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Source estimation for first order time-varying hyperbolic systems

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***Abstract*— This paper proposes a new method for the estimation of an in-domain source of a first order time-varying hyperbolic system. This system description is used to model the heat transfer in solar collectors where the source depends on the solar irradiance intensity and the collector's properties. The proposed method is based on the modulating functions approach. With this, the estimation problem is transformed into an algebraic system of equations. A detailed derivation of the resulting source estimation equations is given. Necessary conditions on the modulating functions for the solvability of this system of equations are also provided. Different test cases are presented to assess the performance of the proposed method.**

I. INTRODUCTION

Source estimation for Distributed Parameters Systems (DPS) is a very important topic in various application fields. An example of applications that motivated this study are thermal solar energy systems. In particular, we are interested in the distributed parabolic-trough solar collector fields, which have attracted a lot of interest from both scientific and industrial parts. For an efficient operation of the solar collectors, a point of interest is to provide effective control strategies to track a desired outlet temperature by tuning the fluid flow rate [1] under varying environmental disturbances. These disturbances include the solar irradiance and the optical efficiency of the mirrors. On one hand, the solar irradiance can be locally measured by pyrheliometers, but this may not be very useful for large plants and may lead to inefficient controllers [2]. On the other hand, the mirrors' cleanness is usually inhomogeneous and subject to environmental changes. Therefore, it is important to estimate the efficient value of the energy source that includes both the solar irradiance and the mirrors efficiency. These estimates will feed the collector's controller.

The heat transfer in distributed solar collectors has been modeled by a first order hyperbolic Partial Differential Equations (PDE) whose in-domain source term includes the received solar irradiance and the mirrors efficiency parameters [1]. There has been a big interest in developing effective estimation methods for different types of PDEs' sources. Various methods have been proposed in the literature which can be classified into optimization methods and recursive methods such as observers [3]. However, these approaches are usually computationally heavy and might not be efficient for online estimation in an adaptive control strategy where the estimation of the source during the system's operation is crucial.

Recently, an algebraic approach based on the modulating functions has been proposed for the online estimation of time-varying sources for hyperbolic systems [4]. Modulating functions have been introduced in the fifties for parameter estimation of differential equations [5] and have been extended to parameter and source estimation for PDE in [6] and more recently in [7]. This approach allows to transform the estimation problem into an algebraic system of equations. It also reduces the computational burden and allows for a non asymptotic estimation of the source term. Moreover, it does not require the knowledge of the PDE's initial conditions and avoids the computation of the measurement's derivatives, which are usually noisy. In [4], polynomial modulating functions have been proposed to estimate the source of the first order hyperbolic equation where a sliding window has been used to allow the estimation of time-varying sources. However, in addition to the outlet temperature data, this approach requires measuring the space derivative of the temperature at the outlet which is not feasible in practice.

To avoid the use of extra measurements and to ensure better performance of the modulating functions method, this paper proposes an effective method for the estimation of a time-varying source term for a first order hyperbolic system. The approach uses signal model generated modulating functions that are solutions of a trajectory planning problem for a signal model. This is inspired by [8], [9], where a similar approach is proposed for fault diagnosis for parabolic systems.

In the next section the estimation problem is formulated. Subsequently in Section III the modulating function operator is applied, and the PDE is mapped to algebraic equations, who's solvability is discussed. In Section IV, the modulating functions are derived and summed. Numerical examples are presented for both, synthetic and realistic data in Section V. Finally a conclusion summarizes the obtained results.

II. PROBLEM FORMULATION

Considering a first order bilinear hyperbolic system

$$\partial_t w(z,t) = -au(t)\partial_z w(z,t) + es(t), \tag{1a}$$
$$w(0,t) = d(t), \qquad t > 0 \tag{1b}$$
$$y(t) = w(L,t), \qquad t \geq 0 \tag{1c}$$
$$w(z,0) = w_0(z), \qquad z \in \Omega \tag{1d}$$

with $(z,t) \in \Omega \times \mathbb{R}^+, \Omega = (0,L]$ and the initial profile $w_0(z) \in \mathbb{R}$. Herein, $w(z,t) \in C^1(\Omega \times \mathbb{R}^+)$ is the system

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variable at the position z and at the time t. The parameter $u \in C^1(\mathbb{R}^+)$ is time-varying and bounded by

$$0 < u_{min} \leq u(t) \leq u_{max}. \tag{2}$$

The boundary input $d(t) \in \mathbb{R}$ and the boundary measurement $y(t) \in \mathbb{R}$ are assumed to be known. The source term $s(t) \in \mathbb{R}$ is unknown and is expected to be described piecewise by a Taylor polynomial

$$s(t) = \vartheta_{t^*}^{\mathsf{T}} \varphi_{t^*}(t), \quad t \in \mathrm{I}^* = [t^* - T, t^*] \tag{3}$$

of order $K-1$ with the evaluation point t^* and the monomials

$$\varphi_{t^*}(t) = \begin{bmatrix} 1 & t - t^* & \dots & (t-t^*)^{K-1} \end{bmatrix}. \tag{4}$$

The unknown coefficients are given by the vector $\vartheta_{t^*} = \left[\vartheta_{t^*}^k\right] \in \mathbb{R}^K$. The constant model parameters $a, e \in \mathbb{R}$ are assumed to be known.

