23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

H_∞ Optimal Adaptive Control for First Order Systems

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***Abstract*— Given a dynamic plant with parametric uncertainty, we present a causal state feedback law, that minimizes the worst case input-output ℓ_2-gain. The control law is adaptive in the sense the past data is used to estimate model parameters for prediction of future dynamics. The given formula recovers standard H_∞ optimal state feedback when the parametric uncertainty shrinks to zero.**

I. Introduction

The history of adaptive control dates back at least to aircraft autopilot development in the 1950s. Later on, computer control and system identification lead to a surge of research activity during the 1970s. Following the landmark paper [2], a long sequence of contributions to adaptive control theory derived conditions for convergence, stability, robustness and performance under various assumptions. For example, [10] analysed adaptive algorithms using averaging, [6] derived an algorithm that gives mean square stability with probability one, while [8] gave conditions for the optimal asymptotic rate of convergence. On the other hand, conditions that may cause instability were studied in [5], [9] and [13]. Altogether, hundreds (maybe thousands) of papers have been written on adaptive control, followed by numerous textbooks, such as [3], [7], [12], [14] and [1]. In this presentation, we focus on robustness to unmodelled dynamics in terms of the ℓ_2-gain, as discussed in [4], [15], [11].

For linear plants with known parameters, H_∞ optimization leads to linear feedback controllers. However, this is not the case when the plant includes uncertain parameters. Instead, minimization of the input-output ℓ_2-gain tends to give controllers that are nonlinear and adaptive: They first collect information about the uncertain parameters, then exploit that knowledge to determine the optimal feedback rule. Our optimal feedback law will be computed by "adaptive dynamic programming", a term that here denotes an extension of standard dynamic programming to the case of parametric uncertainty.

II. Adaptive dynamic programming

Consider the dynamical system

$$x_{t+1} = f_\theta(x_t, u_t) + w_t \tag{1}$$

with the uncertain parameter $\theta \in \Theta$. Given the sequences u and w, define the cost function

$$J_\theta(x_0) = \sum_{t=0}^{T} \ell(x_t, u_t, w_t). \tag{2}$$

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where ℓ is some step cost x is given by (1). Let μ specify the control policies

$$u_t = \mu_t(x_0, \dots, x_t, u_0, \dots, u_{t-1}). \tag{3}$$

for $t = 1, \dots, T$. Define

$$V_\mu(x_0) = \sup_{\theta, w} J_\theta(x_0), \tag{4}$$

Our first result is the following:

Theorem 1 (Adaptive Dynamic Programming): Assume (1)-(3). Then

$$\inf_\mu V_\mu(x_0) = \inf_{u_0} \sup_{w_0} \inf_{u_1} \dots \sup_{w_T} \sup_\theta J_\theta(x_0).$$

Furthermore, suppose that all minima and maxima exist and define

$$\begin{aligned}&\mu_t^*(x_0, \dots, x_t, u_1, \dots, u_{t-1})\\ &\quad = \arg\min_{u_t} \left(\max_{w_t} \min_{u_{t+1}} \dots \max_{w_T} \max_\theta J_\theta(x_0) \right).\end{aligned}$$

Then $\inf_\mu V_\mu(x_0) = V_{\mu^*}(x_0)$.

Theorem 1 gives an explicit formula for an optimal feedback policy μ^*, but several important questions remain. For example: 1) Is it computable? 2) Is it adaptive?

Deferring the first question to section IV, let's start with the second one. The formula can be re-written as follows. To compute u_t from $x_0, \dots, x_t, u_0, \dots, u_{t-1}$, we need to find the minimizing argument u_t for

$$\min_{u_t} \max_{w_t} \min_{u_{t+1}} \dots \max_{w_T} \max_{\theta \in \Theta} \left.\sum_{k=t}^{T} \ell_\theta(x_k, u_k, w_k) \quad \right\} \text{worst case future cost}$$
$$\left. + \sum_{k=0}^{t-1} \ell_\theta(x_k, u_k, w_k) \quad \right\} \text{past cost}$$

For small values of t, the controller needs to guard against all possible values of the paramater θ, but as t grows the terms of the past will play a more and more important role and the controller can focus on the θ-values that tend to maximize the cost of the past. This makes the control law adaptive in the sense that it learns from past data.

