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Convolutional codes over bursty erasure channels with low decoding delay

D. Napp[1] and V. Requena[2]

***Abstract*— In this preliminary work we investigate the burst erasure-correction capabilities of convolutional codes. In particular, we focus on the class of encoders that admit the shortest possible decoding delay to recover all bursts of erasures of a given length for a given rate. This class of codes are simple and required only binary fields as opposed to previous constructions where non-binary fields were used.**

I. INTRODUCTION

Many packet streaming communications applications, like video conferencing in the internet, suffer from congestion as routers fail to cope with sudden increases in buffering requirements. This results in losses of clusters of packets. As the packets typically include a sequence number we can regard such losses as erasures and thus the channel considered in here is a bursty erasure channel.

Although burst erasure-correction codes have been studied for many years, there has been a recent interest in streaming applications where the data is transmitted sequentially in real-time under strict delay constraints [1], [2], [3], [4], [5], [6], [7]. This is due to the fact that in many multimedia applications long delays are usually unacceptable and so transmissions must be reproduced sequentially and with minimal delay at the destination. These constrains led to new problems in coding theory and classical error correction codes, such as Maximum Distance Separable (MDS) and rateless codes, are far from ideal in such situations.

Martinian et al. studied in [3] the construction of convolutional encoders suited for streaming applications. They established a fundamental tradeoff bound between the decoding delay and the burst erasure length, for a given rate. A class of encoders, called Short codes, that attained such a bound with equality were presented. Recently, another construction, called Midas codes, was presented in [2]. Midas codes are also burst erasure-correction codes with low decoding delay but a layer was added in order to deal also with isolated erasures.

We continue this line of research and present a simple class of encoders that are optimal with respect to the rate, decoding delay and burst length, i.e., they admit the shortest possible decoding delay to correct bursts of erasures of a given length. Previous constructions were based on MDS Reed-Solomon block codes and m-MDS convolutional codes and therefore the underlying field sizes are required to be relatively large. Our codes are defined over the binary field and therefore are also optimal with respect to the field size. Moreover, the decoding turns out to be straightforward.

[1]D. Napp is with the Department of Mathematics, University of Aveiro, Portugal diego@ua.pt

[2] V. Requena is with Department of Mathematics, University of Alicante, Spain jcliment@ua.es

II. BURSTY ERASURE CHANNELS AND CONVOLUTIONAL CODES

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of size q and $\mathbb{F}[D]$ be the ring of polynomials with coefficients in $\mathbb{F}$. As opposed to block codes, convolutional encoders take a stream of information bits and converts it into a stream of transmitted bits (by means of shift registers) and therefore they are very suitable for streaming applications. If we introduce a variable D, usually called the *delay operator*, to indicate the time instant in which each information arrived or each codeword was transmitted, then we can represent the sequence message $(\boldsymbol{v}_0, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_\mu)$ as a polynomial sequence $v(D) = \boldsymbol{v}_0 + \boldsymbol{v}_1 D + \cdots + \boldsymbol{v}_\mu D^\mu$.

A **convolutional code** $\mathcal{C}$ of rate k/n is an $\mathbb{F}[D]$-module of $\mathbb{F}[D]^n$ of rank k of the form

$$\mathcal{C} = \mathrm{im}_{\mathbb{F}[D]} G(D) = \{u(D)G(D) \mid u(D) \in \mathbb{F}^k[D]\}$$

where $G(D) \in \mathbb{F}[D]^{k\times n}$ is a right invertible matrix called an encoder of $\mathcal{C}$. The degree δ of $\mathcal{C}$ is defined as the maximum degree of the full size minors of $G(D)$.

If $G(D) = \sum_{i=0}^{m} G_i D^i$, then, m is called the *memory* of $G(D)$ and the associated **sliding matrix** of $G(D)$ is

$$G_j^c = \begin{pmatrix} G_0 & G_1 & \cdots & G_j \\ 0 & G_0 & \cdots & G_{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_0 \end{pmatrix}$$

with $G_j = 0$ when $j > m$, $j \in \mathbb{N}$.

In **burst erasure channels** each symbol $\boldsymbol{v}_i$ of the codeword $v(D)$ either arrives correctly or is completely lost and moreover losses occur in bursts. Here, we are primarily interested in building encoders that allow to recover these type of losses as soon as possible. We follow previous approaches and regard the symbols $\boldsymbol{v}_i$ as packets and consider that losses occur on a packet level. Assume that we have been able to correctly decode up to an instant i and a burst of length L is received at time instant i, i.e., one or more packets are lost from the set $(\boldsymbol{v}_i, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_{i+L-1})$. Then, we say that the **decoding delay** is T if the decoder can reconstruct each source packet with a delay of T source packets, i.e., we can recover $\boldsymbol{u}_{i+j}$ (for $j \in \{0, 1, \ldots, L-1\}$) once $\boldsymbol{v}_{i+L}, \boldsymbol{v}_{i+L+1}, \ldots, \boldsymbol{v}_{i+j+T}$ are received. In [3] the following result on the trade-off between delay and redundancy was derived.

