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L_∞-gain Analysis of Discrete-Time Positive Periodic Systems*

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Abstract— **This paper investigates the L_∞-gain of discrete-time positive periodic systems. By applying the lifting approach, the positive periodic system is transformed into a linear time-invariant positive system. Then based on the L_∞-gain characterization of a time-invariant positive systems, an equivalent condition to describe the L_∞-gain of positive periodic systems is given. Furthermore, a state-feedback periodic controller to guarantee stability and L_∞-gain of the system is obtained by solving linear inequalities. Finally, numerical examples are given to illustrate the theoretical results.**

Index Terms— **L_∞-gain; Periodic systems; Positive systems.**

I. INTRODUCTION

Positive systems, with trajectories always residing in nonnegative orthant under any nonnegative initial condition, have drawn much attention due to their nice theoretical properties [4], [7] and wide applications [1], [5]. As a special kind of positive systems, positive periodic systems are those whose system matrices are periodic. Recently, there have been some works about the stability of positive periodic systems. By applying the lifting approach to positive periodic systems, the stability of both continuous-time and discrete-time positive periodic systems is discussed in [6]. Furthermore, the control synthesis problem for positive periodic systems is solved via linear programming [2]. A necessary and sufficient condition to guarantee both the stability and positivity of the periodic systems is given.

Motivated by the above works, in this extended abstract, a method developed for analyzing the L_∞-gain performance of positive periodic systems is discussed. We characterise the L_∞-gain of a positive periodic system in terms of the L_∞-gain of a linear time-invariant system that derived by applying the lifting approach. Based on the L_∞-gain characterization of a discrete-time positive system, linear inequalities to characterize the L_∞-gain of a positive periodic system are given. Linear inequalities to stabilize the system and minimize the L_∞-gain of the system via state-feedback control is provided.

Notation: $\mathbb{R}^n$ denotes the n-dimensional real vector space, $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices, 1_n denotes an n-dimensional column vector with each entry equals 1, $\mathbb{N}=\{0,1,2,\ldots\}$. The product of n matrices $M_{j_1},M_{j_2},\ldots,M_{j_n}$ is denoted by $\prod_{j=j_1}^{j_n} M_j = M_{j_n}\times M_{j_{n-1}}\times\cdots\times M_{j_1}$. In addition, $\|v\|_\infty=\max_{i\in\{1,2,\ldots,n\}}|v_i|$ stands for the ∞-norm of a vector v, $\|\omega\|_\infty=\sup_{k\geq 0}\|\omega(k)\|_\infty$ stands for the L_∞-norm of a function ω. $v\succeq(\succ)0$ or $v\in\mathbb{R}^n_{0,+}\left(\mathbb{R}^n_+\right)$ means a real vector v is a nonnegative (positive) vector whose entries are all nonnegative (positive). $A\succeq(\succ)0$ or $A\in\mathbb{R}^{m\times n}_{0,+}\left(\mathbb{R}^{m\times n}_+\right)$ means a real matrix $A\in\mathbb{R}^{m\times n}$ is a nonnegative (positive) matrix. For two vectors v_1 and v_2, $v_1\succeq(\succ)v_2$ means v_1-v_2 is a nonnegative (positive) vector. For two matrices A and B, $A\succeq(\succ)B$ means $A-B$ is a nonnegative (positive) matrix.

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II. MAIN RESULTS

A. L_∞-gain Analysis

Considering a linear discrete-time periodic system given as

$$\begin{aligned} x(k+1) &= A(k)x(k)+B_\omega(k)\omega(k),\\ z(k) &= C(k)x(k)+D_\omega(k)\omega(k), \end{aligned} \tag{1}$$

where $x(k)\in\mathbb{R}^{n_x}$ and $z(k)\in\mathbb{R}^{n_z}$ are the state vector and output vector, respectively, and $\omega(k)\in\mathbb{R}^{n_\omega}$ is disturbance. $A(k)=A_{\sigma(k)}$, $B_\omega(k)=B_{\omega,\sigma(k)}$, $C(k)=C_{\sigma(k)}$, and $D_\omega(k)=D_{\omega,\sigma(k)}$, where σ is a cyclic permutation of $\{0,1,\ldots,T_p-1\}$. $T_p>0$ is the fundamental period.

