23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Realizations of NC Rational Functions Around a Matrix Centre: A Generalization of the Fliess-Kronecker Theorem*

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1. Introduction

If $\mathfrak{r}$ is a nc rational function of d non-commuting variables $x_1,\ldots,x_d$, that is regular at 0 (or any $d-$tuple of scalars), it is known that $\mathfrak{r}$ admits a minimal realization and using the minimal realization the (stable extended) domain of $\mathfrak{r}$ can be fully described (this is Theorem 1). In addition, the set of all of nc rational functions regular at 0 can be characterized in terms of their power series expansions (this is Theorem 2). However, there are nc rational functions which do not contain $d-$tuples of scalars. For example, the nc rational function

$$\mathfrak{r}(x_1,x_2)=(x_1x_2-x_2x_1)^{-1}$$

is not regular at any $(y_1,y_2)\in\mathbb{C}^2$; however, $\mathfrak{r}$ is regular at some $(Y_1,Y_2)\in(\mathbb{C}^{2\times 2})^2$, for example at

$$(Y_1,Y_2)=\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\in(\mathbb{C}^{2\times 2})^2.$$

We will not limit ourself for functions regular at 0. In this talk we work in the more general settings of all nc rational functions, i.e., $\mathfrak{r}$ is regular at some Y which is a $d-$tuple of $s\times s$ matrices, say $Y=(Y_1,\ldots,Y_d)\in(\mathbb{C}^{s\times s})^d$, and we prove generalizations of both Theorems 1 and 2 for such an $\mathfrak{r}$.

1.1. Characterizations of NC Rational Functions Regular at 0

The next theorem gives a full characterization of nc rational functions which are regular at 0, and their (stable extended) domains of regularity, in terms of their minimal realizations (for the proofs, see [1, 3, 4, 5, 6]).

Theorem 1 *If $\mathfrak{r}$ is a nc rational function of $x_1,\ldots,x_d$ and $\mathfrak{r}$ is regular at 0, then*

1. *$\mathfrak{r}$ admits a unique (up to similarity) minimal state space realization*

$$\mathfrak{r}(x_1,\ldots,x_d)=D+C(I_L-\sum_{i=1}^{d}A_ix_i)^{-1}(\sum_{i=1}^{d}B_ix_i),$$

where $A_1,\ldots,A_d\in\mathbb{C}^{L\times L},B_1,\ldots,B_d\in\mathbb{C}^{L\times 1},C\in\mathbb{C}^{1\times L},D=\mathfrak{r}(0)$ and $L\in\mathbb{N}$.

2. *For all $m\in\mathbb{N}$: $X=(X_1,\ldots,X_d)\in(\mathbb{C}^{m\times m})^d$ is in the stable extended domain of $\mathfrak{r}$ $\iff$*

$$\det(I_{mL}-X_1\otimes A_1-\ldots-X_d\otimes A_d)\neq 0.$$

The following theorem provides a full characterization of nc rational functions regular at 0 in terms of their power series expansions (for the proof, see [1]).

Theorem 2 (Fliess-Kronecker Theorem) *A nc power series*

$$\sum_{\omega\in\mathcal{G}_d}\mathfrak{r}_\omega x^\omega\in\mathbb{C}\langle\langle x_1,\ldots,x_d\rangle\rangle$$

is the power series expansion of a nc rational function at a 0 $\iff$ the infinite $\mathcal{G}_d\times\mathcal{G}_d$ Hankel matrix $\mathbb{H}=[\mathfrak{r}_{uv}]_{u,v\in\mathcal{G}_d}$ has a finite rank. ($\mathcal{G}_d$ is the free monoid generated by d generators $g_1,\ldots,g_d$).

2. Main Results:

What is a realization in our settings? A realization centered at $Y=(Y_1,\ldots,Y_d)\in(\mathbb{C}^{s\times s})^d$, is a function of the form

$$I_m\otimes D+(I_m\otimes C)\Lambda(X_1,\ldots,X_d)^{-1}\sum_{i=1}^{d}\mathbf{B}_i(X_i-I_m\otimes Y_i) \tag{1}$$

where

$$\Lambda(X_1,\ldots,X_d)=I_{Lm}-\sum_{i=1}^{d}\mathbf{A}_i(X_i-I_m\otimes Y_i),$$

*These results are based on joint work with Daniel Alpay (Ben Gurion University of the Negev) and Victor Vinnikov (Ben Gurion University of the Negev).

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$\mathbf{A}_1,...,\mathbf{A}_d : \mathbb{C}^{s\times s} \to \mathbb{C}^{L\times L}$ and $\mathbf{B}_1,...,\mathbf{B}_d : \mathbb{C}^{s\times s} \to \mathbb{C}^{L\times s}$ are linear mappings, $C \in \mathbb{C}^{s\times L}, D \in \mathbb{C}^{s\times s}, L \in \mathbb{N}$ and

$$X = (X_1,...,X_d) \in (\mathbb{C}^{sm\times sm})^d$$

for every $m \in \mathbb{N}$. Similarly to the classical theory of realizations, modified definitions for observability, controllability and minimality of realizations of the form (1) are introduced. In particular, we obtain a Kalman decomposition and show that a realization is minimal (in the sense that L is the smallest integer for which such a realization exists) if and only if it is both controllable and observable. The following theorem is a generalization of Theorem 1.

