23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Constructing flat inputs for two-output systems*

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***Abstract*— In this paper, we study the problem of constructing flat inputs for two-output dynamical systems. The notion of flat inputs has been introduced by Waldherr and Zeitz in [22], [23] and can be seen as dual to that of flat outputs. In the single-output case, a flat input can be constructed if and only if the original system together with its output is observable. In the multi-output case, the observability is not necessary for the existence of flat inputs. We start by discussing the case when the dynamical system together with the given output is observable and we present a generalization of the results of [23] by relating them with the notion of minimal differential weight. Then, we give our main theorems. We consider the unobservable case for which we study the local and global problems. We completely describe the local case, discuss the issue of the minimal modification of the original system, and propose a solution for the global problem. Finally, we explain how our results can be applied to secure communication.**

I. Introduction

Consider the following nonlinear observed dynamics:

$$\Sigma : \dot{x} = f(x), \tag{1}$$

where x is the state defined on $\mathbb{R}^n$ (or more generally, on an n-dimensional manifold X), together with $y = h(x) \in \mathbb{R}^2$ the measurements. The problem that we are studying in this paper is to find control vector fields g_1 and g_2 (or, equivalently, to place the actuators or the inputs) such that the control-affine system $\Sigma_c : \dot{x} = f(x)+g_1(x)u_1+g_2(x)u_2$, associated to Σ, is flat with the original measurements (h_1, h_2) being a flat output.

The notion of flatness was introduced in control theory in the 1990's, by Fliess, Lévine, Martin and Rouchon [3], [4], see also [10], [11], [14], [19], and has attracted a lot of attention because of its multiple applications in the problem of constructive controllability and motion planning (see, e.g., [15] and references therein). Flat systems form a class of control systems whose set of trajectories can be parametrized by m functions and their time-derivatives, m being the number of controls. More precisely, the control system $\Xi : \dot{x} = F(x, u)$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, is *flat* if we can find m functions, $\varphi_i(x, u, \ldots, u^{(l)})$ such that

$$x = \gamma(\varphi, \ldots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(s)}), \tag{2}$$

for certain integers l and s, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ is called *a flat output*. Therefore the time-evolution of all state and control variables can be determined from that of flat outputs without integration and all trajectories of the system can be completely parameterized. If all functions φ_i depend on the state only, then the system is called x-flat and this will always be the case in our study.

The construction of a flat output can be seen as a problem of sensor placement in order to achieve flatness of the resulting input-output system. Dual to this, one can consider the problem of an actuator placement (i.e., of finding control vector fields g_1 and g_2) in order to achieve the same property. This dual problem has been recently introduced by Waldherr and Zeitz [22], [23] who call inputs u_1 and u_2 multiplying, respectively, g_1 and g_2, as *flat inputs* (which are objects dual to flat outputs). One of the motivations to construct a flat input for a given output is that with such an input, the tracking problem for that output can be solved with no need to calculate the zero dynamics (see, e.g., [6]), but constructing flat inputs may be useful for other problems as well: in this paper, we explain how it can be applied to secure communication.

In the single-output case, a flat input can be constructed if and only if the system Σ together with its output h is observable and the control vector field associated to the flat input can be computed via a system of linear algebraic equations, see [22]. In the multi-output case, observability is not necessary for the construction of flat inputs. The observable case has been discussed in [23] and the control vector fields (there are as many as the number of outputs) associated to the flat inputs can be determined in a similar way as for the SISO case. The goal of this paper is to treat the unobservable case. It is crucial to distinguish the observability (or unobservability) of controlled systems from observability of uncontrolled ones (recall that for nonlinear systems, the observability property depends on the control [5], [7]). Here we deal with unobservable uncontrolled system that become at least locally weakly observable due to a suitable design of flat inputs. We are interested in local and global results. We give a complete solution for the local problem: we show that locally there always exist control vector fields g_1 and g_2 such that the control-affine system Σ_c is flat with h being a flat output. The link between the observed (via the given outputs h_i) subsystem and the unobserved one is made with the help of flat inputs and some linking terms. We discuss how these linking terms can be chosen and address the problem of the minimal modification of the initial dynamical system Σ (the measure of modification being the number of equations that we have to change by adding inputs). Finally, following the local results, we present a solution for the global problem.

The paper is organized as follows. In Section II, we recall the definition of flatness and the notion of differential weight

*Research partially supported by the Natural Science Foundation of China (61573192).

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of a flat system. We discuss (from the point of view of minimal differential weight) the case when the dynamical system Σ together with a given output h is observable and present a generalization of the results of [23]. In Section III, we give our main theorems. We consider the unobservable case for which we study the local and global problems. We completely describe the local case, discuss the issue of the minimal modification of Σ and, finally, propose a solution for the global problem. We explain how our results can be applied to secure communication in Section IV and provide proofs in Section V.

II. Definitions and problem statement

Consider the nonlinear control systems $\Xi : \dot{x} = F(x,u)$, where $x \in X$ and $u \in U$, with X (resp., U) an open subset of $\mathbb{R}^n$ (resp., of $\mathbb{R}^m$), or more generally, an n-dimensional manifold X and an m-dimensional manifold U, respectively. Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\overline{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set U^{-1} is empty and $\overline{u}^{-1}$ in an empty sequence.

Definition 1: The system $\Xi : \dot{x} = F(x,u)$ is *flat* at $(x_0, \bar{u}_0^l) \in X \times U^l$, for $l \geq -1$, if there exist a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property: there exist an integer s and smooth functions γ_i, $1 \leq i \leq n$, and δ_j, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

for any C^{l+s}-control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output*.

If all functions φ_i depend on the state only, then the system is called x-flat and this will be the case for all systems considered in the paper.

The minimal number of derivatives of components of a flat output, needed to express x and u, will be called the differential weight of that flat output and is formalized as follows. By definition, for any flat output φ of Ξ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{array}{rcl} x & = & \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u & = & \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}). \end{array} \tag{3}$$

Moreover, we can choose $(s_1, \ldots, s_m)$, γ and δ such that (see [20]) if for any other m-tuple $(\tilde{s}_1, \ldots, \tilde{s}_m)$ and functions $\tilde{\gamma}$ and $\tilde{\delta}$, we have

$$\begin{array}{rcl} x & = & \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}) \\ u & = & \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(\tilde{s}_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(\tilde{s}_m)}), \end{array}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^{m}(s_i + 1) = m + \sum_{i=1}^{m} s_i$ the differential weight of φ. A flat output of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ. We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output. The differential weight of Ξ is at least $n + m$, since we have to express n states and m independent controls and in order to do that, we need at least $n + m$ derivatives (taking into account also those of order zero).