The subject of this contribution is the reconstruction of the unknown source term s. Therefor, the signal model based modulating functions method is applied to reconstruct the polynomial coefficients. This approach leads to an algebraic expression that is directly implementable. For this, only the known system variables u, y and d are required.

III. Modulating functions method

The proposed approach for the reconstruction of the unknown source term is based on the results for the detection of time-varying faults for parabolic systems presented in [9]. According to the literature, a function $m(z,t) \in \mathbb{R}$, $(z,t) \in (\Omega \times \mathrm{I}_0)$, for the horizon $\mathrm{I}_0 = [0, T]$ with

$$m(z,t)|_{t \in \{0,T\}} = 0, \tag{5}$$

is called a modulating function. Considering the basic steps for the modulating functions method, the functional

$$\mathcal{M}[h](t) = \int_0^T \int_0^L h(z, \tau + t - T) m(z,\tau) \,\mathrm{d}z \,\mathrm{d}\tau \tag{6}$$

for $t > T$ with the fixed interval I_0 is introduced. For convenience, the abbreviation

$$\mathcal{M}[h](t) = \langle h, m \rangle_{\Omega, \mathrm{I}_0} \tag{7}$$

for (6) and the notation, $\langle \cdot, \cdot \rangle_{\mathrm{I}_0}$ for the integration w. r. t. time and $\langle \cdot, \cdot \rangle_{\Omega}$ for the integration w. r. t. space are used. In contrast to [4], the modulating function is a function w. r. t. time and location. Thus, both boundary measurements (1b) and (1c) can be taken into account. With that, all known system variables contribute to the reconstruction of the unknown coefficients ϑ_t^* and no further measurements are required for the source estimation.

Applying (7) to the PDE (1a) leads to

$$\langle \partial_\tau w, m \rangle_{\Omega, \mathrm{I}_0} = -\langle au\partial_z w, m \rangle_{\Omega, \mathrm{I}_0} + \langle es, m \rangle_{\Omega, \mathrm{I}_0}. \tag{8}$$

Thereby, $\partial_t w(z, \tau+t-T) = \partial_\tau w(z, \tau+t-T)$ is considered in the left hand side to allow the use of integration by parts. Following the modulating functions method, the derivatives are transferred to the modulating function utilizing integration by parts. Applying it w. r. t. time, the left-hand side in (8) reads as

$$\langle \partial_\tau w, m \rangle_{\Omega, \mathrm{I}_0} = [\langle w, m \rangle_\Omega]_0^T - \langle w, \partial_\tau m \rangle_{\Omega, \mathrm{I}_0}. \tag{9}$$

Considering the compact support of the modulating function (5), the initial and end values

$$\begin{aligned} &[\langle w(\tau + t - T), m(\tau) \rangle_\Omega]_0^T \\ &\quad = \langle w(t), m(T) \rangle_\Omega - \langle w(t-T), m(0) \rangle_\Omega \end{aligned} \tag{10}$$

vanish and (9) simplifies to

$$\langle \partial_\tau w, m \rangle_{\Omega, \mathrm{I}_0} = -\langle w, \partial_\tau m \rangle_{\Omega, \mathrm{I}_0}. \tag{11}$$

Applying integration by parts w. r. t. location on the right-hand side in (8),

$$\begin{aligned} &\langle au\partial_z w, m \rangle_{\Omega, \mathrm{I}_0} \\ &\quad = [\langle auw, \partial_z m \rangle_{\mathrm{I}_0}]_0^L - \langle auw, \partial_z m \rangle_{\Omega, \mathrm{I}_0} \end{aligned} \tag{12}$$

follows. Considering the boundary condition (1b) and measurement (1c) in the boundary value terms $[\langle auw, \partial_z m \rangle_{\mathrm{I}_0}]_0^L$ of (12), the expression

$$\begin{aligned} &[\langle auw(z), \partial_z m(z) \rangle_{\mathrm{I}_0}]_0^L \\ &\quad = \langle auy, m(L) \rangle_{\mathrm{I}_0} - \langle aud, m(0) \rangle_{\mathrm{I}_0} \end{aligned} \tag{13}$$

is obtained. In this, all the system variables are known. Thus, only the unknown distributed variable w in (11) and (12) has to be eliminated. For this, (9)–(13) is utilized in (8), yielding

$$\begin{aligned} -\langle w, \partial_\tau m \rangle_{\Omega, \mathrm{I}_0} = &- \langle auy, m(L) \rangle_{\mathrm{I}_0} + \langle aud, m(0) \rangle_{\mathrm{I}_0} \\ &+ \langle auw, \partial_z m \rangle_{\Omega, \mathrm{I}_0} + \langle es, m \rangle_{\Omega, \mathrm{I}_0}. \end{aligned} \tag{14}$$