Definition 1: A periodic system (1) is said to be a discrete-time positive periodic system if for any cyclic permutation $\sigma(k)$, initial condition $x(0)\succeq 0$, and input $\omega(k)\succeq 0$, we have $x(k)\succeq 0$ and $z(k)\succeq 0$ for all $k\in\mathbb{N}$.

In the following, some useful lemmas which will be used in the sequel are introduced.

Lemma 1: A periodic system (1) is positive if and only if A_i $B_{\omega,i}$, C_i and $D_{\omega,i}$ are nonnegative matrices for all $i\in\{0,1,\ldots,T_p-1\}$.

In the sequel, 'system (1)' means a periodic system with state matrix satisfying the condition in Lemma 1.

Lemma 2: For a discrete-time periodic system (1) the following statements are equivalent.

i) System (1) is asymptotically stable.

ii) $A_{T_p-1}\cdots A_1A_0$ is Schur.

According to [3], we use the lifting approach and transform system (1) into a time-invariant system as follows:

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}\bar{x}(k)+\bar{B}_\omega\bar{\omega}(k),\\ \bar{z}(k) &= \bar{C}\bar{x}(k)+\bar{D}_\omega\bar{\omega}(k), \end{aligned} \tag{2}$$

where $\bar{x}(k)=x(kT_p)$,

$$\bar{\omega}(k)=\begin{bmatrix}\omega(kT_p) & \omega(kT_p+1) & \cdots & \omega\left((k+1)T_p-1\right)\end{bmatrix}^T,$$

$$\bar{z}(k)=\begin{bmatrix}z(kT_p) & z(kT_p+1) & \cdots & z\left((k+1)T_p-1\right)\end{bmatrix}^T.$$

According to (1), the system matrices of linear time-invariant system (2) are given in (3). Since $\bar{\omega}(k)$ and $\bar{z}(k)$ contain all the vectors of $\omega(k)$ and $z(k)$, $\|\bar{z}\|_\infty=\|z\|_\infty$ and $\|\bar{\omega}\|_\infty=$

$$\begin{aligned}
\bar{A} &= \prod_{i=0}^{T_p-1} A_{\sigma(i)}, \\
\bar{B}_\omega &= \left[\left(\prod_{i=1}^{T_p-1} A_{\sigma(i)}\right) B_{\omega,\sigma(0)} \quad \left(\prod_{i=2}^{T_p-1} A_{\sigma(i)}\right) B_{\omega,\sigma(1)} \quad \cdots \quad B_{\omega,\sigma(T_p-1)} \right], \\
\bar{C} &= \left[C_{\sigma(0)} \quad C_{\sigma(1)} A_{\sigma(0)} \quad \cdots \quad C_{\sigma(T_p-1)} \prod_{i=0}^{T_p-2} A_{\sigma(i)} \right]^T, \\
\bar{D}_\omega &= \begin{bmatrix} D_{\omega,\sigma(0)} & 0 & \cdots & 0 \\ C_{\sigma(1)} B_{\omega,\sigma(0)} & D_{\omega,\sigma(1)} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ C_{\sigma(T_p-1)} \prod_{i=1}^{T_p-2} A_{\sigma(i)} B_{\omega,\sigma(0)} & \cdots & C_{\sigma(T_p-1)} B_{\omega,\sigma(T_p-2)} & D_{\omega,\sigma(T_p-1)} \end{bmatrix}.
\end{aligned} \tag{3}$$

$\|\omega\|_\infty$. Therefore, the L_∞-gain analysis for a positive periodic system (1) can be turned into L_∞-gain analysis for a linear time-invariant system (2). According to [4] and [10], the L_∞-gain of a discrete-time time-invariant system is equal to solving a linear programming problem. For a time-invariant positive system (2), the L_∞-gain can be exactly characterized by the following lemma.

