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On the Polymatroidal Structure of Quasi-Uniform Codes with Applications to Heterogeneous Distributed Storage

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***Abstract*— Recent research on distributed storage systems (DSSs) has revealed interesting connections between locally repairable codes (LRCs) and their associated matroids and polymatroids. In this paper we define L-polymatroids — polymatroids with an added length function — in order to consider completely general LRCs in that they are defined as subsets of $\mathbb{A}^E = \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, where each $\mathbb{A}_i$ is some arbitrary finite set. Earlier research in this area has only considered codes over non-mixed alphabets, *i.e.*, $\mathbb{A}_1 = \cdots = \mathbb{A}_n$.**

We generalize the notions of locality and availability to L-polymatroids, and a Singleton-type bound for L-polymatroids is given. This result implies a corresponding bound on LRCs and generalizes earlier Singleton-type bounds given on LRCs. Moreover, the necessary structural conditions are given for L-polymatroids achieving the bound, yielding also the corresponding necessary conditions for LRCs. Finally, implications of our results for quasi-uniform codes and in particular quasi-uniform codes from a construction built on cosets of groups are examined.

I. Introduction

In distributed storage systems (DSSs), the data is stored on multiple storage nodes instead of just one big server. Recent work has studied various properties of locally repairable codes (LRCs) with availability, and new Singleton-type bounds have been proved for these new settings. See [1] for a comprehensive summary, relevant references, as well as for an introduction to a matroidal perspective to the topic.

In this paper, we generalize the typical homogeneous storage setting to a more heterogeneous one, where the system can accommodate nodes storing data over different alphabets. This is potentially useful in, *e.g.*, peer-to-peer and device-to-device networks, where different (user) devices may have different storage capabilities, different subscriptions to bandwidth etc. Entropy can be used to give a measure on the amount of information in a code and the amount of information in the ambient space of the code. Hence, using entropy, we are able to compare properties such as rate and distance for codes constructed over different alphabets. These properties can be captured by the new concept of entropic L-polymatroids associated to these codes, both for codes over mixed and non-mixed alphabets. More generally, the typical parameters (n, k, d, r, δ, t) of an LRC (cf. Def. 2.3) can be generalized to L-polymatroids, which allows us to use L-polymatroid theory in order to analyze fully general LRCs.

A. Related work and contributions

It was shown in [2] that $(r, 2)$-locality of linear LRCs over finite fields is a matroid invariant. The connection between matroids and linear LRCs over finite fields was further developed in [3], where the main matroidal concept used was the notion of cyclic flats. This approach was then generalized in [4], where a link between polymatroid theory and LRCs over non-mixed alphabets, *i.e.*, a code is a subset of $\mathbb{A} \times \ldots \times \mathbb{A}$ where $\mathbb{A}$ is some finite set, was established and the parameters of LRCs were generalized to polymatroids. A generalized Singelton bound was given for polymatroids and LRCs.

Since $\log_q(|\mathbb{A}^A|) = |A|$ when $|\mathbb{A}| = q$ for non-mixed alphabets, the approach of using $|A|$ for defining rate and cyclic flats works perfectly in [4] and [3], as $|A|$ captures the joint entropy properties of the ambient subspaces $\mathbb{A}^A$ for non-mixed alphabets. However, this approach does not work for mixed alphabets. Here, we take one step further and define the concept of L-polymatroids, *i.e.*, polymatroids with an added length function. This enables us to establish a link to LRCs with storage nodes that can store data over different alphabets. To the best of the authors' knowledge, this is the first time LRCs are considered in such a fully general setting. We give a Singleton-type bound for L-polymatroids. This result implies a corresponding bound on LRCs and generalizes earlier Singleton-type bounds given on LRCs. Moreover, the necessary structural conditions are given for L-polymatroids achieving the bound, yielding also the corresponding necessary conditions for LRCs.

Quasi-uniform codes were introduced in [5] as a broad class of codes satisfying some particular entropy properties. Constructions of quasi-uniform codes are examined in [5], [6]. Applications of quasi-uniform codes are discussed in [5], and one of the discussed applications considers distributed storage. The connection between distributed storage and quasi-uniform codes was also studied in [7], [8]. In this paper, we give explicit formulas for the rank and length function of the entropic L-polymatroids associated to quasi-uniform codes. In particular, we explicitly give the formulas for a group theoretical construction of quasi-uniform codes given in [5], [6]. These formulas can be used with the

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Singleton-type bound and the structure theorem given for L-polymatroids here.

Using entropy to analyze LRCs has, for example, been used in [9], [10], [11]. Further, combinatorial methods other than matroids and polymatroids have been used for LRCs, *e.g.*, by the concept of regenerating sets in [11].

II. Codes, entropy and distributed storage

Throughout the paper, we will use the following notation: E is a finite set, $2^E = \{A : A \subseteq E\}$, $B - A = \{b \in B : b \notin A\}$ for $A, B \subseteq E$, and $[\ell] = \{1, 2, \ldots, \ell\}$. For ease of notation, we will often use the notation e instead of $\{e\}$ for elements $e \in E$. All logarithms will be using $q > 1$ as a base throughout this paper.

Remark 2.1: As explained in the introduction, we will have two different length concepts in this paper. One is the usual *size* of the code vector in number of coordinates, which we will denote by n'. However, as we will consider mixed alphabets, this does not capture the amount of information (entropy) the coordinates carry, as the coordinates can be over different alphabets. Therefore, n is reserved to denote another *length* concept that does capture this property, and hence makes computation of rates fair. See Sec. II-A, Def. 2.1, Rem. 2.2, and Sec. III for more details.