To factorize the terms dependent on the distributed system variable w, the auxiliary variable $u(\tau + t - T) = \bar{u}(\tau, t)$ is introduced. Thereby, the third term in (14) changes to

$$\langle auw, \partial_z m \rangle_{\Omega, \mathrm{I}_0} = \langle w, a\bar{u}\partial_z m \rangle_{\Omega, \mathrm{I}_0}. \tag{15}$$

Considering this in (14), the unknown w can be factorized to

$$\begin{aligned} \langle es, m \rangle_{\Omega, \mathrm{I}_0} = &\langle auy, m(L) \rangle_{\mathrm{I}_0} - \langle aud, m(0) \rangle_{\mathrm{I}_0} \\ &- \langle w, \partial_\tau m + a\bar{u}\partial_z m \rangle_{\Omega, \mathrm{I}_0}. \end{aligned} \tag{16}$$

Based on this, the unknown system variable w is eliminated by the requirement

$$\partial_\tau m(z,\tau) + a\bar{u}(\tau, t)\partial_z m(z,\tau) = 0 \tag{17}$$

with $(z,\tau) \in \Omega \times \mathrm{I}_0$. Thus, (16) simplifies to

$$\langle es, m \rangle_{\Omega, \mathrm{I}_0} = \langle auy, m(L) \rangle_{\mathrm{I}_0} - \langle aud, m(0) \rangle_{\mathrm{I}_0}. \tag{18}$$

This expression only depends on known system variables, except the wanted source term $s(t)$.

A. Source estimation equations

The unknown source term is estimated on the basis of its polynomial form (3). A system of equations is derived that allows the reconstruction of the polynomials coefficients ϑ_{t^*}. Before this, a simple choice for the polynomial's evaluation point is discussed. Regarding (6) and considering (3), the left hand side in (18) reads as

$$\begin{aligned}&\langle es, m\rangle_{\Omega,\mathrm{I}_0}\\&= \int_0^T \int_0^L e\vartheta_{t^*}^{\mathsf{T}}[(\tau+t-T-t^*)^{k-1}]m(z,\tau)\,\mathrm{d}z\,\mathrm{d}\tau.\end{aligned} \tag{19}$$

Choosing $t = t^*$ eliminates the dependence of the time t in (19), thus it is chosen for the following derivations. For $t \in \mathrm{I}^*$, the polynomial coefficients are constant and can be written outside of the integration. With this, the left hand side in (18) is

$$\langle es, m\rangle_{\Omega,\mathrm{I}_0} = \langle e\varphi_t^{\mathsf{T}}, m\rangle_{\Omega,\mathrm{I}_0}\vartheta_t. \tag{20}$$

For the clarity of the further steps, the abbreviation

$$\langle \bar{y}, \bar{m}\rangle_{\mathrm{I}_0} = \langle auy, m(L)\rangle_{\mathrm{I}_0} - \langle aud, m(0)\rangle_{\mathrm{I}_0} \tag{21}$$

with $\bar{y}^{\mathsf{T}}(t) = \begin{bmatrix}au(t)y(t) & au(t)d(t)\end{bmatrix}$ and $\bar{m}^{\mathsf{T}}(t) = \begin{bmatrix}m(L,t) & m(0,t)\end{bmatrix}$ is introduced. Using (20) and (21) in (18), the first reconstruction equation

$$\langle e\varphi_t^{\mathsf{T}}, m\rangle_{\Omega,\mathrm{I}_0}\vartheta_t = \langle \bar{y}, \bar{m}\rangle_{\mathrm{I}_0} \tag{22}$$

is obtained. However, for K unknown coefficients further $K-1$ equations are needed. These are obtained by choosing K modulating functions m_k, $k \in \mathbb{K} = \{1, 2, \ldots, K\}$ that hold the requirements (5) and (17). Combining this to the system of equations

$$M\vartheta_t = \left[\langle \bar{y}, \bar{m}_k\rangle_{\mathrm{I}_0}\right] \tag{23}$$

with

$$M = \left[\langle e\varphi_t, m_k\rangle_{\Omega,\mathrm{I}_0}\right], \tag{24}$$

an expression for the reconstruction of the polynomial's coefficients ϑ_t is achieved. In this paper, the notation

$$[\langle \bar{y}, \bar{m}_k\rangle_{\mathrm{I}_0}] = \begin{bmatrix}\langle \bar{y}, \bar{m}_1\rangle_{\mathrm{I}_0}\\ \langle \bar{y}, \bar{m}_2\rangle_{\mathrm{I}_0}\\ \vdots \\ \langle \bar{y}, \bar{m}_K\rangle_{\mathrm{I}_0}\end{bmatrix} \tag{25}$$

and

$$\begin{aligned}&\left[\langle e\varphi_t, m_k\rangle_{\Omega,\mathrm{I}_0}\right]\\&= \begin{bmatrix}\langle e\varphi_t^1, m_1\rangle_{\Omega,\mathrm{I}_0} & \ldots & \langle e\varphi_t^K, m_1\rangle_{\Omega,\mathrm{I}_0}\\ \vdots & \ddots & \vdots\\ \langle e\varphi_t^1, m_K\rangle_{\Omega,\mathrm{I}_0} & \ldots & \langle e\varphi_t^K, m_K\rangle_{\Omega,\mathrm{I}_0}\end{bmatrix}\end{aligned} \tag{26}$$

is used, so that $M \in \mathbb{R}^{K\times K}$.