Flatness is a property of the state-space dynamics $\dot{x} = F(x,u)$ of a control system. It can also be described as a property of the input-output map for a dummy output y. In fact, x-flatness is equivalent to the existence of an $\mathbb{R}^m$-valued dummy output $y = \varphi(x)$ that renders the system $\dot{x} = F(x,u)$ observable and left-invertible [21]. Indeed, expressing the state as $x = \gamma(\varphi, \dot{\varphi}, \ldots, \varphi^{(s-1)})$ and the control as $u = \delta(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$ corresponds, respectively, to observability and left invertibility.

Let us now consider the dynamical system Σ, given by (1), together with an output $y = h(x) \in \mathbb{R}^2$. In order to emphasize the fact that the system is observed[1] we will use the notation (Σ, h). The problem that we are studying in this paper is to construct control vector fields g_1 and g_2 (whose inputs u_1 and u_2 are called *flat inputs*) such that the control-affine system Σ_c associated to Σ, and given by

$$\Sigma_c : \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2,$$

is x-flat with respect to the the original output (h_1, h_2). In that case, we will say that the pair (Σ_c, h) is x-flat.

As we have already noticed flatness is closely related to observability. Thus, for the problem of constructing flat inputs, it is natural to start by checking observability of Σ with respect to h, see [8], [13]. We denote by $\mathcal{H}(x)$ the codistribution $\mathcal{H}(x) = \text{span } \{\mathrm{d}L_f^j h_i(x), j \geq 0, 1 \leq i \leq 2\}$ associated to the output h. We will actually distinguish the observable and unobservable case. We will need the notion of observability quasi-indices:

Definition 2: The observed system (Σ, h) is said to have observability quasi-indices (ρ_1, ρ_2) at $x_0 \in \mathbb{R}^n$ if $\rho_i \geq 1$, for $1 \leq i \leq 2$, $\rho_1 + \rho_2 = n$ and $\text{span } \{\mathrm{d}L_f^j h_i(x_0), 0 \leq j \leq \rho_i - 1, 1 \leq i \leq 2\} = \mathbb{R}^n$.

Of course, if (Σ, h) has observability quasi-indices, then it is observable. A related concept is that of observability indices, see [13], which are the largest (in the lexicographic ordering) quasi-indices such that $\rho_1 \leq \rho_2$. Contrary to the observability indices (which are unique), the observability quasi-indices (ρ_1, ρ_2) are no longer unique and we do not assume any order relation between them. Since observability quasi-indices may depend on a point, we say that quasi-indices (ρ_1, ρ_2) are uniform in a subset $\mathcal{X}$ of $\mathbb{R}^n$ if (ρ_1, ρ_2) form quasi-indices at any $x \in \mathcal{X}$.

The following theorem states that the observed system (Σ, h) can be made flat of differential weight $n + 2$ (which is the minimal possible) if and only if it admits observability quasi-indices and moreover, gives a system of algebraic equations (whose coefficients are calculated in terms of the Lie derivatives of the outputs) to be solved in order to construct flat inputs.

Theorem 1: Consider the observed system (Σ, h) around $x_0 \in \mathbb{R}^n$ and assume that rk $\mathcal{H}(x_0) = n$. *The following conditions are equivalent:*

$(\mathcal{O}1)$ There exist observability quasi-indices (ρ_1, ρ_2) at x_0;

[1]When we say that a dynamical system Σ is observed, this does not mean that Σ is necessarily observable with respect to the output h.

(O2) There exist g_1 and g_2 such that the system (Σ_c, h) is x-flat at x_0 of differential weight $n+2$, with h being a minimal flat output;

Moreover, under (O1), g_1 and g_2 are given by

(i) The distribution $\mathcal{G} = \operatorname{span}\{g_1, g_2\}$ satisfies

$$\mathcal{G}^{\perp} = \operatorname{span}\{\mathrm{d}L_f^j h_i, 0 \le j \le \rho_i - 2, 1 \le i \le 2\},$$

or, equivalently, by

(ii) The decoupling matrix $(L_{g_j} L_f^{\rho_i - 1} h_i(x_0))$, for $1 \le i, j \le 2$, of (Σ_c, h), is of full rank (equal to 2).

If (ρ_1, ρ_2) are uniform quasi-indices on $\mathcal{X}$ (in particular, on $\mathcal{X} = \mathbb{R}^n$), then the vector fields g_1 and g_2 exist globally on $\mathcal{X}$, on which are given by (ii), and they yield the global system Σ_c on $\mathcal{X}$ that is locally flat around any $x \in \mathcal{X}$ with the help of the globally defined flat output $(\varphi_1, \varphi_2) = (h_1, h_2)$.

Remark 1: Theorem 1 generalizes the result of [23] according to which if a system has observability quasi-indices (ρ_1, ρ_2) and the vector fields g_1 and g_2 satisfy (ii), then the control system (Σ_c, h) is flat. We also prove such g_i's (and only such) lead to a flat system of minimal weight.

The above conditions are local and valid around a nominal point x_0 that can be equilibrium or not. If quasi-indices are uniform on $\mathbb{R}^n$, then the control vector fields g_1 and g_2 exist globally and the control system is x-flat on $\mathbb{R}^n$ with a flat output being globally defined as $\varphi_i = h_i$, $1 \le i \le 2$, nevertheless representation (3) is, in general, local only. In particular, the map $(x, u) \mapsto (h_1, \dot{h}_1, \ldots, h_1^{(s_1)}, h_2, \dot{h}_2, \ldots, h_2^{(s_2)})$ need not be globally injective. Notice that there can be many observability quasi-indices and different observability quasi-indices (ρ_1, ρ_2) lead to different control vector fields g_1 and g_2 and thus to different flat control systems (Σ_c, h).

