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The Moment problem for the real polynomial algebra in infinitely many variables

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The univariate moment problem is an old problem with origins tracing back to work of Stieltjes. Given a sequence $(s_k)_{k\geq 0}$ of real numbers one wants to know when there exists a Radon measure μ on $\mathbb{R}$ such that

$$s_k = \int x^k d\mu \;\forall\; k \geq 0.^2$$

Since the monomials x^k, $k \geq 0$ form a basis for the polynomial algebra $\mathbb{R}[x]$, this problem is equivalent to the following one: Given a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, when does there exist a Radon measure μ on $\mathbb{R}$ such that $L(f) = \int f d\mu \;\forall\; f \in \mathbb{R}[x]$. One also wants to know to what extent the measure is unique, assuming it exists. Akhiezer 1965 and Shohat-Tamarkin 1943 are standard references.

The multivariate moment problem has been considered more recently. For $n \geq 1$, $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables $x_1, \ldots, x_n$. Given a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ and a closed subset Y of $\mathbb{R}^n$ one wants to know when there exists a Radon measure μ on $\mathbb{R}^n$ supported on Y such that $L(f) = \int f d\mu \;\forall\; f \in \mathbb{R}[\underline{x}]$. Haviland, 1936 proved that such a measure exists if and only if $L(\mathrm{Pos}(Y)) \subseteq [0, \infty)$, where $\mathrm{Pos}(Y) := \{f \in \mathbb{R}[\underline{x}] : f(x) \geq 0 \quad \forall x \in Y\}$.

Again, one also wants to know to what extent the measure is unique, assuming it exists. Berg 1987, Fuglede 1983 are general references. A major motivation here is the close connection between the multivariate moment problem and real algebraic geometry; as revealed in Schmüdgen 1991, see Marshall 2008 for an excellent exposition.

The infinite-variate moment problem is dealing with the moment problem in infinitely many variables, mainly where the linear functional in question is continuous for a certain topology. Albeverio-Herzberg 2008 apply the Positivstellensatz to represent L^1-continuous linear functionals on the vector space of polynomials of Brownian motion as integration with respect to probability measures on the Wiener space of $\mathbb{R}$. Berezansky-Kondratiev 1995, Berezansky-Sifrin 1971, Borchers-Yngvason 1975, Hegerfeldt 1975, Infusino-Kuna-Rota 2014, consider linear functionals on the symmetric algebra of a nuclear space. Ghasemi-Kuhlmann 2013, Lasserre 2013, Ghasemi-Kuhlmann-Marshall 2014 study continuous linear functionals on topological real algebras in general. This is applied in Ghasemi-Infusino-Kuhlmann-Marshall 2018 to linear functionals on the symmetric algebra of a locally convex space (V, τ) which are continuous with respect to the finest locally multiplicatively convex topology extending τ.

In this talk, based on Ghasemi-Kuhlmann-Marshall 2016, we deal with the general case systematically, without any continuity assumptions. The main new concept that we introduce and develop to this end is that of a *constructibly Radon measure* on the (infinite dimensional) real vector space $\mathbb{R}^\Omega$. After that, due to time constraints, we will focus on presenting the following (infinite dimensional) generalization of Haviland's theorem:

Theorem 1: Let $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, the ring of polynomials in an arbitrary number of variables x_i, $i \in \Omega$ with coefficients in $\mathbb{R}$. Suppose $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$ and is described by countably many inequalities (i.e., there exists a countable $S \subset A_\Omega$ such that $Y = \{\alpha \in \mathbb{R}^\Omega \mid \hat{g}(\alpha) \geq 0 \;\forall\; g \in S\}$.) Then there exists a constructibly Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L(f) = \int \hat{f} d\nu \;\forall\; f \in A_\Omega$.

We note that the condition imposed on Y is always satisfied for countable Ω and we deduce the following version of Haviland in the countable dimensional case:

Corollary 2: Suppose Ω is countable, $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$. Then there exists a Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L(f) = \int \hat{f} d\nu \;\forall\; f \in A_\Omega$.

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[2] All Radon measures considered are assumed to be positive.

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