Lemma 3: For a linear time-invariant positive system (2), the system is asymptotically stable and the L_∞-gain is less than γ if and only if there exists a vector $\lambda \in \mathbb{R}_+^{n_x}$ such that

$$\left(\bar{A} - I\right)\lambda + \bar{B}_\omega 1_{T_p n_\omega} \prec 0, \tag{4}$$

$$\bar{C}\lambda + \bar{D}_\omega 1_{T_p n_\omega} \prec \gamma 1_{T_p n_z}. \tag{5}$$

Due to the presence of the product of A_i, it will affect the design of state-feedback controller for system (1). In order to reduce the nonlinearity of the conditions in Lemma 3, an equivalent condition to characterize the asymptotic stability and L_∞-gain of system (1) is given in Theorem 1.

Theorem 1: For a positive periodic system (1), the system is asymptotically stable and the L_∞-gain is less than γ if and only if there exist vectors $\lambda_i \in \mathbb{R}_+^{n_x}$ such that

$$A_i \lambda_i + B_{\omega,i} 1_{n_\omega} \prec \lambda_{i+1}, \tag{6}$$

$$C_i \lambda_i + D_{\omega,i} 1_{n_\omega} \prec \gamma 1_{n_z}, \tag{7}$$

where $\lambda_{T_p} = \lambda_0$, for all $i = 0, 1, \ldots, T_p - 1$.

B. Controller Design

By applying the state-feedback controller $u(k) = K(k)x(k)$, the closed-loop system is given as follows:

$$\begin{aligned} x(k+1) &= (A(k) + B_u(k)K(k))\,x(k) + B_\omega \omega(k), \\ z(k) &= (C(k) + D_u(k)K(k))\,x(k) + D_\omega \omega(k), \end{aligned} \tag{8}$$

where $B_u(k) = B_{u,\sigma(k)}$, $D_u(k) = D_{u,\sigma(k)}$, and $K(k) = K_{\sigma(k)}$. For system (8), our goal is to design the periodic controller $K(k)$ such that it not only guarantees the positivity of the system but also guarantees the asymptotic stability and L_∞-gain performance of the system. According to Theorem 1, we can derive a new theorem for the closed-loop systems.

Theorem 2: For the closed-loop positive periodic system (8), the system is asymptotically stable and the L_∞-gain is less than γ if and only if there exist a set of vectors $\lambda_i \in \mathbb{R}_+^{n_x}$ and matrices $Y_i \in \mathbb{R}^{n_u \times n_x}$ such that the following inequalities hold:

$$\left(A_i \mathrm{diag}(\lambda_i) + B_{u,i} Y_i\right) 1_{n_x} + B_{\omega,i} 1_{n_\omega} \prec \lambda_{i+1}, \tag{9}$$

$$\left(C_i \mathrm{diag}(\lambda_i) + D_{u,i} Y_i\right) 1_{n_x} + D_{\omega,i} 1_{n_\omega} \prec \gamma 1_{n_z}, \tag{10}$$

$$A_i \mathrm{diag}(\lambda_i) + B_{u,i} Y_i \succeq 0, \tag{11}$$

$$C_i \mathrm{diag}(\lambda_i) + D_{u,i} Y_i \succeq 0, \tag{12}$$

where $\lambda_{T_p} = \lambda_0$, for all $i = 0, 1, \ldots, T_p - 1$, and matrix $K_i = Y_i \mathrm{diag}(\lambda_i)^{-1}$.

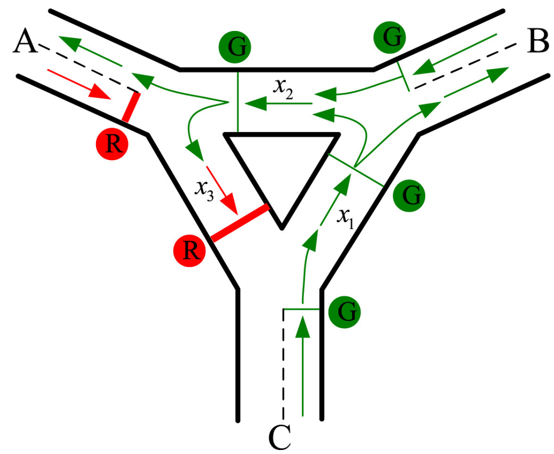

Fig. 1. Triangular intersection

III. ILLUSTRATIVE EXAMPLES

A. Example 1.

A model of traffic control of cross way [8], [9], which can be seen as a positive periodic system is introduced. A triangluar intersection governed by traffic lights with three main roads is shown in Fig. 1, where x_1, x_2 and x_3 denote the number of vechicles waiting in A, B and C lane, respectively. The traffic system in Fig. 1 can be expressed by