B. Brief discussion of solvability

To use (23) for the reconstruction of the coefficients its solvability has to be ensured. Thus, the further requirement

$$\det M \neq 0 \tag{27}$$

for the modulating functions m_k has to hold. In accordance with [7], a necessary condition for the solvability of (23) is the linear independence of the modulating functions m_k. A sufficient and necessary condition to hold (27) is the linear independence of the rows of M, i. e.,

$$\sum_{k=1}^{K} c_k[\langle e\varphi_t, m_k\rangle_{\Omega,\mathrm{I}_0}] \neq 0. \tag{28}$$

has to hold for any set $c_k \in \mathbb{R}$, $k \in \mathbb{K}$, except $[c_k] = 0$. Commuting the order of the summation and integration in (28),

$$[\langle e\varphi_t, \sum_{k=1}^{K} c_k m_k\rangle_{\Omega,\mathrm{I}_0}] \neq 0 \tag{29}$$

follows. As this can not hold if the modulating functions are linear dependent on $(z,t) \in \Omega \times \mathrm{I}_0$, i. e.,

$$\sum_{k=1}^{K} c_k m_k(z,t) = 0, \quad (z,t) \in \Omega \times \mathrm{I}_0, \tag{30}$$

the linear independence of the modulating functions m_k is a necessary condition. Furthermore, for (27) also the columns of M have to be linear independent. This can be seen by following the approach for the necessary condition for linear independent rows for the linear independence of the columns of M. As the basis functions φ_t^k are monomials of different order $k-1$, this directly holds.

With this, only a necessary but not a sufficient condition is given. So far, this is only known for two special cases yet. For the particular case of a constant source term, i. e., $K = 1$, $\varphi_t^1 = 1$, the necessary and sufficient condition for the source term estimation is

$$\langle e, m_1\rangle_{\Omega,\mathrm{I}_0} \neq 0. \tag{31}$$

This can be easily ensured by an appropriate choice of the modulating function. For the second particular case with time-varying source terms of polynomial type but constant transportation speed u, the approach proposed in [9] for the fault diagnosis of time-varying faults can directly be transferred to the source term estimation and guarantees the solvability of (23). However, for time-varying source term and time-varying transportation speed, a sufficient condition still has to be found and will be regarded in future work.

IV. Signal model

In the literature, several types of modulating functions are used. A brief review can be found in [10]. All of these approaches have modulating functions based on a certain function type. In contrast to this, the modulating functions in this contribution are based on a signal model.

According to (17), a modulating function m_k in (23) has to be the solution of a transport equation with time-varying transportation speed. To obtain a well-posed DPS for each modulating function m_k the boundary condition

$$m_k(0,\tau) = n_k(\tau), \quad \tau \in \mathrm{I}_0 \tag{32}$$

with a given function $n_k(\tau) \in \mathbb{R}$ as input signal is added. With (5), an initial condition is given. The resulting system (5), (17), and (32) is called the signal model. Furthermore, from (5) also an end condition for $\tau = T$ is obtained. Thus, the determination of the modulating function is considered as the trajectory planning problem to find an input n_k, so that

$$m_k(z,\tau) \not\equiv 0, \quad (z,\tau) \in \Omega \times \mathrm{I}_0 \tag{33a}$$

$$m_k(z,0) = 0 \rightarrow m_k(z,T) = 0 \tag{33b}$$

with $k \in \mathbb{K}$ holds for the signal model. Furthermore, the additional requirement (27) has to be regarded.

A. Transformation of the signal model

Although, an analytic solution for the bilinear signal model (5), (17), and (32) is known, see e. g. [3], the determination of the modulating functions is simplified by the time transformation

$$\xi(\tau,t) = \int_0^\tau a\bar{u}(\zeta,t)\,\mathrm{d}\zeta, \tag{34}$$

as proposed in [11]. Because of the time dependency of $\bar{u}(\tau,t)$, also the transformation (34) is time-varying in t. With $\tilde{\bar{u}}(\xi,t) = \bar{u}(\tau,t)$ and according to [11], the inverse transformation

$$\tau(\xi) = \int_0^\xi \frac{1}{a\tilde{\bar{u}}(\zeta,t)}\,\mathrm{d}\zeta \tag{35}$$