In particular, to any choice of observability quasi-indices (ρ_1, ρ_2), there correspond flat inputs u_1, u_2 giving (3) with $s_i = \rho_i$. An interesting remark is that in some practical applications, it may be interesting to use more derivatives of a particular output component to decrease the number of derivatives of another (more sensitive) component. Notice also that for each pair (ρ_1, ρ_2), the resulting control system is static feedback linearizable with (h_1, h_2) playing the role of the linearizing output. It is well known that systems linearizable via invertible static feedback are flat and their description (3) uses the minimal possible, which is $n+2$, number of time-derivatives of the components of the flat output (see [16] where that property is discussed). So, according to Theorem 1, if the pair (Σ, h) is observable, then there always exist g_1 and g_2 such that the associated control system Σ_c is flat with h being a flat output and, moreover, g_1 and g_2 can be chosen such that Σ_c is static feedback linearizable (see [12], [9] for the latter property) and can be calculated via a system of algebraic equations. The goal of this paper is thus to solve the problem of finding flat inputs for the unobservable two-output case. In that case the corresponding flat control system will not be static feedback linearizable, its differential weight is thus greater than $n+2$ and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system.

Similarly to the definition of observability quasi-indices, we introduce the notion of unobservability quasi-indices:

Definition 3: The observed system (Σ, h) is said to have unobservability quasi-indices (ρ_1, ρ_2) if $\dim \operatorname{span}\{\mathrm{d}L_f^j h_i(x), 0 \le j \le \rho_i - 1, 1 \le i \le 2\} = \dim \operatorname{span}\{\mathrm{d}L_f^j h_i(x), j \ge 0, 1 \le i \le 2\} = \rho_1 + \rho_2 = k < n$, for all $x \in \mathbb{R}^n$.

According to the above definition, if the system (Σ, h) has unobservability quasi-indices, then it is unobservable and, in addition, the associated codistribution $\mathcal{H}(x) = \operatorname{span}\{\mathrm{d}L_f^j h_i(x), j \ge 0, 1 \le i \le 2\}$ is of constant rank, equal to k, for all $x \in \mathbb{R}^n$. This means that only k directions can be observed (and this is valid around any point). The above definition is more restrictive than the lack of observability at a point: we require the system to be nowhere observable on $\mathbb{R}^n$ and, moreover, its observability defect to be constant.

Assumption 1: From now on, unless stated otherwise, we assume that all ranks involved are constant in a neighborhood of a given $x_0 \in \mathbb{R}^n$.

Without the above assumption (which we will apply to the ranks of certain observability codistributions), the results of this paper are actually valid on an open and dense subset of $\mathbb{R}^n$.

III. Main results

Throughout, we assume that the system (Σ, h) is not observable at x_0, the point around which we work, and that it has unobservability quasi-indices (ρ_1, ρ_2). By introducing the coordinates $w_i^j = L_f^{j-1} h_i$, for $1 \le j \le \rho_i$, $1 \le i \le 2$, the system Σ can be transformed into the following observed-unobserved form

$$\begin{aligned} \dot{w}_i^j &= w_i^{j+1}, \qquad 1 \le j \le \rho_i - 1, \\ \dot{w}_i^{\rho_i} &= a_i(w), \qquad 1 \le i \le 2, \end{aligned} \tag{4}$$

$$\dot{z} = b(w, z) \tag{5}$$

with $\dim w = k < n$ and $\dim w + \dim z = n$, where z consists of any coordinates completing w to a coordinate system, and $(h_1, h_2) = (w_1, w_2)$. The w-coordinates of the above form are the states observed with the help of the output h and its time derivatives and there are k of them (the rank of the codistribution $\mathcal{H}$). The z-coordinates correspond to the unobserved directions, there are $n-k$ (which is the observability defect) of them and they complete w to a coordinate system. We denote by a (resp., by b) the drift associated to the observed (resp., unobserved) subsystem.

A. Local results

The main result of the paper is given by the following theorem according to which even in the unobservable case, we can always, locally, construct control vector fields g_1 and g_2 such that the associated control system (Σ_c, h) is x-flat with flat output $\varphi = h$. Theorem 2 below also states that the associated flat control system is of differential weight at least $n + m + (n-k)$, where $n - k \ge 1$ is the observability defect (and, in particular, it is never static feedback linearizable). The nature of the nominal point around which we work (equilibrium or not) plays an important role in our study. For each case, a normal form for the control system Σ_c is presented.

Theorem 2: Consider the observed system (Σ, h) around $x_0 \in \mathbb{R}^n$ and assume that $\operatorname{rk} \mathcal{H}(x) = k < n$, *in a neighborhood of x_0. We have:*

($\mathcal{U}1$) If there exist g_1 and g_2 such that (Σ_c, h) is x-flat at $(x_0, \bar{u}_0^l) \in \mathbb{R}^n \times \mathbb{R}^{2(l+1)}$, for a certain $l \geq -1$, then the differential weight of Σ_c is at least $n + m + (n-k)$;

($\mathcal{U}2$) If $f(x_0) \neq 0$, then there locally exists a change of z-coordinates that transforms the unobserved subsystem (5) *into*

$$\begin{array}{rcl} \dot{z}^q & = & 0, \quad 1 \leq q \leq n-k-1, \\ \dot{z}^{n-k} & = & 1, \end{array} \tag{6}$$

and for this form, we can always locally construct g_1 and g_2 such that (Σ_c, h) is x-flat at $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^2$ of differential weight $n + m + (n-k)$, and is given by:

$$NF1 \begin{cases} \dot{w}_1^j = w_1^{j+1} & \dot{w}_2^j = w_2^{j+1} \\ \dot{w}_1^{\rho_1} = a_1(w) + u_1 & \dot{w}_2^{\rho_2} = a_2(w) + z^1 u_1 \\ & \dot{z}^q = z^{q+1} u_1, \\ & \dot{z}^{n-k} = 1 + u_2, \end{cases}$$

where $1 \leq j \leq \rho_s - 1$, for $1 \leq s \leq 2$, $1 \leq q \leq n-k-1$, and $(h_1, h_2) = (w_1^1, w_2^1)$.