III. H_∞ Optimal Adaptive Control

Given a plant of the form

$$x_{t+1} = a x_t + u_t + w_t \qquad x_0 = 0 \tag{5}$$

we will study feedback laws of the form (3), such that the closed loop system satisfies

$$\sum_{t=0}^{T} (q x_t^2 + r u_t^2) \le \gamma^2 \sum_{t=0}^{T} w_t^2. \tag{6}$$

The previous section can then be applied with $\theta = a$, $f_\theta(x,u) = ax + u$ and $\ell(x,u,w) = qx^2 + ru^2 - \gamma^2 w^2$. When the model parameter a is known, standard theory for H_∞ optimization shows existence of an optimal control law that gives u_t as a linear function of x_t. This is no longer the case when a is unknown. However, the following result can be derived from Theorem 1:

Theorem 2: Given $\bar{a}, \gamma > 0$, the following are equivalent:

(i) There exists a feedback law of the form (3), such that the closed loop system with (5) satisfies (6) whenever $a \in [-\bar{a}, \bar{a}]$.

(ii) The Riccati equation

$$p = q + \bar{a}^2(p^{-1} + r^{-1} - \gamma^{-2})^{-1}$$

has a solution $p \in (0, \gamma^2]$ with

$$\gamma^2 \geq q + \bar{a}^2(p^{-1} - \gamma^{-2})^{-1}.$$

Remark 1. Let γ_* denote the minimal gain γ with $q = r = 1$. It can be verified that $\bar{a}^2 + 0.5 \leq \gamma_* \leq \bar{a}^2 + 1.11$. This can be compared with the optimal gain $\sqrt{1+\bar{a}^2}$ for the case that a is known to be $\bar{a}$ and no adaptation is needed.

To explain the main idea behind Theorem 2, a slightly modified problem will be discussed in the next section.

IV. A Priori Information

A priori estimates of the unknown system parameters can often be used to get a better gain bound. Such situations motivate the following modified problem statement where $\widehat{a}$ and $\widehat{b}$ denote nominal values for the unknown parameters a and b: Find a feedback law of the form (3), such that

$$\sum_{k=0}^{T}(x_k^2 + u_k^2) \leq \gamma^2 \sum_{k=0}^{T} w_k^2 + \left[\gamma^2 + C(a-\widehat{a})^2 + C(b-\widehat{b})^2\right] x_0^2$$

for all solutions to $x_{t+1} = ax_t + bu_t + w_t$ with $a \in [\underline{a}, \bar{a}]$, $b \in [\underline{b}, \bar{b}]$. Using the same approach as in section II, we find that an optimal feedback law can be defined as

$$\begin{aligned}
&\min_{u_t} \dots \max_{x_{T+1}} \max_{(a,b)} \\
&\left.\sum_{k=t}^{T}\left[x_k^2 + u_k^2 - \gamma^2(ax_k + bu_k - x_{k+1})^2\right]\right\} \text{worst case future cost} \\
&\left.+\sum_{k=0}^{t-1}\left[x_k^2 + u_k^2 - \gamma^2(ax_k + bu_k - x_{k+1})^2\right]\right\} \text{past cost} \\
&\left.-\gamma^2 x_0^2 - C(a-\widehat{a})^2 x_0^2 - C(b-\widehat{b})^2 x_0^2\right\} \text{a priori info}
\end{aligned}$$

However, defining is one thing, computing is another. To simplify the formula, the last terms are very helpful, since a large enough C makes the total expression concave in (a,b) even after the maximizations over x_k. This makes it possible to switch the order of maximizations and minimizations and handle the quadratic expressions first, using standard theory for H_∞ optimization. The resulting controller is adaptive of certainty equivalence type, computing control actions based on the worst possible parameter values with respect to the sum of future and past costs.

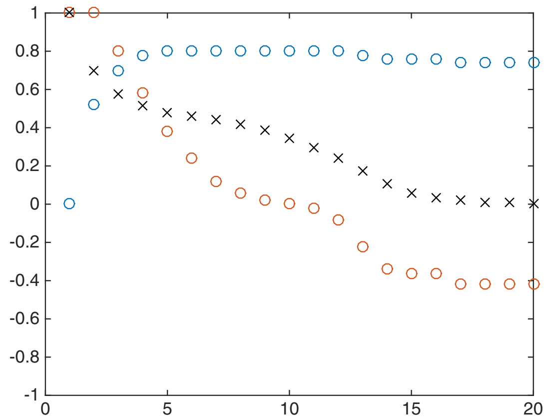

Fig. 1. The circles show how the estimates of a and b propagate from nominal values $\widehat{a} = 0$ and $\widehat{b} = 1$ towards the correct values $a = 0.7$ and $b = -0.4$. In fact, they never really get there, since worst case disturbances create a bias in the estimates. The crosses show how the state x_t tends towards zero.

V. Acknowledgement

Financial support from the Swedish Research Council, the Linnaeus center LCCC and the excellence center ELLIIT is gratefully acknowledged.

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