A. Codes and entropy

Let $E = \{j_1, \ldots, j_{n'}\}$ and let $\mathbb{A}_i$ be a finite set for each $i \in E$. Define $\mathbb{A}^E$ as the cartesian product

$$\mathbb{A}^E = \mathbb{A}_{j_1} \times \ldots \times \mathbb{A}_{j_{n'}}.$$

A *code* in $\mathbb{A}^E$ is a subset C of $\mathbb{A}^E$. The set $\mathbb{A}^E$ is the *ambient space* of C. For $\boldsymbol{a}, \boldsymbol{z} \in \mathbb{A}^E$, *supports of* $\boldsymbol{a}$ *and* C *relative to* $\boldsymbol{z}$ are defined as

$$\begin{aligned}\operatorname{supp}(\boldsymbol{a}, \boldsymbol{z}) &= \{i \in E : a_i \neq z_i\},\\ \operatorname{supp}(C, \boldsymbol{z}) &= \textstyle\bigcup_{\boldsymbol{c} \in C} \operatorname{supp}(\boldsymbol{c}, \boldsymbol{z}),\end{aligned}$$

respectively. Further, for $B = \{i_1, \ldots, i_j\} \subseteq E$ and $\boldsymbol{v} \in \mathbb{A}^B$, the *puncturings* of $\boldsymbol{a}$ and C into $\mathbb{A}^B$, and *shortening* of C by A relative to $\boldsymbol{v}$ are defined as

$$\begin{aligned}\boldsymbol{a}_{|B} &= (a_{i_1}, \ldots, a_{i_j}),\\ C|B &= \{\boldsymbol{c}_{|B} : \boldsymbol{c} \in C\},\\ C/B(\boldsymbol{v}) &= \{\boldsymbol{c}_{|\overline{B}} : \boldsymbol{c} \in C, \, \boldsymbol{c}_{|B} = \boldsymbol{v}\},\end{aligned}$$

respectively. For technical reasons, we let $C|\emptyset = \{\emptyset\}$ with $|C|\emptyset| = 1$.

Let $p_C = \{p_{\boldsymbol{c}}\}_{\boldsymbol{c} \in C}$ be a probability distribution on C, *i.e.*, $p_{\boldsymbol{c}} > 0$ for $\boldsymbol{c} \in C$ and $\sum_{\boldsymbol{c} \in C} p_{\boldsymbol{c}} = 1$. Further, let $\boldsymbol{Z} = \{Z_i\}_{i \in E}$ be a random vector governed by p_C. The *joint entropy* of C on $A \subseteq E$ is defined as

$$H_C(A) = \sum_{\boldsymbol{z} \in C|A} \Pr(\boldsymbol{Z_A} = \boldsymbol{z}) \log_q \left(\frac{1}{\Pr(\boldsymbol{Z_A} = \boldsymbol{z})} \right),$$

where $H_C(\emptyset) = 0$ and $\boldsymbol{Z}_A = \{Z_i\}_{i \in A}$. If C has a *uniform probability distribution*, *i.e.*, $p_{\boldsymbol{c}} = \frac{1}{|C|}$ for all $\boldsymbol{c} \in C$, then

$$H_C(A) = \sum_{\boldsymbol{z} \in C|A} \frac{|C/A(\boldsymbol{z})|}{|C|} \log_q \left(\frac{|C|}{|C/A(\boldsymbol{z})|} \right).$$

B. Global and local parameters for distributed storage via entropy

Definition 2.1: Let C be a code in $\mathbb{A}^E$ with probability distribution p_C. Then $n = \log_q(|\mathbb{A}^E|)$, $k = H_C(E)$, $\mathcal{R} = \frac{k}{n}$, and $d = |E| - \max\{|A| : A \subseteq E \text{ and } H_C(A) < k\}$.

Remark 2.2: The definitions of $n, k, \mathcal{R}, d$ above correspond to the definitions of length, dimension, rate and minimum Hamming distance for linear codes over finite fields, when q is taken to be the field size and the probability distribution on the code is assumed to be uniform. Moreover, $\mathcal{R} = -\sum_{\boldsymbol{c} \in C} p_{\boldsymbol{c}} \log_{|\mathbb{A}^A|} p_{\boldsymbol{c}}$. Consequently, n and k are dependent on the value of q, while $\mathcal{R}$ and d are not. Hence, the rate should, in general, be used instead of n and k individually when comparing the performance of two codes.

Definition 2.2: A code C is *non-degenerate* if $H_C(E) - H_C(E - e) < \|e\|$ and $H_C(e) > 0$ for each coordinate $e \in E$.

Definition 2.3: An LRC in $\mathbb{A}^E$ is a non-degenerate (n, k, d)-code C. A coordinate $e \in E$ has *locality* (r, δ), $\delta \geq 2$, and availability t if there are t subsets $R_1, \ldots, R_t$ of E, called *recovering sets* of e, such that for $i, j \in [t]$,

$$\begin{aligned}&(i) && e \in R_i,\\ &(ii) && |R_i| \leq r + \delta - 1,\\ &(iii) && d(C|R_i) \geq \delta,\\ &(iv) && i \neq j \quad \Rightarrow \quad R_i \cap R_j = \{e\}.\end{aligned}$$

If every element $e \in S \subseteq E$ has locality (r, δ) and availability t in C, then C has $(n, k, d, r, \delta, t)_S$-availability.

Remark 2.3: The common definition given in the literature of an (n, k, d, r, δ, t)-LRC C over $\mathbb{A}$, where δ denotes the local distance of a recovering set, corresponds to the special case where C has a uniform probability distribution, $\mathbb{A}_e = \mathbb{A}$ for each $e \in E$ and $|\mathbb{A}| = q$. Further, for $(n, k, d, r, \delta, r)_S$-LRCs, information-symbol locality, denoted by $(n, k, d, r, \delta, t)_i$, corresponds to the case that there is a subset S of E such that $H_C(S) = H_C(E)$. All-symbol locality, denoted by $(n, k, d, r, \delta, t)_a$, corresponds to the case $S = E$.

III. (n, k, d, r, δ, t)-L-polymatroids

A. L-polymatroids

The concept of polymatroids was introduced in [12]. A polymatroid can equivalently be considered as a set-combinatorial object with an associated rank function or as a special class of convex polytopes. We will here use the set-combinatorial approach. For more on polymatroids, see for example [13].