exists, due to the positive and bounded u, see (2). Applying (34) to the bilinear signal model and introducing $\tilde{m}_k(z,\xi) = m_k(z,\tau)$ and $\tilde{n}_k(\xi) = n_k(\tau)$ the linear time-invariant DPS

$$\partial_\xi \tilde{m}_k(z,\xi) = -\partial_z \tilde{m}_k(z,\xi), \qquad (z,\xi) \in \Omega \tag{36a}$$

$$\tilde{m}_k(0,\xi) = \tilde{n}_k(\xi), \qquad \xi > 0 \tag{36b}$$

$$\tilde{m}_k(z,0) = 0, \qquad z \in \Omega \tag{36c}$$

on the reconstruction time domain $\mathrm{J}_0 \in (0,\Xi)$, $\Xi = \xi(T,t)$ with the end condition

$$\tilde{m}_k(z,\Xi) = 0, \qquad z \in \Omega \tag{36d}$$

is obtained. With the transformation (34), also the reconstruction equations can be written in the time ξ. For this the transformed variables $\tilde{y}(\xi+\zeta-\Xi) = y(\tau+t-T)$, $\tilde{d}(\xi+\zeta-\Xi) = d(\tau+t-T)$ are introduced. Substituting (34) into (23), the reconstruction equation reads as

$$\tilde{M}\vartheta_t = \Big[\langle a\tilde{y}, \tilde{m}_k(L)\rangle_{\mathrm{J}_0} - \langle a\tilde{d}, \tilde{m}_k(0)\rangle_{\mathrm{J}_0}\Big] \tag{37}$$

with

$$\tilde{M} = \left[\int_0^\Xi \int_0^L e\tilde{\varphi}_t(\xi)\frac{\tilde{m}_k(z,\xi)}{\tilde{\bar{u}}(\xi,t)}\,\mathrm{d}z\,\mathrm{d}\xi\right] \tag{38}$$

and $\tilde{\varphi}_t(\xi) = \varphi_t(\tau)$. Contrary to the reconstruction equation in time t, the right hand side in (37) is not directly dependent of the time-varying transportation speed u, but the reconstruction matrix $\tilde{M}$ is directly dependent on it. Choosing a reconstruction interval J_0 with constant length Ξ leads to a time-varying length $T(t)$ for the reconstruction. The determination of the modulating functions in time ξ for a reconstruction interval J_0 with constant length, has the advantage that the same modulating functions, independent of the time-varying parameter u, can be used for the reconstruction of the source term. Thus, it can be computed only once a priori. The draw back is that the reconstruction interval T in time t is dependent on u and therefore time-varying. Regarding the inverse time transformation (35) the reconstruction time $T = \int_0^\Xi \frac{1}{a\tilde{\bar{u}}(\zeta,t)}\,\mathrm{d}\zeta$ can be computed.

B. Determination of the modulating functions

For the determination of the modulating functions, the analytic solution of (36)

$$m_k(z,\xi) = n_k(\xi - z) \tag{39}$$

derived from the method of characteristics is taken into account. From the initial condition (36c), $n_k(-z) = 0, z \in \Omega$ and from (36d), $n_k(\Xi - z) = 0, z \in \Omega$ follows. Hence, the input of the signal model has to satisfy

$$n_k(\xi) = \begin{cases} \phi_k(\xi) & : 0 < \xi < \Xi^* \\ 0 & : \text{otherwise} \end{cases} \tag{40}$$

with a basic variable $\phi_k(\xi) \in \mathbb{R}$ and $\Xi^* = \Xi - L > 0$. On the basis of (40), a modulating function can be determined by a suitable reference trajectory $\phi_{d,k}(\xi) \in \mathbb{R}$ for the basic variable. To achieve the required linear independence of the modulating functions, linear independent reference trajectories $\phi_{d,k}(\xi) \in \mathbb{R}$ have to be chosen. This follows directly from applying (39) and (40) in (30). After this, the necessary condition on the reference trajectory $\phi_{d,k}$

$$\sum_{k=1}^{K} c_k \phi_{d,k}(\xi) \neq 0, \quad \exists \xi \in (0,\Xi^*) \tag{41}$$

for the solvability of (23) is obtained.

From the right hand side in (37), it can be seen, that the system variables in the reconstruction equation are multiplied with the modulating function m_k, evaluated at $z = 0$ and $z = L$. Thus, only the system data for $\xi \in (0,\Xi^*)$ and $\xi \in (L, L+\Xi^*)$ are considered, as

$$m_k(0,\xi) \equiv 0, \quad \xi \notin (0,\Xi^*) \tag{42}$$

and

$$m_k(L,\xi) \equiv 0, \quad \xi \notin (L, L+\Xi^*). \tag{43}$$

Hence, the parameter Ξ^* determines the length of the system data that contribute to the reconstruction of the solar irradiance.

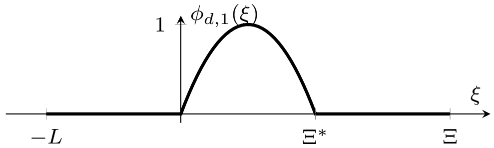

Fig. 1. Reference trajectory $\phi_{d,1}(\xi)$ using (44) with $l_1 = 1$, $\Xi^* = L$ and $c_1 = (0.5\,\Xi^*)^{-2}$ for the determination of n_1.