Theorem 1: If a rate R encoder enables correction of all erasure bursts of length L with decoding delay at most T, then,

$$\frac{T}{L} \leq \max\left[1, \frac{R}{1-R}\right]. \tag{1}$$

A generalization of this result was later presented in [2] taking into account not only bursts of erasures but isolated erasures as well.

Convolutional codes with large column distance are very appealing for sequential decoding and have excellent error correction capabilities, but they require, in general, large finite fields [8] and long delays. Even though these type of codes have been proposed for applications that consider erasure channels, see [9], they do not generally achieve the best trade-off between delay, redundancy, field size and burst correction. The notion of column span was introduced in [3] as an indicator of the error-burst-correction capabilities of an encoder.

Definition 2: The column span of G_j^c is defined as

$$CS(j) = \min_{\boldsymbol{u}=(\boldsymbol{u}_0,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_j),\ u_0\neq 0} \text{span } (\boldsymbol{u}G_j^c)$$

where the span of a vector equals $j-i+1$, where j is the last index where the vector is nonzero and i is the first such index.

It is not difficult to see that if a burst of maximum length L occurs within a window of length $W+1$, then, it can be corrected if and only if $CS(W) > L$, see also [5, Lemma 1.1] for a similar result. In this paper we are interested in convolutional codes with high erasure-burst-correcting capabilities but that also admit low latency decoding.

In addition to Theorem 1, it is also worth mentioning the upperbounds given in [5], [6] on the maximum correctable burst length in terms of the encoder parameters n, k and m. In this preliminary work we shall focus on low delay decoding under bursts of erasures, and so consider only the bound given in (1) without taking into consideration the memory of the encoder or isolated losses.

III. A Simple Systematic Encoder for Burst Erasure Correction with Low Delay

Suppose that in the channel only burst of erasures of length L occur. We first consider the case $k > (n-k)$, say, $(n-k)\lambda + \gamma = k$ for some integer λ and $\gamma < n-k$. Let $G(D) = [I_k \quad \widehat{G}(D)] \in \mathbb{F}^{k\times n}$, $\widehat{G}(D) = \sum_{j\geq 0} \widehat{G}_j D^j$ be a systematic encoder given by

$$\widehat{G}_{Li} = \begin{pmatrix} O_{(i-1)(n-k)} \\ I_{(n-k)} \\ O_{k-i(n-k)} \end{pmatrix}$$

for $i = 1, \ldots, \lambda$ where I_s stands for the identity matrix of size s and O_s is the null matrix of size $s \times (n-k)$ and if $(n-k) \nmid k$, i.e., $\gamma \neq 0$, then, we also define

$$\widehat{G}_{(\lambda+1)L} = \begin{pmatrix} O_{k-\gamma} \\ I_\gamma \mid O \end{pmatrix}.$$

The remaining coefficients $\widehat{G}_i$ of $\widehat{G}(D)$ are null matrices.

Suppose that a burst of erasure of length L occurs at time j. Then, one can verify that at time instant $j+L+Li$, we recover $n-k$ coordinates of $\boldsymbol{u}_j$ for $i = 1, \ldots, \lambda - 1$, and wait until time $j + (\lambda+1)L$ to retrieve the remaining part of $\boldsymbol{u}_j$, if necessary. Then, the delay to recover $\boldsymbol{u}_j$ is $T = \lceil \frac{k}{n-k} \rceil$. Furthermore, due to the Toeplitz structure of the sliding matrix it follows that T is also the delay for decoding all the remaining erasures of $\boldsymbol{v}_s$, $s = j+1, \ldots, j+L-1$. Assume now for simplicity that $\gamma = 0$ to show that the bound in (1) is met with equality. First note that $\frac{R}{1-R} = \lambda$ for the selected parameters $(n-k)\lambda = k$ and $R = k/n$. On the other hand it is easy to verify that $T = \lambda L$ and therefore $\frac{T}{L} = \lambda = \frac{R}{1-R}$.

Thus, the proposed construction admits an optimal delay decoding when only bursts of erasures of length up to L occur. Note that this construction requires only binary entries whereas previous contributions require larger finite fields and then the decoding is computationally more efficient. The case $k \leq n-k$ readily follows by considering

$$\widehat{G}_L = \begin{pmatrix} O & I_k \end{pmatrix}$$

and the remaining coefficients of $G(D)$, G_j, $j \notin \{0, L\}$ null matrices.

IV. Conclusions

In this preliminary work we have investigated convolutional encoders with a low delay decoding with optimal burst erasure-correcting capabilities. We have introduced a class of encoders which, to the best of our knowledge, is new in this context. They have the singularity that its construction and decoding is very simple. An interesting avenue for further research is the construction of new classes of optimal encoders suited for less restricted streaming channels, e.g., channels that allow isolated both errors and erasures. In this direction, codes for bursty channels can be considered as a first step towards developing more general classes of convolutional codes, see [2] where a burst erasure-correcting code is constructed and then an addition layer is added in order to deal with more general errors. Some interesting problems in this area remain open, e.g., the problem of constructing optimal streaming codes when the source and channel rates are unequal and the channel introduces both burst and isolated erasures [2].

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