Definition 3.1: A pair $P = (E, \rho)$ is a *polymatroid* on E with *rank function* $\rho : 2^E \to \mathbb{R}$ if ρ satisfies the following three conditions for all subsets $A, B \subseteq E$:

$$\begin{aligned}&(R1) && \rho(\emptyset) = 0,\\ &(R2) && A \subseteq B \Rightarrow \rho(A) \leq \rho(B),\\ &(R3) && \rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B).\end{aligned}$$

A *matroid* is a polymatroid which additionally satisfies the following two conditions for all $A \subseteq E$:

$$\begin{aligned}&(R4) && \rho(A) \in \mathbb{Z},\\ &(R5) && \rho(A) \leq |A|.\end{aligned}$$

We will now introduce the concept of L-polymatroids, which is a polymatroid with an added length function. This concept is inspired by [14], where a function, which indeed is a length function, is used in order to define a dual of a polymatroid.

Definition 3.2: A *polymatroid with a length function*

$$\|\cdot\| : 2^E \to \mathbb{R},$$

a so called *L-polymatroid*, is a triple $P = (E, \rho, \|\cdot\|)$ where (E, ρ) is a polymatroid and the function $\|\cdot\|$ satisfies the following conditions for $e \in E$ and $A \subseteq E$:

$$\begin{array}{ll} (L1) & \|e\| > 0, \\ (L2) & \|e\| \geq \rho(e), \\ (L3) & \|A\| = \sum_{e \in A} \|e\|. \end{array}$$

B. Entropic L-polymatroids

Fujishige showed in [15] that letting $\rho(A) = H_C(A)$ for any code $C \subseteq \mathbb{A}^E$, probability distribution p_C and subset $A \subseteq E$ makes (E, ρ) a polymatroid. A polymatroid which can be represented by entropy in such a way is called *entropic*. This concept is naturally extended to L-polymatroids by the following definition.

Definition 3.3: Let $C \subseteq \mathbb{A}^E$ be a code with probability distribution p_C. Then $P_C^{ent} = (E, \rho, \|\cdot\|)$ denotes the associated entropic L-polymatroid of C, where

$$\rho(A) = H_C(A) \text{ and } \|A\| = \log_q(|\mathbb{A}^A|) \text{ for } A \subseteq E.$$

As an almost direct consequence of the log sum inequality, for $A \subseteq E$,

$$\begin{array}{l} H_C(A) \leq \log_q(|C|A|) \text{ with equality iff} \\ \Pr(\boldsymbol{Z}_A = \boldsymbol{z}) = \frac{1}{|C|A|} \text{ for all } \boldsymbol{z} \in C|A. \end{array} \quad (1)$$

We remark that if C has a uniform probability distribution and equals its ambient space $\mathbb{A}^E$, then $H_C(A) = \log_q(|\mathbb{A}^A|) \geq H_{C'}(A)$ for all $A \subseteq E$ and for all codes C' in $\mathbb{A}^E$. Hence, in the perspective that $H_C(A)$ gives a measure on the amount of information in $C|A$, the rate $\mathcal{R} = \frac{H_C(A)}{\log_q(|\mathbb{A}^A|)}$ gives a measure on how far $C|A$ is from having the maximum amount of information.

C. (n, k, d, r, δ, t)-L-polymatroids

Since $\rho(A) = H_C(A)$ and $\|A\| = \log_q(|\mathbb{A}^A|)$ in the definition of entropic L-polymatroids above, we directly obtain that the parameters $(n, k, d, r, \delta, t)_S$ of an LRC C are captured by its L-polymatroid P_C^{ent}. Consequently, the notion of $(n, k, d, r, \delta, t)_S$-LRCs can be directly generalized to $(n, k, d, r, \delta, t)_S$-L-polymatroids (both entropic and non-entropic), by letting $\rho(A)$ correspond to $H_C(A)$ and $\|A\|$ correspond to $\log_q(|\mathbb{A}^A|)$ in the definitions in Section II-B. Hence, in the context of L-polymatroid theory for (n, k, d, r, δ, t)-L-polymatroids we obtain that

$$\begin{array}{l} n = \|E\|, \\ k = \rho(E), \\ d = |E| - \max\{|A| : A \subseteq E,\ \rho(A) < k\}. \end{array} \quad (2)$$

A subset $R \subseteq E$ is an (r, δ)-recovering set if

$$|R| \leq r + \delta - 1 \text{ and } \rho(A) = \rho(R) \quad (3)$$

for all $A \subseteq R$ where $|A| \geq |R| - \delta + 1$. Also, an L-polymatroid is non-degenerate if $\rho(E) - \rho(E - e) < \|e\|$ and $\rho(e) > 0$ for all $e \in E$.

IV. A Singleton-type bound for L-polymatroids and cyclic flats

We first remark, as discussed in particular for LRCs earlier in Section II-B, that when comparing how good different parameters $(n, k.d, r, \delta, t)_S$ are for L-polymatroids, one should compare the rate $\mathcal{R} = \frac{k}{n}$ and not the parameters n and k individually.

A. Cyclic flats

By using $\|A\|$ instead of $|A|$ in the definition of cyclic flats given in [4], we obtain the following definition for L-polymatroids.

Definition 4.1: Let $P = (E, \rho, \|\cdot\|)$ be an L-polymatroid and $A \subseteq E$. Then,

- (i) A is a *flat* if $\rho(A) < \rho(A \cup e)$ for $e \in (E - A)$,
- (ii) A is a *cyclic set* if $\rho(A) - \rho(A - a) < \|a\|$ for $a \in A$,
- (iii) A is a *cyclic flat* if A is both a flat and a cyclic set.

The collections of flats, cyclic set,s and cyclic flats of P are denoted by $\mathcal{F}$, $\mathcal{U}$, and $\mathcal{Z}$, respectively.

By the definition above and the connection between codes and entropic L-polymatroids, we obtain that a cyclic flats of a code are subsets A of the coordinates so that $H_C(A) < H_C(A \cup e)$ and $H_C(A) - H_C(A - a) < \log_q(|\mathbb{A}^a|)$ for all $e \in E - a$ and $a \in A$. Informally, A is a cyclic flat of an LRC if adding a coordinate to the projection $C|A$ always increases the amount of information, and deleting a coordinate results in a possible loss of information which is less than the maximum amount of information $\log_q(|\mathbb{A}^a|)$.

