bar{s}}^r u^+_\rho \, d\rho\right] \\ &= \left(1 - \tfrac{\hat{\beta}}{\bar{\beta}}\right)\bar{\xi}_r + \left(\tfrac{\hat{\beta}}{\bar{\beta}}\right)\xi^+_r. \end{aligned} \tag{30}$$

Applying (29) and (30) in (28), we see

$$[(S(\hat{\beta}, \bar{z})\bar{v})_r]_k = \frac{1}{m} \left[\int_r^t (\xi^+_\rho - \bar{\xi}_\rho)^T \mathcal{V}^k\left(\left(1 - \tfrac{\hat{\beta}}{\bar{\beta}}\right)\bar{\xi}_\rho + \left(\tfrac{\hat{\beta}}{\bar{\beta}}\right)\xi^+_\rho\right)\bar{w}_\rho \, d\rho \right]_k \tag{31}$$

B. Compactness of $S(\hat{\beta}, \bar{z})$

We fix any $\hat{\beta} \in [0, \bar{\beta}]$ and $\bar{z}$ with $|\bar{z}| = 1$, and for compactness of notation, we let $S = S(\hat{\beta}, \bar{z})$ throughout this section. Defining

$$\mathcal{Y} \doteq \left\{ y \in C[\bar{s}, t] \,\middle|\, \exists v \in \mathcal{U}_{\bar{s}} \text{ s.t. } y_r = \int_r^t \int_{\bar{s}}^\rho v_\sigma \, d\sigma \, d\rho \;\forall r \in [\bar{s}, t] \right\}, \tag{32}$$

it will be shown that $S : \mathcal{Y} \to \mathcal{Y}$ is a compact linear operator. We begin with a technical lemma; in the interests of space, the proof is not included.

Lemma 12: Suppose there exists $\hat{C} < \infty$ such that $|\bar{\xi}_r|, |\xi^+_r|, |\bar{u}_r|, |u^+_r| \le \hat{C}$ for all $r \in [\bar{s}, t]$. There exists $\bar{C} < \infty$ such that for any $h \in (0, 1]$, any s_1, s_2 such that $\bar{s} + h \le s_1 \le s_2 \le t - h$ and any $v \in \mathcal{Y}$,

$$\int_{s_1}^{s_2} \left| \left[\tfrac{d}{dr} Sv\right]_{r+h} - \left[\tfrac{d}{dr} Sv\right]_r \right|^2 dr \le \bar{C}\|v\|^2 h^2.$$

Lemma 13: Suppose there exists $\hat{C} < \infty$ such that $|\bar{\xi}_r|, |\xi^+_r|, |\bar{u}_r|, |u^+_r| \le \hat{C}$ for all $r \in [\bar{s}, t]$. Let $R < \infty$. There exists $\bar{C} < \infty$ such that for any $h \in (0, 1]$, any s_1, s_2 such that $\bar{s} + h \le s_1 \le s_2 \le t - h$ and any $v \in B_R(0) \subset \mathcal{Y}$,

$$\int_{s_1}^{s_2} \left| \left[\tfrac{d}{dr} Sv\right]_{r+h} - \left[\tfrac{d}{dr} Sv\right]_r \right|^2 dr \le \bar{C} R^2 h^2.$$

The two lemmas above assume C^1 $\bar{\xi}$ and ξ^+ trajectories. We indicate that these conditions are satisfied for two standard forms of ϕ. We note here that throughout, in the interests of space, proofs have been omitted.

Proposition 14: There exists $\hat{C} < \infty$ such that $|\bar{\xi}_r|, |\xi^+_r|, |\bar{u}_r|, |u^+_r| \le \hat{C}$ for all $r \in [\bar{s}, t]$.

Combining Lemma 13 and Proposition 14, we obtain

Lemma 15: Let $R < \infty$. There exists $\bar{C} < \infty$ such that for any $h \in (0, 1]$, any s_1, s_2 such that $\bar{s} + h \le s_1 \le s_2 \le t - h$ and any $v \in B_R(0) \subset \mathcal{Y}$,

$$\int_{s_1}^{s_2} \left| \left[\tfrac{d}{dr} Sv\right]_{r+h} - \left[\tfrac{d}{dr} Sv\right]_r \right|^2 dr \le \bar{C} R^2 h^2.$$

Lemma 16: $S \in L_2(\mathcal{Y}; \mathcal{Y})$. That is, S is a bounded linear operator from $\mathcal{Y}$ into $\mathcal{Y}$.

Lemma 17: Suppose $v^n \in \mathcal{Y}$ for all $n \in \mathbb{N}$, and that there exists $R < \infty$ such that $\|v^n\| \le R$ for all $n \in \mathbb{N}$. Then, there exists $\dot{\tilde{y}} \in C[\bar{s}, t]$ and subsequence $\{v^{n_k}\}_{k \in \mathbb{N}}$ such that $\frac{d}{dr} S v^{n_k} \to \dot{\tilde{y}}$ uniformly on $[\bar{s}, t]$.

Lemma 18: Suppose $v^n \in \mathcal{Y}$ for all $n \in \mathbb{N}$, and that there exists $R < \infty$ such that $\|v^n\| \le R$ for all $n \in \mathbb{N}$. Let $\dot{\tilde{y}}$ and $\{v^{n_k}\}$ be as indicated in Lemma 17, and let $\tilde{y}_r \doteq -\int_r^t \dot{\tilde{y}}_\rho \, d\rho$ for all $r \in [\bar{s}, t]$. Then, as $k \to \infty$, $S v^{n_k} \to \tilde{y}$ uniformly on$[\bar{s}, t]$ and $\|S v^{n_k} - \tilde{y}\| \to 0$.