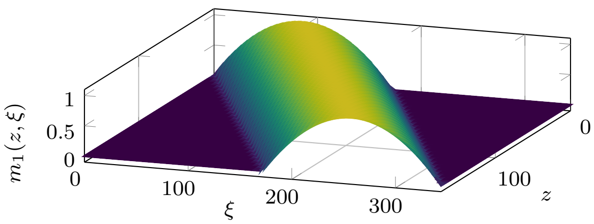

Fig. 2. Modulating function resulting from the signal model input shown in Figure 1.

C. A reference trajectory

A simple choice, for the reference trajectory is

$$\phi_{d,k}(\xi) = c_k\left((\Xi^* - \xi)\xi\right)^{l_k}, \quad \xi \in [0, \Xi^*] \tag{44}$$

with a constant gain $c_k \in \mathbb{R}$. It is based on the polynomial type modulating functions used in [4]. From this, the K linear independent reference trajectories can be constructed with

$$\phi_{d,k}(\xi) = \mathrm{d}_\xi^{k-1}\phi_{d,1}(\xi), \quad k = 2, 3, \ldots, K. \tag{45}$$

Therein $\mathrm{d}_\xi^k\phi_{d,1}(\xi)$ describes the k-th derivative of $\phi_{d,1}$ w. r. t. ξ. By choosing $\phi_{d,1}$ with $l_1 \geq K-1$, the requirement (33a) holds as $\phi_{d,k} \not\equiv 0, \forall k \in \mathbb{K}$. Because the reference trajectories $\phi_{d,k}$ are polynomials of different order $2l_1 - k + 1$, the linear independence for the K reference trajectories $\phi_{d,k}$ is directly obtained.

A signal model input n_1, planned with a reference trajectory $\phi_{d,1}$ based on (44) with $l_1 = 1$, $\Xi^* = L$ and $c_1 = (0.5\,\Xi^*)^{-2}$ is depicted in Figure 1. The resulting modulating function is shown in Figure 2. As this modulating function is non negative and the transportation speed $\tilde{\tilde{u}}$ is always positive (see (2))

$$\langle e, \frac{\tilde{m}_1}{\tilde{\tilde{u}}}\rangle_{\Omega,\mathrm{J}_0} > 0 \tag{46}$$

holds. With this, the sufficient condition (31) for the guaranteed source estimation holds independent of the transportation speed u. Thus, the modulating function shown in Figure 2 is an appropriate choice for the estimation of a constant source term.

V. Simulation results

According to [12], system (1) describes the heat transfer of a solar thermal collector. A common construction for this

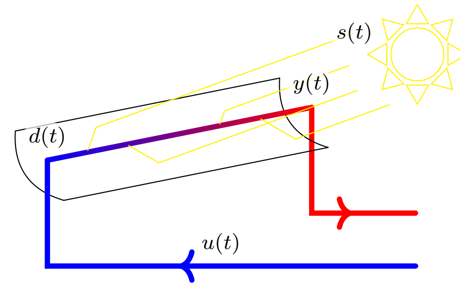

Fig. 3. Sketch of a parabolic trough as example for a solar collector.

TABLE I
Parameters for the model of the solar collector.

Parameter	Value	Unit
a	$1.6667 \cdot 10^3$	m^{-2}
e	0.0014	$\mathrm{m}^2\,\mathrm{K}\,\mathrm{W}^{-1}\,\mathrm{s}^{-1}$
u_{min}	0.001	$\mathrm{m}^3\,\mathrm{s}^{-1}$
u_{max}	0.012	$\mathrm{m}^3\,\mathrm{s}^{-1}$

is a parabolic trough as shown in Figure 3. It uses a trough-shaped parabolic reflector to concentrate the solar irradiance s, given in $\mathrm{W\,m^{-2}}$, on an insulated tube. A fluid with the flow rate u given in $\mathrm{m^3\,s^{-1}}$ flows through this tube. By absorption of the concentrated solar irradiance, the fluid is heated and transfers the thermal energy to the outlet. The inlet temperature d and the outlet temperature y are given in K. The further parameters for the example system are given in Table I and are taken from [4]. If the solar irradiance s is assumed to be slowly varying, it can be described by a piecewise polynomial (3). Thus, the presented approach is applied for its reconstruction.

A. Discretization of the hyperbolic system

For the simulation of the solar collector, the bilinear hyperbolic system is discretized in space using the first order backward finite differences scheme and in time by a first order forward finite differences scheme. For the spatial discretization a fixed step size is chosen and for the temporal discretization an adaptive step size, dependent on the fluid flow rate u and the spatial step size, is employed. For the following simulations a discretization order of $N = 5000$ for the spatial domain is chosen.