Let $\mathrm{cl} : 2^E \to 2^E$ and $\mathrm{cyc} : 2^E \to 2^E$ denote the closure and cyclic operators of an L-polymatroid $P = (E, \rho, \|\cdot\|)$, *i.e.*,

$$\begin{array}{l} \mathrm{cl}(A) = A \cup \{e \in E - A : \rho(A \cup e) = \rho(A)\}, \\ \mathrm{cyc}(A) = A - \{e \in A : \rho(A) - \rho(A - e) = \|e\|\}, \end{array}$$

for $A \subseteq E$. The following results can be proved by using $\|A\|$ instead of $|A|$ in the proofs of the corresponding results for matroids, see *e.g.* [16]. The union of two cyclic sets is a cyclic set and the intersection of two flats is a flat. The closure of a set, $\mathrm{cl}(A)$, is a flat and is minimal with respect to inclusion, *i.e.*, $A \subseteq F \in \mathcal{F} \Rightarrow \mathrm{cl}(A) \subseteq F$, and $\rho(A) = \rho(\mathrm{cl}(A))$. Further, $\mathrm{cyc}(A)$ is a cyclic set and is maximal with respect to inclusion, *i.e.*, $A \supseteq U \in \mathcal{U} \Rightarrow \mathrm{cyc}(A) \supseteq U$, and $\rho(A) = \rho(\mathrm{cyc}(A)) + \|A - \mathrm{cyc}(A)\|$. Moreover, $\mathrm{cyc}(F), \mathrm{cl}(U) \in \mathcal{Z}$ for any flat F and cyclic set U. Consequently, as $\emptyset$ is a cyclic set and E is a flat, we obtain that $0_{\mathcal{Z}} = \mathrm{cl}(\emptyset)$, $1_{\mathcal{Z}} = \mathrm{cyc}(E) \in \mathcal{Z}$ and $0_{\mathcal{Z}} \subseteq Z \subseteq 1_{\mathcal{Z}}$ for all cyclic flats $Z \in \mathcal{Z}$. Moreover, we may also conclude that $\mathcal{Z}$ is a (poset) lattice under inclusion, where $Z_1 \vee Z_2 = \mathrm{cl}(Z_1 \cup Z_2)$ and $Z_1 \wedge Z_2 = \mathrm{cyc}(Z_1 \cap Z_2)$ for $Z_1, Z_2 \in \mathcal{Z}$.

By using the results above we can now state the following results, which show how to examine (n, k, d, r, δ, t)-L-polymatroids via cyclic flats. First we observe, by the definition of non-degeneracy, that an L-polymatroid $P = (E, \rho, \|\cdot\|)$ is non-degenerate if and only if $0_{\mathcal{Z}} = \emptyset$ and $1_{\mathcal{Z}} = E$.

Theorem 4.1: Let $P = (E, \rho, \|\cdot\|)$ be a non-degenerate (n, k, d)-L-polymatroid. Then $n = \|1_{\mathcal{Z}}\|$, $k = \rho(1_{\mathcal{Z}})$ and $d = |1_{\mathcal{Z}}| - \max\{|Z| + \gamma(Z) : Z \in \mathcal{Z} - 1_{\mathcal{Z}}\}$, where $\gamma(Z) = \max\{|A| : A \subseteq E - Z \text{ and } \rho(Z) + \|A\| < k\}$.

Proof: The formulas for n and k follow directly as P is non-degenerate. We know that $d = |E| - \max\{|A| : A \subseteq E, \rho(A) < \rho(E)\}$. Assume $d = |E| - |F|$, then F is a flat. Consequently, $Z_F = \text{cyc}(F) \in \mathcal{Z}$ and $\rho(F) = \rho(Z_F) + \|F - Z_F\|$. ■

B. A Singleton-type bound and structure results for L-polymatroids

We need some notations before we can state a Singleton-type bound and a structure theorem.

Let $P = (E, \rho, \|\cdot\|)$ be an $(n, k, d, r, \delta, t)_S$-L-polymatroid. Define the integers a_i as follows:

(i) $a_0 = 0$,

(ii) $a_i - a_{i-1} = \begin{cases} r + \delta - 1 & \text{if} \quad t | (i-1), \\ r + \delta - 2 & \text{if} \quad t \nmid (i-1). \end{cases}$

For any subset $R = \{e_1, \ldots, e_l\} \subseteq E$, we order its elements such that $\|e_1\| \geq \ldots \geq \|e_l\|$ and let $e_I = \cup_{i \in I} e_i$ for $I \in [l]$. Let $s = s_R$ be the smallest integer for which $a_s \geq l$, and define the sets $U_1^R, \ldots U_{s-1}^R$ by $U_i^R = \{e_{a_{i-1}+1}, \ldots e_{a_i}\}$ and $U_s^R = R - U_{[s-1]}^R$, where

$$U_T^E = \cup_{i \in T} U_i^E$$

for any $T \subseteq [s]$. Further, let

(iii) $A_i^R = e_I$, where $I = \{a_{i-1} + \delta, \ldots, a_i\}$.

Note that A_s^R may be an empty set and that $A_T^R = \cup_{i \in T} A_i^R$ for $T \subseteq [s]$. Now, by setting $R = E$, let

(iv) $\alpha = \max\{j : \|A_{[j]}^E\| < \rho(E)\}$,

(v) $\beta = \max\{j : \|e_J\| < \rho(E) - \|A_{[\alpha]}^E\|\}$, where $J = \{|U_{[\alpha]}^E| + 1, \ldots, |U_{[\alpha]}^E| + j\}$,

(vi) σ denotes the index such that $e_{|E|-d+1} \in U_{\sigma+1}^E$,

(vii) $\Gamma = \min\{\|A_{\sigma+1}^E\|, \|e_J\|\}$, where $J = \{|U_{[\sigma]}^E| + 1, \ldots, |E| - d + 1\}$.

We remark that when the upper bound for d is given in the theorem below, then the parameter $k = \rho(E)$ is assumed to be fixed. Similarly, when the upper bound for k is given, then the parameter d is assumed to be fixed.