Lemma 19: Suppose $v^n \in \mathcal{Y}$ for all $n \in \mathbb{N}$, and that there exists $R < \infty$ such that $\|v^n\| \le R$ for all $n \in \mathbb{N}$. Let $\tilde{y}$ and $\{v^{n_k}\}$ be as indicated in Lemma 18. Then, $\tilde{y} \in \mathcal{Y}$.

Theorem 20: $S \in L_2(\mathcal{Y}; \mathcal{Y})$ is compact.

C. Main Results on Isolation

Theorem 21: For $\beta \in (0, d)$, let $S(\beta, \bar{z}) : \mathcal{Y} \to \mathcal{Y}$ be given by (28) (equivalently, (27)). Suppose there exists $\bar{\beta}_0 \in (0, d)$ and $\bar{K} < \infty$ such that for all $\beta \in (0, \bar{\beta}_0)$ and $|\bar{z}| = 1$, $S(\beta, \bar{z})$ is bijective and such that $\|S(\beta, \bar{z})\| \le \bar{K}$. Then, there exists $\bar{\beta}_1 \in (0, \bar{\beta}_0]$ such that for all $\beta \in (0, \bar{\beta}_1]$, $|\bar{z}| = 1$ there does not exist $\bar{v} \in \mathcal{Y}$, $\bar{v} \neq 0$ such that

$$\left[\mathcal{I} - G_{uu}(\bar{s}, \bar{x} + \beta\bar{z}, u_\beta^+)\right]\bar{v} = 0, \tag{33}$$

where $u_\beta^+ \in \operatorname{argstat}_{u\in\mathcal{G}} J(\bar{s}, \bar{x} + \beta\bar{z}, u)$.

Corollary 22: Let $\mathcal{Y}$, $\mathcal{S}(\cdot,\cdot)$ and $\bar{\beta}_0$ be as in Theorem 21. There exists $\bar{\beta}_1 \in (0, \bar{\beta}_0]$ such that for all $\tilde{x} \in B_{\bar{\beta}_1}(\bar{x}) \setminus \{\bar{x}\}$, $[\mathcal{I}u - G_u(\bar{s}, x, u)] = 0$ uniquely defines differentiable $\breve{u} : \mathcal{O} \to \mathcal{U}_s$ on some neighborhood, $\mathcal{O}$, of $\tilde{x}$, with $\breve{u}(\tilde{x}) \in \operatorname{argstat}_{u\in\mathcal{G}} J(\bar{s}, \tilde{x}, u)$.

The corollary indicates that points where uniqueness of the stationary-action solution fails are isolated.

VI. Gravitational Systems

In this section, we consider the case of gravitational systems associated to some set of spherical masses, that is, the classic N-body problem. We will show that the conditions of Theorem 21 (and hence Corollary 22) are satisfied.

Note that for spherical, uniform density bodies, the associated gravitational potential satisfies $V \in C^1(\mathbb{R}^3)$, cf. [18], [24]. This may be mollified to a C^∞ form such that the mollification and the derivatives approximate the original potential arbitrarily closely (cf. [1] among many others), and such that the mollification is identical to the original potential outside a sphere with radius $\tilde{\epsilon} > 0$ greater than the body radius. Henceforth, we assume that V is such a mollified potential, and consequently satisfies Assumption $(A.1)$ and $(A.4)$.

The following is immediate from (28) (cf. [26], Section 5.7).

Lemma 23: For all $\beta \in (0, d)$ and $|\bar{z}| = 1$, $S(\beta, \bar{z})$ is compact.

We will find it useful to denote by $(\xi_r^+ - \bar{\xi}_r)^T \mathcal{V}(\tilde{\xi}_r)$ the matrix given component-wise by

$$\left[(\xi_r^+ - \bar{\xi}_r)^T \mathcal{V}(\tilde{\xi}_r)\right]_{j,k} \doteq \sum_{i=1}^n [\xi_r^+ - \bar{\xi}_r]_i [\mathcal{V}^k(\tilde{\xi}_r)]_{i,j}.$$

With this notation, we write

$$(S(\hat{\beta}, \bar{z})\bar{v})_r = \tfrac{1}{m}\int_r^t (\xi_\rho^+ - \bar{\xi}_\rho)^T \mathcal{V}(\tilde{\xi}_r)\bar{w}_\rho \, d\rho \tag{34}$$

$$= \tfrac{1}{m}\int_r^t (\xi_\rho^+ - \bar{\xi}_\rho)^T \mathcal{V}\Big(\big(1 - \tfrac{\hat{\beta}}{\beta}\big)\bar{\xi}_\rho + \big(\tfrac{\hat{\beta}}{\beta}\big)\xi_\rho^+\Big)\bar{w}_\rho \, d\rho \tag{35}$$

Using (34), one easily proves the uniform boundedness criterion of Theorem 21 and Corollary 22. Specifically, we have the following.

Lemma 24: Then, there exist $\bar{K} < \infty$ such that $\|S(\beta, \bar{z})\| \leq \bar{K}$ for all $\beta \in (0, \bar{\beta}_0)$ and $|\bar{z}| = 1$.

The bijectivity required by Theorem 21 and Corollary 22 remains to be obtained. Recall that by Theorem 20, $S(\beta, \bar{z}) \in L_2(\mathcal{Y}; \mathcal{Y})$ is compact. Hence, by standard results (cf. [26], Section 5.7), S is surjective if and only if it is injective. Consequently, it only remains to demonstrate that $S(\beta, \bar{z}) \in L_2(\mathcal{Y}; \mathcal{Y})$ is injective. In the two-body case, this is easily demonstrated, and the computations are not included. The n-body case is more technical, and the computations are delayed to a later effort.

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