B. Constant source term

The solar collector is simulated with the inlet temperature d shown in Figure 4, the fluid flow rate u shown in Figure 5, and a constant solar irradiance $s(t) = 700\,\mathrm{W\,m^{-2}}$. The resulting output y is shown in Figure 6. The input temperature d and the fluid flow rate u are chosen as an arbitrary profile to show that they do not affect the estimation of the source.

For the reconstruction of the solar irradiance s the modulating function shown in Figure 2 is used. With (31), the solvability of the source estimation equation is ensured. In accordance with Section IV-B, also $l_1 = 0$, i. e., a

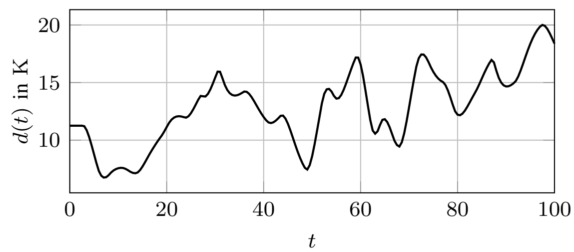

Fig. 4. Inlet temperature d for the simulation of the solar collector.

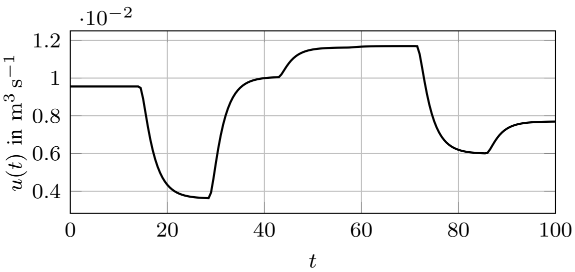

Fig. 5. Fluid flow rate u for the simulation of the solar collector.

piecewise constant modulating function, would be possible. However, the discontinuity leads to numerical problems, thus the continuous reference trajectory (44) with $l_1 = 1$ is chosen for the reconstruction. The numerical problems, occurring for discontinuous modulating functions can be traced back to the numerical integration that is implemented with the trapezoidal rule.

In Figure 7, the difference $\Delta s(t) = s(t) - \hat{s}(t)$ between the simulated solar irradiance s and the reconstructed $\hat{s}$ obtained by the proposed approach is depicted. Because of the required reconstruction time Ξ the first estimation result is obtained for $t = 28.7\,\mathrm{s}$. Notice, this value is dependent of the flow rate u. As can be seen in Figure 7, a deviation between the estimated source term $\hat{s}$ and the simulation input s is still remaining. For the visualization, that this deviation is due to numerical errors, the solar collector is simulated using different discretization orders N_k for the spatial domain. Because of the adaptive step size of the temporal grid, also the discretization in time changes

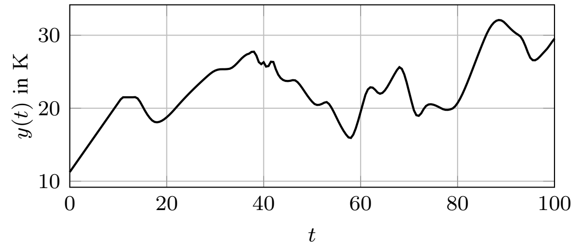

Fig. 6. Outlet temperature y for the simulation of the solar collector with the input temperature shown in Figure 4, the time-varying flow rate shown in Figure 5 and the constant solar irradiance $s = 700\,\mathrm{W\,m^{-2}}$.

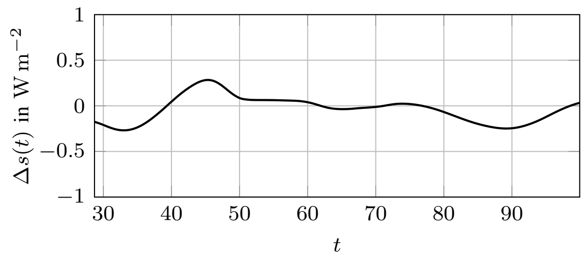

Fig. 7. Difference Δs between the constant simulated solar irradiance s and the reconstructed solar irradiance $\hat{s}$.

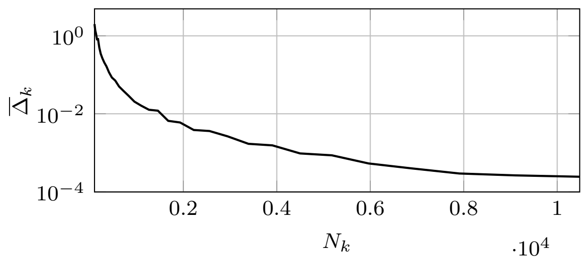

Fig. 8. Mean value $\overline{\Delta}_k$ of the reconstruction error Δs_k for different discretization orders N_k.

for different N_k. From the resulting estimation $\hat{s}_k$ for a simulation with the discretization order N_k, the estimation error $\Delta s_k(t) = s(t) - \hat{s}_k(t)$ and from this the mean value

$$\overline{\Delta}_k = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \left(\Delta s_k(\tau)\right)^2 \, \mathrm{d}\tau \tag{47}$$

is computed. At this, t_0 is the time of the first estimation and t_1 the time of the last. From the mean errors $\overline{\Delta}_k$ for different discretization orders N_k, shown in Figure 8, it can be seen that this goes to zero with higher discretization orders. Thus, the remaining error in Figure 7 is assumed to be caused by the required discretization of the system for simulation.