Theorem 4.2 (Singleton-type bound): Let $\rho(S) = \rho(E)$ and let $P = (E, \rho, \|\cdot\|)$ be an $(n, k, d, r, \delta, t)_S$-L-polymatroid. Then

$$(i) \quad t(\delta - 1) + 1 \leq d \leq |E| - |U_{[\alpha]}^E| - \beta,$$
$$(ii) \quad \mathcal{R} = \tfrac{k}{n} \leq \tfrac{\|A_{[\sigma]}^E\| + \Gamma}{\|E\|}.$$

Before we prove the theorem, we will give an example on how it can be used.

Example 4.1: Let $E = \{e_1, \ldots, e_{42}\}$, with $\|e_1\| = \ldots = \|e_{23}\| = 2$ and $\|e_{24}\| = \ldots = \|e_{42}\| = 1$. For $r = 4, \delta = 3, t = 2$ we have the following associated partition,

$$\begin{array}{ll} U_1^E = \{e_1, \ldots, e_6\}, & U_2^E = \{e_7, \ldots, e_{11}\}, \\ U_3^E = \{e_{12}, \ldots, e_{17}\}, & U_4^E = \{e_{18}, \ldots, e_{22}\}, \\ U_5^E = \{e_{23}, \ldots, e_{28}\}, & U_6^E = \{e_{29}, \ldots, e_{33}\}, \\ U_7^E = \{e_{34}, \ldots, e_{39}\}, & U_8^E = \{e_{40}, \ldots, e_{42}\}. \end{array}$$

Further,

$$\begin{array}{ll} A_1^E = \{e_3, \ldots, e_6\}, & A_2^E = \{e_9, \ldots, e_{11}\}, \\ A_3^E = \{e_{14}, \ldots, e_{17}\}, & A_4^E = \{e_{20}, \ldots, e_{22}\}, \\ A_5^E = \{e_{25}, \ldots, e_{28}\}, & A_6^E = \{e_{31}, \ldots, e_{33}\}, \\ A_7^E = \{e_{36}, \ldots, e_{39}\}, & A_8^E = \{e_{42}\}, \end{array}$$

with

$$\begin{array}{l} \|A_1^E\| = \|A_3^E\| = 8, \ \|A_2^E\| = \|A_4^E\| = 6, \\ \|A_5^E\| = \|A_7^E\| = 4, \ \|A_6^E\| = 3, \ \|A_8^E\| = 1. \end{array}$$

Then for $k = 31$, we obtain that $\alpha = 4$ and $\beta = 1$. Consequently,

$$5 \leq d \leq 42 - 22 - 1 = 19.$$

Moreover, for $d = 19$, we found that

$$\sigma = 4, \ \Gamma = 3.$$

Consequently,

$$\mathcal{R} \leq \frac{31}{65}.$$

Proof: [Theorem 4.2] We will prove the Singleton-type bound here, by showing that there must be a union of recovering sets R and some added elements $D \subseteq E - R$ such that $\rho(R \cup D) < \rho(E)$ and $|R \cup D| \geq \gamma$. Using (2), this proves that $d \leq |E| - \gamma$.

Let $R_1^e, \ldots, R_t^e$ denote the t associated recovering sets for $e \in S$, $R_I^e = \cup_{i \in I} R_i^e$ for $I \subseteq [t]$, $R^e = R_{[t]}^e$ and $R_T = \cup_{e \in T} R^e$ for $T \subseteq S$. Moreover, let $\mathcal{T}_S$ be the collection of all unions of recovering sets R, where $R = R_T \cup R_I^f$ for $T \subseteq S$, $f \in S - T$ and $I \subseteq [t]$ such that

$$\rho(R) < \rho(E) \text{ and } \rho(R \cup R') = \rho(E)$$

for some recovering set R'. The existence of such a set R follows from the assumption that $\rho(S) = \rho(E)$.

By (R3), $\rho(B \cup R_i^e) \leq \rho(B) + \rho(R_i^e) - \rho(B \cap R_i^e)$ for $B \subseteq E$ and $e \in S$. Consequently, using (3) and (R2),

$$\begin{array}{l} \rho(B \cup R_i^e) = \rho(B) \text{ if } |B \cap R_i^e| \geq |R_i^e| - \delta + 1, \\ \rho(B \cup R_i^e) \leq \rho(B) + \|B_i^e\| \text{ if } |B \cap R_i^e| \leq |R_i^e| - \delta, \end{array} \quad (4)$$

where B_i^e is a subset of $R_i^e - B$ such that $|B_i^e| = |R_i^e - B| - \delta + 1$ and $\|B_i^e\| = \min\{\|A\| : A \subseteq (R_i^e - B), |A| = |R_i^e - B| - \delta + 1\}$. Hence, for any $R \in \mathcal{T}_S$, each recovering set $R_i^{e'}$ of R contributes at most with $\|B_i^{e'}\|$, where $B = R_{[i-1]}^e$, to the total rank of R. Consequently, using the notation given above the theorem,

$$\rho(R) \leq \|A_{[s]}^R\| \leq \|A_{[\alpha]}^E\|.$$

Hence,

$$\begin{array}{rl} & |R| + \min\{|C| : C \subseteq E - R, \rho(R \cup C) = \rho(E)\} - 1 \\ \geq & |R| + \min\{|C| : C \subseteq E - R, \rho(R) \cup \|C\| \geq \rho(E)\} - 1 \\ \geq & |U_{[\alpha]}^E| + \beta. \end{array}$$

By (2), this implies that $d \leq |E| - \|U^E_{[\alpha]}\| - \beta$. The inequality $d \geq t(\delta-1)+1$ follows from the fact that $A \subseteq R^e_i$, and $|A| \geq |R^e_i| - (\delta-1)$ implies that A can recover R^e_i. Hence, we have to delete e and at least $\delta-1$ elements in $R^e_i - e$ for each $i \in [t]$ to obtain a set $T \subseteq E$ such that $\rho(T) < \rho(E)$.