($\mathcal{U}3$) There locally exists (independently of whether $f(x_0) = 0$ or $f(x_0) \neq 0$) a change of z-coordinates that transforms the unobserved subsystem (5) *into*

$$\begin{array}{rcl} \dot{z}_i^q & = & \dot{z}_i^{q+1}, \quad 1 \leq q \leq \mu_i - 1, \\ \dot{z}_i^{\mu_i} & = & b_i(w, \bar{z}_i), \quad 1 \leq i \leq \ell, \end{array} \tag{7}$$

where $\mu_i \geq 1$, for $1 \leq i \leq \ell$, $\sum_{i=1}^{\ell} \mu_i = n-k$, and $\bar{z}_i$ denotes $\bar{z}_i = (z_1^1, \ldots, z_1^{\mu_1}, \ldots, z_i^1, \ldots, z_i^{\mu_i})$, and for this form, we can always locally construct g_1 and g_2 such that (Σ_c, h) is x-flat at $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^2$, and is given by:

$$NF2 \begin{cases} \dot{w}_1^j = w_1^{j+1} & \dot{w}_2^j = w_2^{j+1} \\ \dot{w}_1^{\rho_1} = a_1(w) + u_1 & \dot{w}_2^{\rho_2} = a_2(w) + z_1^1 u_1 \\ & \dot{z}_i^q = \dot{z}_i^{q+1} \\ & \dot{z}_i^{\mu_i} = b_i(w, \bar{z}_i) + z_{i+1}^1 u_1 \\ & \dot{z}_\ell^{\mu_\ell} = b_\ell(w, z) + u_2, \end{cases}$$

where $1 \leq j \leq \rho_s - 1$, for $1 \leq s \leq 2$, $1 \leq q \leq \mu_i - 1$, for $1 \leq i \leq \ell - 1$, and $(h_1, h_2) = (w_1^1, w_2^1)$.

Since a flat system is observable (with respect to its flat output and for any inputs), we have to render the original system (Σ, h) observable with the help of the control vector fields and, in order to do that, we need to add extra linking terms between the w- and the z-subsystems. For the case $f(x_0) \neq 0$, corresponding to the normal form $NF1$, only one linking term is needed and the variable z^1 plays the role of that linking term. The link between the observed variables w and the unobserved variables z is made with the help of the control vector field g_1 only. The remaining input (flatness implies that there are necessarily two inputs, which is the number of outputs) appear at the last level of the z-chain. Since the z-chain is of length $n-k$, the input u_1 needs to be differentiated $(n-k)$ times in order to express all states and the remaining control u_2 as a function of h and its time-derivatives, so the differential weight is $n + m + (n-k)$.

If $f(x_0) \neq 0$, then the drift corresponding to the z-variables, hat we denote by b in (5), can be rectified (even if originally the non vanishing component of f is in the w-part): indeed, in well chosen z-coordinates, we have $b(w,z) = \frac{\partial}{\partial z^{n-k}}$. It follows that, if x_0 is not an equilibrium point, the drift after rectification plays no role in choosing the linking term (that can be any) and, moreover, it has no impact on the triangular structure of the z-part, which is completely determined by g_1. Notice that the vector fields g_1, g_2 depend on the unobserved (with respect to the original output h) states only and they are designed to be in the chained form. This particular form guarantees that the differential weight of Σ_c is minimal and equal to $n + m + (n-k)$. It is clear that $NF1$ is not linearizable via invertible static feedback, however, $NF1$ becomes static feedback linearizable after pre-integrating $n-k$ times the first control u_1 (thus after the application of a dynamical precompensator of dimension $n-k$). In order to compute the z-coordinates for the normal form $NF1$, we have to solve the partial differential equation $L_f\psi(x) = 0$. Among $n-1$ first integrals of f obtained this way, we chose (any) $n-k-1$ of them (independent modulo $\mathcal{H}$) : these are the variables z^j, for $1 \leq j \leq n-k-1$. For z^{n-k} , we have to solve again a partial differential equation: $L_f\psi(x) = 1$.

If x_0 is an equilibrium point of f, the drift can no longer be rectified and we need to respect its own triangular structure. In that case the system can be transformed into $NF2$. This normal form actually works around any x_0, equilibrium or not, and in order to compute the z coordinates we do not have to compute any partial differential equation. For $NF2$, the linking terms are $z_1^1, z_2^1, \ldots, z_\ell^1$. In fact, we choose an arbitrary function ψ_1^1 defining the first linking term z_1^1, we keep differentiating it, and when we loose observability again (the derivative of order μ_1 of z_1^1 depends on w and z_1^q, for $1 \leq q \leq \mu_1$, only), we link it with the remaining still unobservable system via the control u_1 and another (arbitrarily chosen) linking term $\psi_2^1 = z_2^1$. We repeat this process until all states are observable. The integer μ_1 can be interpreted as the unbservability quasi-index of $\psi_1^1 = z_1^1$ (observability being considered modulo $\operatorname{span}\{\mathrm{d}w_i^j, 1 \leq j \leq \rho_i, 1 \leq i \leq 2\}$). Notice that linking terms are far from being unique and their choice may be important.

As for the previous case, the control vector fields g_1 and g_2 depend on z only. Moreover, u_1 appears only in the equations for $\dot{w}_1^{\rho_1}$, $\dot{w}_2^{\rho_2}$ and $\dot{z}_i^{\mu_i}$, $1 \leq i \leq \ell - 1$, and u_2 only in the equation for $\dot{z}_\ell^{\mu_\ell}$ (the number of modified equations is $\ell + 2$).

If the system (Σ, h) is observable at a given x_0, then for the corresponding flat control system (Σ_c, h), even if $\varphi_i = h_i$ yield a local flatness only, representing the state and the control with the help of $\varphi_i = h_i$ and their derivatives is global with respect to u, so we never face singularities in the control space. This is no longer the case if (Σ, h) is unobservable, and both $NF1$ and $NF2$ exhibit singularities in the control space. For both normal forms, the system ceases to be flat with h as a flat output at $u_1 = 0$ (which is a singular control for flatness). If we want to avoid singularities in the control space, we can construct another control system as follows: in the equations for $w_2^{\rho_2}$ and z, replace u_1 by $\exp(u_1)$ (and keep u_1 unchanged for $w_1^{\rho_1}$). We obtain: $\dot{w}_1^{\rho_1} = a_1(w) + u_1$, $\dot{w}_2^{\rho_2} = a_2(w) + z_1^1 \exp(u_1)$, $\dot{z}_i^{\mu_i} = b_i(w, \bar{z}_i) + z_{i+1}^1 \exp(u_1)$, for $1 \le i \le \ell - 1$. We have thus constructed a new control system Σ_c that does no longer display singularities in the control space (representing the control u, with the help of the components $\varphi_i = h_i$ and their derivatives, is global), but the system is nonlinear with respect to u (which is the price for avoiding singularities).