Theorem 3 *If $\mathfrak{r}$ is a nc rational function, regular at some $Y = (Y_1,...,Y_d) \in (\mathbb{C}^{s\times s})^d$, then $\mathfrak{r}$ admits a unique (up to similarity) minimal realization of the form*

$$I_m \otimes D + (I_m \otimes C)\Lambda(X_1,...,X_d)^{-1}\left(\sum_{i=1}^{d} \mathbf{B}_i(X_i - I_m \otimes Y_i)\right)$$

as in (1). Moreover, for any minimal realization of $\mathfrak{r}$ of the form (1), the stable extended domain of $\mathfrak{r}$ is equal to

$$\bigcup_{m=1}^{\infty} \{(X_1,...,X_d) \in (\mathbb{C}^{sm\times sm})^d : \det(\Lambda(X_1,...,X_d)) \neq 0\},$$

which is the invertibility set of Λ.

The proof is obtained by steps: first, we show the existence of a realization of the form (1), centered at Y, then using a modified Kalman decomposition we can choose the realization to be a minimal one and finally, to obtain the stable extended domain, we use the minimality of the realization and the difference-differential operator, as in [3, 4]. In [2], the authors proved that a generalized nc formal power series expansion

$$\sum_{\omega\in\mathcal{G}_d} \mathfrak{r}_\omega (X - I_m \otimes Y)^{\odot_s \omega} \tag{2}$$

is a nc function if and only if the linear mappings $(\mathfrak{r}_\omega)$ satisfy the so called lost-Abbey conditions; we found necessary and sufficient conditions on the coefficients $(\mathfrak{r}_\omega)$, so that the power series in (2) is a nc rational function regular at Y. The following is a generalization of Theorem 2.

Theorem 4 (Generalized Fliess-Kronecker Theorem) *A generalized nc formal power series of the form*

$$\sum_{\omega\in\mathcal{G}_d} \mathfrak{r}_\omega (X - I_m \otimes Y)^{\odot_s \omega}, \quad X \in (\mathbb{C}^{sm\times sm})^d$$

centered at $Y \in (\mathbb{C}^{s\times s})^d$ is the power series expansion of a nc rational function $\mathfrak{r}$ that is regular at Y $\iff$

1. *$(\mathfrak{r}_\omega)$ satisfy the lost-Abbey conditions, and*
2. *the Hankel matrix $\mathbb{H}_{(\mathfrak{r})}$ given by*

$$\mathbb{H}_{(\mathfrak{r})} = \left[\mathbb{H}^{(\mathfrak{r})}_{T_1,T_2}\right]_{T_1,T_2\in\mathbb{I}}$$

has a finite (column) rank.

In the above theorem, the entries of the Hankel matrix are given explicitly by

$$\mathbb{H}^{(\mathfrak{r})}_{T_1,T_2} = (E_{i_1j_1},...,E_{i_\ell j_\ell},E_{p_1q_1},...,E_{p_tq_t})\mathfrak{r}_{\omega\nu}$$

for each $T_1 = (\omega,((i_1,.j_1),...,(i_\ell,j_\ell))) \in \mathbb{I}$ and $T_2 = (\nu,((p_1,q_1),...,(p_t,q_t))) \in \mathbb{I}$, where

$$\mathbb{I} = \bigcup_{\ell=0}^{\infty} \left(\mathcal{G}_d^{[\ell]} \times (\{1,...,s\}\times\{1,...,s\})^{\otimes\ell}\right)$$

and $\{E_{ij}\}_{i,j=1}^{s}$ is the standard basis of $\mathbb{C}^{s\times s}$. In the proof of Theorem 4, we use Theorem 3 and more results from the theory of minimal realizations of the form (1). In particular, if $\mathfrak{r}$ is a nc rational function, it admits a minimal realization and one can get the explicit formulas $\mathfrak{r}_\emptyset \equiv D$ and

$$\mathfrak{r}_\omega(Z^1,...,Z^\ell) = C\mathbf{A}_{i_1}(Z^1)...\mathbf{A}_{i_{\ell-1}}(Z^{\ell-1})\mathbf{B}_{i_\ell}(Z^\ell),$$

where $\omega = g_{i_1}...g_{i_\ell}$, for the coefficients $(\mathfrak{r}_\omega)$ of the power series expansion of $\mathfrak{r}$, and also reexpress the lost-Abbey conditions in terms of $\mathbf{A}_1,...,\mathbf{A}_d,\mathbf{B}_1,...,\mathbf{B}_d,C$ and D.

No paper will be submitted to the proceedings.

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