To get the upper bound on the rate $\mathcal{R}$, we will prove an upper bound for $k = \rho(E)$ assuming that we have a fixed d so that $d \geq t(\delta-1)+1$. From the definition of d, we have that $k = \rho(A)$ when $|A| = |E| - d + 1$. Using similar arguments as in the proof of the upper bound for d, we obtain that $\max\{\rho(A) : |A| = |E| - d + 1\} = \|A^E_{[\sigma]}\| + \Gamma$. ■

The above Singleton-type theorem yields the same results as special cases for L-polymatroids, LRCs, matroids, and polymatroids with information-symbo and all-symbol locality. In more detail, the polymatroid Singleton bound given in [4] states that

$$d \leq n - \lceil k \rceil + 1 - \left(\left\lceil \frac{t(\lceil k \rceil - 1) + 1}{t(r-1)+1} \right\rceil - 1 \right)(\delta - 1), \quad (5)$$

for any $(n,k,d,r,\delta,t)_i$-polymatroids. To obtain this result, the rank function of a polymatroid $P = (E, \rho)$ was rescaled in [4] such that $\rho(e) \leq 1$. Hence, these rescaled polymatroids may be considered as L-polymatroids $P = (E, \rho, \|\cdot\|)$ where $\|A\| = |A|$ for $A \subseteq E$. Therefore, using that $\|A\| = |A|$ in Theorem 4.2 gives a Singleton-type bound for polymatroids. Formula (5) follows by the use of the proof of Theorem 4.2 together with similar arguments as those used in the proof of Theorem 2 in [10]. For a summary on how (5) implies other bounds on polymatroids, matroids and LRCs over non-mixed alphabets, see [1].

The following corollary gives an upper bound for the rate for any $(r,\delta,t)_S$-L-polymatroid. This is the same upper bound as given in Theorem 4.2 with $d = t(\delta+1)+1$, but in a more explicit form. We need some more notation before we can state the upper bound on the rate.

Let $P = (E, \rho, \|\cdot\|)$ be an $(r,\delta,t)_S$-L-polymatroid. Let

(viii) $m = \left\lfloor \frac{|E|}{t(r+\delta-2)+1} \right\rfloor$.

Let $V = E - U^E_{[mt]}$. If $|V| \geq t(\delta-1)+1$, then let $V_1, \dots, V_t \subseteq V$ and $V_I = \bigcup_{i \in I} V_i$ for $I \subseteq [t]$, so that $V_1, \dots, V_t$ is the partition of V where

(ix) $e_j \in V_i,\ e_{j'} \in V_{i'},\ i < i' \quad \Rightarrow \quad j < j'$,

(x) $|V_1| = \min\{r+\delta-1, |V| - (t-1)(\delta-1)\}$,

(xi) $|V_i| = \min\{r+\delta-2, |V| - |V_{[i-1]}| - (t-i)(\delta-1)\}$ for $i = 2, \dots, t$,

(xii) $B_i = \{e_i : |E-V| + |V_{[i-1]}| + \delta \leq i \leq |E-V| + |V_{[i]}|\}$ for $i = 1, \dots, t$, ($V_{[0]} = \emptyset$, $B_{[t]} = \cup_{i \in [t]} B_i$).

Corollary 4.1: Let $P = (E, \rho, \|\cdot\|)$ be an $(r,\delta,t)_S$-L-polymatroid with $\rho(S) = \rho(E)$. Then

$$\mathcal{R} = \frac{k}{n} \leq \begin{cases} \frac{\|A^E_{[mt]}\|}{\|E\|} & \text{if} \quad |V| \leq t(\delta-1), \\ \frac{\|A^E_{[mt]}\| + \|B_{[t]}\|}{\|E\|} & \text{if} \quad |V| > t(\delta-1). \end{cases}$$

We will give an example on how the above corollary can be used before we prove it.

Example 4.2: Let us consider Corollary 4.1 in the case of Example 4.1, with the only change that $E = e_1, \dots, e_{40}$. Then we have the following parameters, $m = 3$, $|V| = 7 \geq t(\delta-1)+1 = 5$, $V_1 = \{e_{34}, \dots, e_{38}\}$, $V_1 = \{e_{39}, e_{40}\}$, $B_1 = \{e_{36}, \dots, e_{38}\}$ and $B_2 = \emptyset$. Hence,

$$\mathcal{R} \leq \frac{38}{63}$$

for any $(r,\delta,t)_S$-L-polymatroid $P = (E, \rho, \|\cdot\|)$.

Proof: [Corollary 4.1] Using the notation given in the proof of Theorem 4.2. By (4), for any $e \in S$, $|R^e - (E - V)| \leq t(\delta-1) \Rightarrow \rho(R^e \cup (E-V)) = \rho(E-V)$. Consequently, by similar argument as in the proof of Theorem 4.2,

$$\mathcal{R} \leq \frac{\|A^E_{[mt]}\|}{\|E\|},$$

when $|V| \leq t(\delta-1)$.

Assume that $|V| > t(\delta-1)$, $e \in S \cap V$, $V \subseteq R^e$ and $\rho(E-V) < \rho(E)$. Then $|V_i - e| \leq \delta - 1$ for $i = 1, \dots, t$ and

$$\rho(E) - \rho(E-V) \leq \rho(V) \leq \|B_{[t]}\|,$$

which implies the theorem. ■

We remark, by the use of Theorem 4.2 and Corollary 4.1, we obtain that $t(r+1)+1 \leq d \leq |E|$ and k is larger that 0 and less than or equal the upper bound given in Corollary 4.1, for any (r,δ,t)-L-polymatroid (i.e., we don't assume that d or k are fixed).

The key tool for us when examining and trying to construct (n,k,d,r,δ,t)-L-polymatroids or (n,k,d,r,δ,t) -LRCs with good parameters is a cyclic flat. This is mainly due to the fact that all the recovering sets must be cyclic sets and that $\mathrm{cl}(R)$ of any union R of cyclic sets is a cyclic flat. We recall the notation given in the proof of Theorem 4.2: $R^e = R^e_1 \cup \ldots \cup R^e_t$ and $R_T = \bigcup_{s \in T} R^s$ for $e \in S$ and $T \subseteq S$ where R^e_i is an (r,δ)-recovering set of e. An $(n,k,d,r,\delta,t)_S$-L-polymatroid which achieves the bound given above is called *Singleton-perfect*.