Summing up, locally (and under the constant rank assumption), we can always construct control vector fields g_1, g_2 such that the associated control system (Σ_c, h) is flat of differential weight at least $n + m + (n - k)$, thus, locally, the problem of constructing flat inputs is completely solved.

B. Choice of the linking terms and minimal modification of Σ

In some cases, it may be important to modify the initial dynamical system in a minimal way (the measure of modification being the number of equations for which we add an input). An interesting question arises: how to choose the linking terms in order to obtain a minimally modified system? We answer this question below.

It is clear that in our construction, for both normal forms, the w-subsystem is minimally modified because the control is added only in one equation per w-chain (which is the minimal w-modification, since we cannot achieve flatness by modifying only one w-equation). So the best we can hope for is that only three equations involve the control (we necessarily have one per w-chain involving u_1 and another for the z chain that has to be affected by u_2) and this is always the case if $n - k = 1$. If we want to modify the original system Σ in that minimal way, then we necessarily have to construct the associated flat control system using the procedure of the normal form $NF2$. Indeed, for $NF1$ (unless $n - k = 1$), the z-subsystem is maximally modified since the control appears in each equation.

If we want to get $NF2$ with a minimal modification of the z-equations also in the case $n - k > 1$, we can use only one linking term $z^1 = \psi(w, z)$ and the ideal one would be such that $(\mathrm{d}\psi \wedge \mathrm{d}L_f\psi \wedge \ldots \wedge \mathrm{d}L_f^{n-k-1}\psi)(w_0, z_0) \neq 0 \bmod \mathrm{span}\ \{L_f^{j-1}h_i, 1 \le i \le 2, j \ge 1\}$. In that case, we would obtain $NF2$ with $\mu_1 = n - k$, only the last equation of the z-chain would be modified by adding u_2, and the vector fields g_1 and g_2 would distort three equations only. Thus the existence and construction of a minimal modification is equivalent to the existence of such a function ψ, which around an equilibrium x_0 of f reduces, actually, to a linear problem. Define a linear change of coordinates $(w, z)^T = Px$, where P is any constant invertible matrix, such that $\mathrm{span}\ \{\mathrm{d}w_1, \ldots, \mathrm{d}w_k\} = \mathcal{H}(x_0)$. Represent the system in (w, z)-coordinates $\dot{w} = f^1(w, z)$, $\dot{z} = f^2(w, z)$. Put $(w_0, z_0)^T = P(x_0)$ and if x_0 is an equilibrium, that is $f(x_0) = 0$, then define $A = \frac{\partial f^2}{\partial z}(w_0, z_0)$.

Proposition 1: The following conditions are equivalent:

(i) There exist g_1, g_2 such that (Σ_c, h) is x-flat and only three equations of Σ are modified by g_1 and g_2;

(ii) There exists $\psi(w, z)$ such that $(\mathrm{d}\psi \wedge \mathrm{d}L_f\psi \wedge \ldots \wedge \mathrm{d}L_f^{n-k-1}\psi)(w_0, z_0) \neq 0 \bmod \mathrm{span}\ \{L_f^{j-1}h_i, 1 \le i \le 2, j \ge 1\}$.

Moreover, if $f(x_0) = 0$, then (i) and (ii) are equivalent to the following

(iii) There exits $C \in (\mathbb{R}^n)^*$ such that $C, CA, \ldots, CA^{n-k-1}$ are independent, in which case a solution of (ii) is $\psi(w, z) = Cz$.

Given a matrix A, it is not always possible to achieve observability with the help of one output. In that case, we choose ψ such that the linear approximation is maximally observable.

C. Global results and minimal modification of Σ

Theorem 2 is local and valid around a nominal point x_0 (equilibrium or not) and guarantees a local construction of the vector fields g_1 and g_2. Similarly to the observable case, under some hypothesis, g_1 and g_2 may exist globally on $\mathbb{R}^n$ and the control system may be x-flat on $\mathbb{R}^n$ with the flat output being globally defined as $\varphi_i = h_i$, $1 \le i \le 2$, (but, as explained after stating Theorem 1, with representation (3) which is, in general, local). The following theorem gives conditions allowing the global construction of g_1 and g_2.

Theorem 3: Consider the observed system (Σ, h) and assume $\mathrm{rk}\ \mathcal{H}(x) = k < n$, for all $x \in \mathbb{R}^n$. If there exists a map $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\mathrm{d}L_f^j\psi(x)$, for $0 \le j \le n-k-1$, are everywhere independent modulo $\mathcal{H}(x)$, then there exist g_1 and g_2, defined globally on $\mathbb{R}^n$ such that (Σ_c, h) is locally x-flat around $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^2$, where $x_0 \in \mathbb{R}^n$ may be any and u_0 is such that $u_{10} \neq 0$, and is given by:

$$NF_{min} \begin{cases} \dot{w}_1^j = w_1^{j+1} & \dot{w}_2^j = w_2^{j+1} \\ \dot{w}_1^{\rho_1} = a_1(w) + u_1 & \dot{w}_2^{\rho_2} = a_2(w) + z^1 u_1 \\ & \dot{z}^q = \dot{z}^{q+1} \\ & \dot{z}^{n-k} = b(w, z) + u_2, \end{cases}$$

where $1 \le j \le \rho_s - 1$, for $1 \le s \le 2$, $1 \le q \le n - k - 1$, $(h_1, h_2) = (w_1^1, w_2^1)$ and $z^1 = \psi(x)$. Moreover, the vector fields g_1 and g_2, have together the minimal possible, which is three, number of nonzero components.

Recall that when defining the unobservability quasi-indices we supposed that the rank of the codistribution $\mathcal{H}$ is constant everywhere. It is clear that under that assumption and the conditions of Theorem 3, the functions $w_i^j = L_f^{j-1}h_i$, for $1 \le i \le 2$, $1 \le j \le \rho_i$, and $z^q = L_f^{q-1}\psi(x)$, $1 \le q \le n - k$, are defined globally and form around any x_0 a local system of coordinates. NF_{min} is minimally modified (it has only three nonzero components of g_i's) and, like for $NF1$ and $NF2$, the differential weight equals $n + m + (n - k)$.