C. Polynomial source term

For the demonstration of the reconstruction of time-varying source terms, the solar collector is simulated with a solar irradiance s described by a second order polynomial. This is shown in Figure 9 by (⋯⋯). To obtain continuous modulating functions, a reference trajectory $\phi_{d,1}(\xi)$ with $l_1 = 3$ and $\Xi^* = L$ is planned. In Figure 9, the reconstruction result $\hat{s}$ (——) is shown. From the reconstruction error Δs, depicted in Figure 10, it can be seen, that the numerical error is higher as for the reconstruction of the constant irradiance, but still has reasonable bounds. Although, the solvability can not be ensured for time-varying u, no problems have occurred in the simulations.

D. Solar irradiance profile

To evaluate the proposed approach for the reconstruction of a real solar irradiance profile, the solar collector is simulated with the given solar irradiance s (⋯⋯), shown in

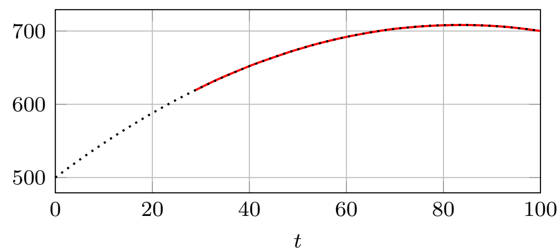

Fig. 9. Simulated polynomial solar irradiance s (······) and the reconstructed solar irradiance $\hat{s}$ (——).

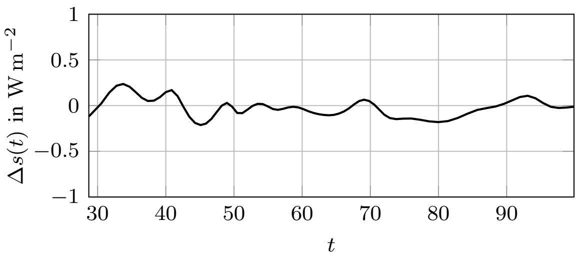

Fig. 10. Difference Δs between the simulated polynomial solar irradiance s and the reconstructed solar irradiance $\hat{s}$.

Figure 11. For its reconstruction the modulating functions are determined with a reference trajectory based on (44), with the order $l_1 = 3$ and $\Xi^* = \frac{L}{4}$. The smaller Ξ^* compared with the previously used Ξ^* is chosen to obtain a shorter reconstruction window J_0. This leads to better estimation results, as the source term has to be approximated by a second order polynomial for each reconstruction window. If the source term is not exactly of the second order polynomial type, an approximation error will occur. To reduce this error, a shorter reconstruction window J_0 is chosen. The estimated source term $\hat{s}$ (——) is depicted in Figure 11. From this it can be seen, that the profile can be reconstructed, although it is not of polynomial type. However, if the solar irradiance has fast changes the approximation of the irradiance profile by a second order polynomial leads to errors. But, as can be seen in Figure 12, the slowly varying parts of the solar irradiance can be reconstructed very precisely.

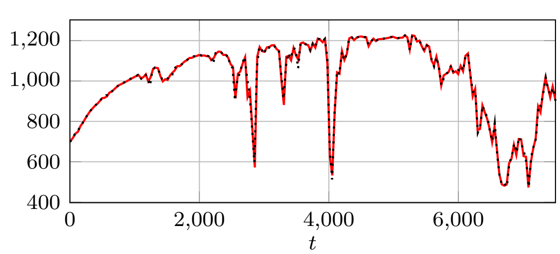

Fig. 11. Simulated experimental data for the solar irradiance s (······) and the reconstructed solar irradiance $\hat{s}$ (——).

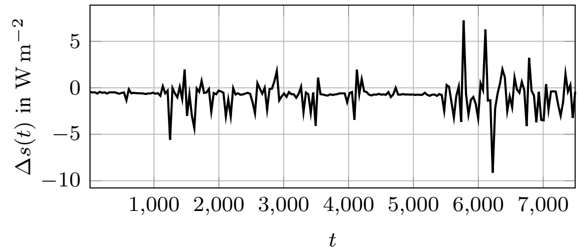

Fig. 12. Difference Δs between the simulated solar irradiance profile s, shown in Figure 11 and the reconstructed solar irradiance $\hat{s}$.

VI. Conclusion

This paper introduced an effective method for time-varying source estimation for a first order hyperbolic PDE. The approach applies the modulating function operator to map the PDE into an algebraic system of equations. The solvability of the system has been studied and the modulating functions equations have been derived.

VII. Acknowledgment

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