Theorem 4.3 (Structure theorem): Let $P = (E, \rho, \|\cdot\|)$ be a Singleton-perfect $(n,k,d,r,\delta,t)_S$-L-polymatroid. Then,

(i) $\rho(R_T) < \rho(E) \quad \Rightarrow \quad R_T$ is a cyclic flat,

(ii) if Z is a cyclic flat, then $\frac{\rho(E)-\rho(Z)}{\|E-Z\|}$ must satisfy the upper bound on the rate given in Theorem 4.2 (ii) on the ground set $E - Z$ and parameters $(d,r,\delta,t)_{S-Z}$.

Proof: Assume that $\rho(R_T) < \rho(E)$. The proof of the upper bound of d in Theorem 4.2 and the property that P is Singleton-perfect implies that there is no element $e \in E - R_T$ so that $\rho(R_T \cup e) = \rho(R_T)$. Consequently, this implies that R_T is a flat and therefore also a cyclic flat since R_T is a cyclic set by definition.

For (ii), assume that $Z \in \mathcal{Z}$ and that $\rho(R_T) = \rho(E)$ for some subset $T \subseteq E$. Then $R^E_i - Z$ are (r,δ)-recovering sets in $(E-Z)$ for $e \in T - Z$. Further, $\rho(R_{T-Z}) \geq \rho(E) - \rho(Z)$ and $d \geq |E| - |Z| - \max\{|A| : A \subseteq E - Z,\ \rho(A) < \rho(E) - \rho(Z)\}$.

■

Informally, the above structure theorem and the proof of Theorem 4.2 show that for Singleton-perfect L-polymatroids or LRCs, most of the recovering sets need to have full size

$r+\delta-1$, the cyclic flats need to have large rank and the intersections of unions of recovering sets of different elements cannot be too large, and $|R^e \cap R^f|$ has to be small for different elements $e, f \in S$.

V. Quasi-uniform codes

Quasi-uniform codes (which include linear and group codes as special cases) were introduced in [5].

A. Classes of quasi-uniform codes

Definition 5.1 ([5]): A code C in $\mathbb{A}^E$ is *quasi-uniform* if $H_C(A) = \log_q(|C|A|)$ for each subset $A \subseteq E$. [1]

By (1), quasi-uniform codes can be considered as codes with maximal amount of information in comparison with the sizes of its code puncturings.

Proposition 5.1 ([5] Proposition 1): A code $C \subseteq \mathbb{A}^E$ with probability distribution p_C is quasi-uniform if and only if p_C is uniform for all $A \subseteq E$, and $|C/A(\boldsymbol{z})|$ is constant for all $\boldsymbol{z} \in C|A$.

By the definition of quasi-uniform codes and entropic L-polymatroids we directly obtain the following characterization of quasi-uniform codes.

Theorem 5.1: A code C in $\mathbb{A}^E$ associated with $P_C^{ent} = (E, \rho, \|\cdot\|)$ is quasi-uniform if and only if $\rho(A) = \log_q(|C|A|)$ for each $A \subseteq E$.

A *group code* is a subgroup C of $\mathbb{A}^E$, where $\mathbb{A}^E$ is a direct sum of finite groups. Let R be a finite ring. A *linear code* is an R-submodule C of $\mathbb{A}^E$, where $\mathbb{A}^E$ is a direct sum of finite R-modules. A *vector-linear code over* R^m is a linear code in $\mathbb{A}^E$ where $\mathbb{A}_i = R^m$ for each $i \in E$. An *R-linear code* is a linear code in $\mathbb{A}^E$ where $\mathbb{A}_i = R$ for each $i \in E$. By basic group theory and Proposition 5.1, one may conclude that every code with uniform probability distribution which is a group code is quasi-uniform. Consequently, the class of quasi-uniform codes is huge and for codes with uniform probability distribution we obtain the following subclasses of quasi-uniform codes:

$\{$linear codes over finite fields$\} \subsetneq \{R$-linear codes$\} \subsetneq$
$\{$vector-linear codes over $R^m\} \subsetneq \{$linear codes$\} \subsetneq$
$\{$group codes$\} \subsetneq \{$quasi-unifom codes$\}$.

Further, if C and C' are two codes with uniform probability distribution where C is a group code and C' a coset of C, then both C and C' are quasi-uniform and with same joint entropies.

B. Constructions of quasi-uniform codes via group theory

Constructions of quasi-uniform codes by the use of group theory were considered in [5], [6].

Construction 5.1 ([5], [6]): Let G be a finite group, and let $G_1, \dots, G_{n'}$ be some (not necessarily distinct) subgroups of G. We denote the left cosets by $G/G_i = \{gG_i : g \in G\}$ for $i \in [n']$, and the intersection $G_A = \bigcap_{i\in A} G_i$ for $A \subseteq [n']$. Then, assuming uniform probability distibution,

(i) $C = \{(gG_1, \dots, gG_{n'}) : g \in G\}$ is a quasi-uniform code in the ambient space $\mathbb{A}^{[n']}$ with $\mathbb{A}_i = G/G_i$ and $|\mathbb{A}_i| = |G|/|G_i|$ for $i \in [n']$,

(ii) $|C|A| = \frac{|G|}{|G_A|}$ for $A \subseteq [n']$.

Obviously, the above code construction also works for right cosets. If $G_1, \dots, G_{n'}$ are normal subgroups, then $\mathbb{A}_1, \dots, \mathbb{A}_{n'}$ are quotient groups and C is a group code of the group $\mathbb{A}^{[n']}$.

C. The entropic L-polymatroid of a quasi-uniform code

Assume that we have a mathematical object with an associated L-polymatroid $P = (E, \rho, \|\cdot\|)$, and that we are able to give explicit expressions for $\rho(A)$ and $\|e\|$ for each $A \subseteq E$ and $e \in E$. Then we can directly formulate the definitions and results given in Section III and Section IV in the context of the mathematical object by using the explicit expressions of $\rho(A)$ and $\|e\|$.