Global results are very useful for applications to secure communication (that partially motivates this work), for which we want to preserve some global properties of the system.

Chaotic models are often used for constructing secure communication systems and the chaotic behavior need to be globally preserved (in fact, it is enough to assume that the chaotic properties are valid on the attraction basin of the strange attractor), see the next section where we explain in more detail how our results can be applied to secure data transmission.

IV. Application to secure communication

We speak about secure communication when two entities are communicating and do not want a third party to listen in. Therefore they need to communicate in a way not susceptible to interception and the contents of the transmitted message have to be protected from being accessed by unauthorized users. Since the work of [18], it is known that the problem of secure communication can be investigated with the help of synchronization of chaotic systems. It is important to distinguish the unidirectional synchronization and the bidirectional one. For secure data transmission the unidirectional synchronization is considered and the problem is equivalent to that of an observer design (see, e.g., [17]). The idea is to use the output of a particular dynamical chaotic system (that masks the message) to drive the response of a, in general, identical system (that recovers the message) so that they oscillate in a synchronized manner.

We will next highlight the interest of our approach for the chaotic data secure transmission (see also [1] for a related approach). The general problem can be summarized as follow. We suppose that two messages $u_1(t)$ and $u_2(t)$ have to be sent to the receiver. We use a transmitter, composed of two independent chaotic systems, a Chua circuit (x_c, y_c, z_c) and a Rössler circuit (x_r, y_r, z_r) of the form:

$$(Ch): \begin{cases} \dot{x}_c &= p(-x_c + y_c - r(x_c)) \\ \dot{y}_c &= x_c - y_c + z_c \\ \dot{z}_c &= -qy_c \\ \dot{x}_r &= -y_r - z_r \\ \dot{y}_r &= x_r + ay_r \\ \dot{z}_r &= b + z_r(x_r - c), \end{cases}$$

where the output $(h_1, h_2) = (y_c, x_c)$ is the masked information transmitted via the communication multiplexed channel, the parameters a, b, c, p and q are constant and the function $r(x_c)$ is equal to $m_1 x_c$, if $|x_c| \le 1$, respectively, to $m_0(x_c - sign(x_c)) + m_1 sign(x_c)$, if $|x_c| > 1$, where m_0 and m_1 are constant. In order to transmit messages $u_1(t)$ and $u_2(t)$, we add to (Ch) two control vector fields g_1 and g_2 whose controls are, respectively, u_1 and u_2 (that is, messages to be sent):

$$(Ch_c): \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x), y_i = h_i(x), 1 \le i \le 2,$$

where g_1 and g_2 are chosen in such a way that (Ch_c) is flat with $\varphi = (h_1, h_2)$ being a flat output, and f is the right-hand side of (Ch). The chaotic behavior (depending on the values of the constant parameters) is crucial and has to be preserved by the modifications applied on the system.

It is clear that with the given output (y_c, x_c) only the Chua variables can be observed and in the observed-unobserved form (4)-(5), using the new w-coordinates (which are valid everywhere) $w_1^1 = y_c$, $w_1^2 = L_f y_c$, $w_2^1 = x_c$, the Chua circuit is equivalently given by: $\dot{w}_1^1 = w_1^2$, $\dot{w}_1^2 = p(-w_2^1 + w_1^1 - r(w_2^1)) - qw_1^1 - w_1^2$ and $\dot{w}_2^1 = p(-w_2^1 + w_1^1 - r(w_2^1))$. Notice that, here, the unobserved subsystem, described by the Rössler circuit, is completely independent of the observed one. Define the linking term $z^1 = \psi(w, x_r, y_r, z_r)$ as $\psi = y_r$ and compute its successive time-derivatives. We get $L_f\psi = x_r + ay_r$ and $L_f^2\psi = -(y_r + z_r) + a(x_r + ay_r)$. It is clear that $\mathrm{d}\psi$, $\mathrm{d}L_f\psi$ and $\mathrm{d}L_f^2\psi$ are independent everywhere, so $z^j = L_f^{j-1}\psi$, $1 \le j \le 3$, (together with the w-coordinates) define a global change of coordinates. According to our results, we can globally construct control vector fields g_1 and g_2 such that the corresponding control system is flat with $(w_1^1, w_2^1) = (y_c, x_c)$ being a flat output. The following control system is obtained:

$$(Ch_c): \begin{cases} \dot{w}_1^1 = w_1^2 \\ \dot{w}_1^2 = p(-w_2^1 + w_1^1 - r(w_2^1)) - qw_1^1 - w_1^2 + k_1 u_1 \\ \dot{w}_2^1 = p(-w_2^1 + w_1^1 - r(w_2^1)) + k_2 z^1 \mathrm{e}^{u_1} \\ \dot{z}^1 \ = z^2 \\ \dot{z}^2 \ = z^3 \\ \dot{z}^3 \ = -b - z^2 - az^3 - (z^3 - z^1 + az^2) \times \\ \qquad (z^2 - az^1) + k_3 u_2. \end{cases}$$

In order to avoid possible singularities in the control space, we rather work with $\exp(u_1)$ instead of u_1 (see the comments at the end of Section III-A, where we discuss flatness singularities in the control space). Notice also that control gains k_1, k_2 and k_3 are used: they play an important role in preserving the chaotic behavior of each subsystem. The controlled system in the original coordinates is:

$$(Ch_e): \begin{cases} \dot{x}_c = p(-x_c + y_c - r(x_c)) + k_2 y_r e^{u_1} \\ \dot{y}_c = x_c - y_c + z_c \\ \dot{z}_c = -qy_c - k_2 y_r e^{u_1} + k_1 u_1 \\ \dot{x}_r = -y_r - z_r \\ \dot{y}_r = x_r + ay_r \\ \dot{z}_r = b + z_r(x_r - c) - k_3 u_2. \end{cases}$$

The receiver knows the complete dynamics of the flat control system $\dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ as well as the transmitted output $(y_1(t), y_2(t))$ and therefore using the flatness property, the original message $u = (u_1(t), u_2(t))$ can be recovered at the receipt as functions of the received output and its successive time derivatives: $u(t) = \delta(y, \dot{y}, \ldots, y^{(s)})$.