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_\omega(k)\omega(k), \\ z(k) &= C(k)x(k), \end{aligned} \tag{13}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T$, disturbance $\omega(k)$ denotes the number of vechicles entering the road, and output

$z(k)$ denotes the summation of the vechicles in road A, B and C. The system matrices are given as follows:

$A(k) = A_k$, $B_\omega(k) = B_{\omega,k}$ and $C(k) = C_k$, and $A_{k+3} = A_k$, $B_{\omega,k+3} = B_{\omega,k}$ and $C_{k+3} = C_k$, and

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1.1 & 1 & 0 \\ 0 & 0 & 0.3679 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 1.1 \\ 0 & 0.3679 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3679 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1.1 & 1 \end{bmatrix},$$

$$B_{\omega,1} = B_{\omega,2} = B_{\omega,3} = \begin{bmatrix} 1.1 \\ 1.1 \\ 1.1 \end{bmatrix}, \quad C_1 = C_2 = C_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Since the spectral radius of $A_2 A_1 A_0$ is 0.7327, system (13) is stable according to Lemma 2. Based on Theorem 1, the exact L_∞-gain of system (13) is 38.72.

B. Example 2.

A discrete-time positive periodic system is taken into consideration as follows:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_\omega(k)\omega(k), \\ z(k) &= C(k)x(k) + D_\omega(k)\omega(k), \end{aligned} \tag{14}$$

where $T_p = 2$, and

$$A(0) = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 1.4 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0.4 & 0 \\ 1.2 & 0.2 \end{bmatrix},$$

$$B_\omega(0) = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}, \quad B_\omega(1) = \begin{bmatrix} 1.2 \\ 0.7 \end{bmatrix},$$

$$C(0) = \begin{bmatrix} 1.2 & 1.1 \\ 0.4 & 0.6 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 2.1 & 1.4 \\ 0.8 & 1.9 \end{bmatrix},$$

$$D_\omega(0) = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}, \quad D_\omega(1) = \begin{bmatrix} 0.5 \\ 1.7 \end{bmatrix}.$$

The spectral radius of $A(1)A(0)$ is 1.0996. According to Lemma 2, we can find out that system (14) is unstable. Then a state-feedback controller is taken into consideration and the closed-loop system is denoted as follows:

$$\begin{aligned} x(k+1) &= (A(k) + B_u(k)K(k))\,x(k) + B_\omega(k)\omega(k), \\ z(k) &= (C(k) + D_u(k)K(k))\,x(k) + D_\omega(k)\omega(k), \end{aligned} \tag{15}$$

where

$$B_u(0) = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad B_u(1) = \begin{bmatrix} 1.2 \\ 1.3 \end{bmatrix},$$

$$D_u(0) = \begin{bmatrix} 0.7 \\ 0.4 \end{bmatrix}, \quad D_u(1) = \begin{bmatrix} 0.2 \\ 2.4 \end{bmatrix}.$$

Based on Theorem 2, a solution of $K(i)$ is obtained as follows:

$$K(0) = \begin{bmatrix} -0.3 & -1.4 \end{bmatrix}, \quad K(1) = \begin{bmatrix} -1/3 & 0 \end{bmatrix},$$

and the L_∞-gain of the system is 4.1488.

IV. CONCLUSIONS

In this paper, the L_∞-gain analysis and controller synthesis problem of positive periodic systems is studied. By applying the lifting approach, a positive periodic system is transformed into a linear time-invariant positive system. Then based on a necessary and sufficient condition of L_∞-gain of time-invariant positive systems, a new theorem that characterizes the L_∞-gain of positive periodic systems is given. Furthermore, the control synthesis problem is taken into consideration by introducing the state-feedback to the periodic positive system. Linear inequalities to design the periodic state-feedback controller is given. Finally, numerical examples are given to illustrate the theoretical results.

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