This approach is precisely what we have been working with in this paper for general $(n,k,d,r,\delta,t)_S$-LRCs, using the properties that $\rho(A) = H_C(A)$ and $\|e\| = \log_q(|\mathbb{A}_e|)$ when entropic L-polymatroids are considered. The expressions for the rank function can be made even more explicit for quasi-uniform codes, using that $\rho(A) = \log_q(|C|A|)$ by Theorem 5.1. Even more explicitly, when quasi-uniform codes constructed by Construction 5.1 are considered, we can use that

$$\rho(A) = \log_q\left(\frac{|G|}{|G_A|}\right) \quad \text{ and } \quad \|i\| = \log_q\left(\frac{|G|}{|G_i|}\right),$$

for $A \subseteq [n']$ and $i \in [n']$.

Example 5.1: Let C be the quasi-uniform code generated by elementary abelian group $G = C_p \times \ldots \times C_p = (C_p)^v$ with $v \geq 2$, p prime, $C_p = \{0, 1, \dots, p-1\}$, and all subgroups of order p. Since G is an abelian group, according to [6] the minimum distance is $d = n_c - \max_{A \subseteq [n_c], G_A \neq 0} |A|$, where n_c is the number of subgroups. In [8], it was proved that this code has size $|C| = p^v$, length $n_C = \frac{p^v-1}{p-1}$ and minimum distance $d = n_C - 1 = \frac{p^v - p}{p-1}$ since the pairwise intersection of subsets is already trivial. For the locality parameters, every set of coordinates A has minimum distance $d_{C|A} = |A| - 1$, meaning that C has locality $(2, \delta)$ for every $2 \leq \delta \leq d$. We will assume from now on that δ is fixed. For all sets $A \subseteq [n_c]$ with size $|A| = \delta$ and $e \in [n_c] - A$, the set $A \cup e$ has minimum distance equal to δ. Since there are $\lfloor \frac{n_C - 1}{\delta} \rfloor$ disjoint sets of size δ, the availability is $t = \lfloor \frac{n_C-1}{\delta} \rfloor = \left\lfloor \frac{p^v - p}{\delta(p-1)} \right\rfloor$. Thus, C is an $(n, |C| = p^v, n_C - 1, 2, \delta, \lfloor \frac{n_C - 1}{\delta} \rfloor)$-LRC.

To compute the bounds in Theorem 4.2, we first need to construct the partition $U_1, \dots, U_s$ of E, where we omit the superscript to ease the notation. By floor properties, we have $n_C - (\delta - 1) \leq t\delta + 1 \leq n_C$. Since $a_t = a_1 + (t-1)a_2 = t\delta + 1$, we obtain that s is either equal to t or $t+1$ and in the latter case, we have $|E - U_{[t]}| \leq \delta - 1$. Therefore, the partition is $U_1, \dots U_t, U_{t+1}$ with sizes $|U_1| = \delta + 1$, $|U_2| = \ldots = |U_t| = \delta$, and $|U_{t+1}| = |E| - |U_{[t]}| \leq \delta - 1$. The corresponding sets A_i are $A_1 = \{\delta, \delta+1\}$, $|A_2| = \ldots = |A_t| = 1$, and $A_{t+1} = \emptyset$.

[1] We remark that our notation for puncturing and relative shortening differ from the notation used in [5]. The reason for this is that we want to be consistent with the respective notation for restriction and contraction for matroids.

We can now evaluate the parameters α, β, σ and Γ. Remember that $\|e\| = \log_q\left(\frac{|G|}{|G_e|}\right) = (v-1)\log_q(p)$ for all $e \in [n_C]$ and $\rho(E) = \log_q(|C|) = v\log_q(p)$. Since $v \geq 2$, we have $\|A_1\| = 2(v-1)\log_q(p) \geq \rho(E)$ so $\alpha = 0$ and similarly, $\|\{1,2\}\| \geq \rho(E)$ so $\beta = 1$. For σ, we obtain $e_{|E|-d+1} = e_2 \in U_1$ so $\sigma = 0$. Finally, since $\|A_1\|$ is the same as $\|\{1,2\}\|$, we get $\Gamma = \|A_1\|$.

We are now ready to compute the bounds in Theorem 4.2. We have for $e \in [n_c]$:

$$\mathcal{R}_{\max} = \frac{\Gamma}{\|E\|} = \frac{2\|e\|}{n_C\|e\|} = \frac{2}{n_c}$$
$$d_{\max} = |E| - (|U_{[\alpha]}|) - \beta = n_C - 1$$

We can already see that this family of codes achieves the best minimum distance since $d = n_C - 1 = d_{\max}$. However in general the maximal rate is not achieved as demonstrated next. Since $|\mathbb{A}^{[n_C]}| = (p^{v-1})^{n_C}$ and $|C| = p^v$, we have

$$\mathcal{R}_C = \frac{\log_q(|C|)}{\log_q(|\mathbb{A}^{[n_C]}|)} = \frac{v\log_q(p)}{n_C(v-1)\log_q(p)} = \frac{v}{n_C(v-1)}.$$

Then, C achieves the maximal rate if and only if $v = 2$. In [6], the authors proved that codes coming from the group $G = C_p \times C_p$ can be seen as $[p+1, 2, p]$ linear MDS codes over $\mathbb{F}_p$. However, even if these codes achieve the best rate, the locality $(2, \delta)$ is not interesting in the practical sense, the global rank also being $k = 2$.

Nonetheless, for higher v, this family of codes always reaches the highest possible minimum distance and provides an interesting flexibility in terms of locality and availability, since δ can take any value between 2 and d.

VI. Conclusion

We have introduced the concept of L-polymatroids, *i.e.*, polymatroids with an associated length function. This new concept was then utilized to characterize locally repairable codes over mixed alphabets, where different codeword coordinates can arise from different alphabets. Such a property may be useful in scenarios, where the storage network consists of heterogeneous storage nodes. We derived a Singleton-type bound for the global minimum distance of such a code and more generally for L-polymatroids, as well as a bound for the code rate. A quasi-uniform code construction achieving the distance bound was also provided, however not reaching the best possible rate. As future work, one should seek for further quasi-uniform code constructions achieving both the optimal rate and distance or, alternatively, show that there is a tradeoff between the two quantities. This could possibly show that quasi-uniform codes can improve on linear codes for certain parameter values.

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