There are several observations that can be made at this point. Notice that, at first sight, only left invertibility (which is a less strong property than flatness) is needed. But, if we only require left invertibility of (Ch_c) or, equivalently, of (Ch_e), then the zero dynamics have to be known exactly by both the transmitter and the receiver (not only its structure, but also the initialization). Another interesting question is why the unobserved part z is needed. The unobserved subsystem plays an important role for increasing the safety. Indeed, consider the system in (w, z)-coordinates, the message u_1 has a degree of security significantly lower than that of u_2. The z-part can be seen as a second level of security. This motivates even more the use of the quasi-indices: in order to

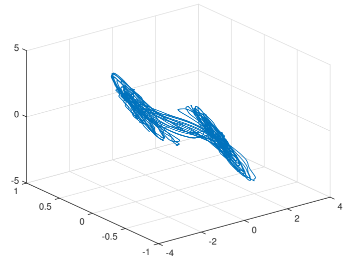

Fig. 1. Chua x_c,y_c, z_c phase plot.

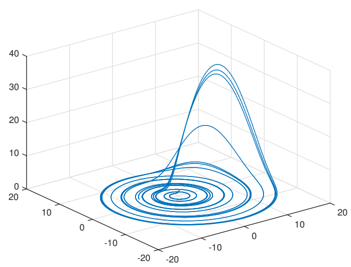

Fig. 2. Rössler x_r, y_r, z_r phase plot.

decode u_1, we may need only one output (if a_1 depends on w_1^j only) but we never need the linking term z^1, however, we always need it for decoding u_2.

A. Simulations

Figures 1 and 2 correspond to the solutions of the control system (Ch_e) and show that the chaotic behavior is preserved after modifying the original dynamical chaotic system (Ch) by adding the inputs. Figures 3 and 4 show that the inputs (which correspond to the confidential messages) u_1 and u_2 are respectively recovered. The parameters for the simulations come from [2] and are given by $a = 0.15$, $b = 0.20$, $c = 10$, $d = 20$, $e = 50$, $f = 40$, $p = 10$, $q = 14.87$, $m_0 = -0.68$, $m_1 = -1.27$, with initial conditions $x_c(0) = y_c(0) = z_c(0) = -0.1$, $x_r(0) = y_r(0) = z_r(0) = 1$ and control gains $k_1 = k_2 = k_3 = 0.1$. The input u_1 is equal to $\sin(10\pi t)$ and u_2 equals $3\sin(16\pi t)$. A simple Euler scheme with a step of $10^{-4}s$ is used.

V. Proofs

A. Proof of Theorem 2

Proof of ($\mathcal{U}1$). Consider the system $\Sigma : \dot{x} = f(x)$, around x_0, together with its output $h = (h_1(x), h_2(x))$ and suppose that there exist g_1, g_2 such that the associated control system $\Sigma_c : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ is x-flat at $(x_0, \bar{u}_0^l)$ with $\varphi = (h_1, h_2)$ a minimal flat output, defined in a neighborhood $\mathcal{X}$ of x_0. Let μ_1 and μ_2 denote the relative degrees of h_1 and h_2, that is, $h_1^{(\mu_1)}$ and $h_2^{(\mu_2)}$ are the lowest time-derivatives depending explicitly on u. Since the components of a flat output and their successive time-derivatives are independent, it follows that we necessarily have $\mu_1 + \mu_2 \leq k$, with k the number of directions of Σ

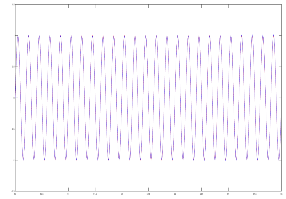

Fig. 3. u_1 in red and estimated u_1 in blue coincide.

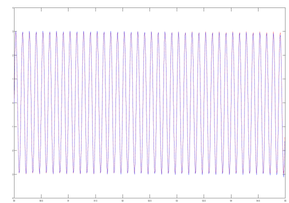

Fig. 4. u_2 in red and estimated u_2 in blue coincide.

observed with the help of h. Consider the decoupling matrix $D_{ij} = (L_{g_j} L_f^{\mu_i - 1} h_i)$, for $1 \leq i, j \leq 2$. By definition of the relative degree, we have $1 \leq \mathrm{rk}\, D(x) \leq 2$ and according to the constant rank assumption, $\mathrm{rk}\, D(x)$ is constant in a neighborhood of x_0. It is easy to see that $\mathrm{rk}\, D(x) = 1$. Indeed, if $\mathrm{rk}\, D(x) = 2$, then Σ_c is either static feedback linearizable (if $\mu_1 + \mu_2 = n$), and in that case the original dynamical system Σ would be observable with respect to h, contradicting our assumption, or not flat (if $\mu_1 + \mu_2 < n$, since we define $h_1^{(\mu_1)} = \tilde{u}_1$, $h_2^{(\mu_2)} = \tilde{u}_2$ and $n - (\mu_1 + \mu_2)$ components of the state x are not expressed via the derivatives $h_1^{(j)}$ and $h_2^{(j)}$), for $j \geq 0$. Put $h_1^{(\mu_1)} = L_{f+gu} L_f^{\mu_1 - 1} h_1 = \tilde{u}_1$, where $gu = u_1 g_1 + u_2 g_2$. With the component h_1 of the flat output we can no longer produce any new state or control, therefore, all the remaining states and the second control have to be expressed with the help of $h_2^{(\mu_2 + j)}$, $j \geq 0$, which depends explicitly on $\tilde{u}_1^{(j)} = h_1^{(\mu_1 + j)}$, so at each step we add one new derivative of $\tilde{u}_1$ (and therefore of the component h_1). In the best case, due to each differentiation we compute one new state, the remaining control being expressed at the last step. The differential weight of h is thus $n + m + (n - (\mu_1 + \mu_2)) \geq n + m + (n - k)$. It follows that the differential weight of the system is the minimal possible (equal to $n + m + (n - k)$) when the relative degrees μ_1 and μ_2 are such that $\mu_1 + \mu_2 = k$, and, in that case, they actually form a pair of unobservability quasi-indices $\rho_i = \mu_i$.

Proof of ($\mathcal{U}2$). Suppose that $f(x_0) \neq 0$ and bring Σ into the observed-unobserved form (4)-(5). Since $f(x_0) \neq 0$, it follows that there exist local z-coordinates (that we continue to denote by z) completing w to a coordinate system in which the drift b of the unobserved subsystem is rectified, i.e., $b = \frac{\partial}{\partial z^{n-k}}$, and the unobserved subsystem (5) takes the form (6). Keeping in mind that the components of a flat output and their successive time-derivatives are independent, we need to connect the observed subsystem and the unobserved one. To this end, we can introduce a control vector field g_1 of the form $g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + g_2^{\rho_2}(w, z)\frac{\partial}{\partial w_2^{\rho_2}} + \sum_{q=1}^{n-k} g^q(w, z)\frac{\partial}{\partial z^q}$, where $g_2^{\rho_2}(w,z)$ has to depend explicitly on z. For the components of g_1 we will take the simplest possible choices: hence we put $g_2^{\rho_2} = z^1$. Again, at each differentiation, we have to be able to express a new z-state and since the drift of the z-part is constant, to compute z^{q+1} from $\dot{z}^q$, we necessarily have to use the control vector field g_1 (through its component g^q). Therefore, we put $g^q = z^{q+1}$. With that construction of g_1, the remaining control necessarily appears at the last differentiation of h_2, which gives the vector field $g_2 = \frac{\partial}{\partial z^{n-k}}$. The corresponding control system is in the form $NF1$, with $(h_1, h_2) = (w_1^1, w_2^1)$, which is clearly x-flat of differential weight $n+m+(n-k)$ with (h_1,h_2) being a minimal flat output.

Proof of (U3). Consider the dynamical system (Σ, h) around x_0 (which can be an equilibrium or not) and transform it locally into the observed-unobserved form (4)-(5). Consider any function $\psi_1^1(w,z)$ such that $\mathrm{d}\psi_1^1(x_0) \neq 0 \bmod \mathrm{span}\,\{\mathrm{d}w_i^j, 1 \le j \le \rho_i, 1 \le i \le 2\}$ (the function ψ_1^1 always exists, since $\dim z \geq 1$). Two cases have to be distinguished: the first case corresponds to the fact that the pair (b, ψ_1^1) is observable modulo $\mathcal{H}(x) = \mathrm{span}\,\{\mathrm{d}w_i^j, 1 \le j \le \rho_i, 1 \le i \le 2\}$, where b is the drift of the unobservable part, and, the second one corresponds to non-observable (b, ψ_1^1).

First, assume that (b, ψ_1^1) is observable. We denote by μ_1 the observability quasi-index of ψ_1^1 modulo $\mathcal{H}(x)$ (i.e., $\mu_1 = n-k$ and $(\mathrm{d}\psi_1^1 \wedge \ldots \wedge \mathrm{d}L_f^{\mu_1-1}\psi_1^1)(x_0) \neq 0 \bmod \mathcal{H}(x_0)$). Introduce new coordinates (that we continue to denote by z) $z_1^q = L_f^{q-1}\psi_1^1$, $1 \le q \le \mu_1$. The unobserved subsystem becomes: $\dot{z}_1^q = z_1^{q+1}$, $1 \le q \le \mu_1 - 1$, $\dot{z}_1^{\mu_1} = b_1(w,z)$. Now, in order to construct a flat control system, we have to link the observed (with respect to the initial output h) w-subsystem and the unobserved one. We thus put $g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + z_1^1 \frac{\partial}{\partial w_2^{\rho_2}}$ and $g_2 = \frac{\partial}{\partial z_1^{\mu_1}}$. The corresponding control system is in the form $NF2$, with $\ell = 1$ and $(h_1,h_2) = (w_1^1, w_2^1)$, which is clearly x-flat with (h_1,h_2) being a minimal flat output of differential weight $n+m+\mu_1 = n+m+(n-k)$.

Let us now consider the case when (b, ψ_1^1) is non-observable. Compute the unobservability quasi-index μ_1 (modulo $\mathcal{H}(x)$) of the function ψ_1^1 and introduce, as above, the new coordinates $z_1^q = L_f^{q-1}\psi_1^1$, and $1 \le q \le \mu_1$. The first μ_1 equations of (5) become: $\dot{z}_1^q = z_1^{q+1}$, $1 \le q \le \mu_1 - 1$, $\dot{z_1}^{\mu_1} = b_1(w, \bar{z}_1)$, where $\bar{z}_i = (z_1^1, \ldots, z_1^{\mu_1}, \ldots, z_i^1, \ldots, z_i^{\mu_1})$. Now, take any function ψ_2^1 verifying $\mathrm{d}\psi_2^1(x_0) \neq 0 \bmod \mathcal{H}(x) + \mathrm{span}\,\{\mathrm{d}z_1^q, 1 \le q \le \mu_1\}$, compute the corresponding unobservability (or observability) quasi-index μ_2 (observability being now considered modulo $\mathcal{H}(x) + \mathrm{span}\,\{\mathrm{d}z_1^q, 1 \le q \le \mu_1\}$), and introduce $z_2^q = L_f^{q-1}\psi_2^1$, $1 \le q \le \mu_2$, to get $\dot{z}_2^q = z_2^{q+1}$, $1 \le q \le \mu_2 - 1$, $\dot{z_2}^{\mu_2} = b_2(w, \bar{z}_2)$. Repeat this process until all states are observed and denote by ℓ the number of functions ψ_i^1 needed to achieve observability. At this point the unobserved subsystem (5) is transformed into (7). As for the above case, in order to construct a flat control system, we have to link all subsystems. In order to go from a subsystem to anotherthe next one in a simplest possible way, we use only one control and at each step we are able to express a new z-state. In this respect, we put $g_1 = \frac{\partial}{\partial w_1^{\rho_1}} + z_1^1 \frac{\partial}{\partial w_2^{\rho_2}} + \sum_{i=1}^{\ell-1} z_{i+1}^1 \frac{\partial}{\partial z_i^{\mu_i}}$ and $g_2 = \frac{\partial}{\partial z_\ell^{\mu_\ell}}$, and obtain the normal form $NF2$, with $(h_1,h_2) = (w_1^1, w_2^1)$, which is clearly x-flat with (h_1,h_2) being a minimal flat output of differential weight $n+m+(n-k)$. □

B. Proof of Theorem 3

The proof follows exactly the same arguments as those of item $(\mathcal{U}3)$ of Theorem 2 in the case (b, ψ_1^1) observable. □

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