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Structured Linear Quadratic Control for Transportation

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***Abstract*— We study a linear quadratic control problem for a transportation problem on a directed line graph. We show that the solution to the Riccati equation associated with this problem is highly structured. The feedback law is almost triangular, and the synthesis of the feedback law is given by a recursion, making it scalable. The structure of the feedback law also allows for an efficient realization of the controller using a local communication scheme.**

I. INTRODUCTION

We study a standard infinite horizon Linear Quadratic (LQ) problem:

$$\begin{aligned} \min_u \quad & \mathbb{E}\left(\sum_{t=0}^{\infty} x[t]^T Q x[t]\right) \\ \text{Subject to} \quad & x[t+1] = \sqrt{\alpha} A x[t] + B u[t] + w, \end{aligned} \tag{1}$$

In the above, A, B, Q are compatibly dimensioned matrices, α a scalar, and w a vector of normally distributed zero mean random variable. Our main contribution, is to show that when A, B, Q have a particular structure, an optimal control u can be obtained from the formula

$$u_k = \beta_k (x_{2k+1} + x_{2k+2}) - (1 - \beta_k) \sum_{i=1}^{2k} x_i.$$

We give a closed form expression of β_k, which is iteratively calculated. Furthermore, when the system is extended to larger size, the β_k's need not be recalculated. Interestingly this means that the resulting controller is inherently structured, and exhibits a closed form solution that is easily updated as the problem shrinks or grows. Moreover, the specific problem has a natural interpretation in terms of transportation of goods under delay. In this context the control loop has a natural scalable interpretation that relies on a simple local communication scheme.

At its heart, this work is another contribution to the field of structured optimal control. Early work include studies on team game problems. In those problems a set of agents have different information and work toward a common goal. See for example [5], [2].

More recently, attempts to formalize the role of structure have been made. In [6], it is shown that subject to satisfying a quadratic constraint, the Youla parameterization inherits the structure of the control, allowing for efficient computation of optimal controllers. [3] presents a class of decentralized controllers for the LQ problems where the controller and plant satisfy the same delay and sparsity constraints. In [7], a poset-causal constraint on the controller is added to the $\mathscr{H}_2$ problem. This constraint is similar to the structure of the controller that we attain, by solving a unconstrained problem.

Examples were the structure is not imposed on the controller, but rather a consequence of the plant includes [1], where it is shown that for spatially invariant systems the optimal controller is localized in space. In [4], an optimal control problem for coordination is solved. The solution is structured, containing a diagonal part and a rank one part.

Our controller allows for a structured controller that solves the unconstrained problem. Furthermore, the controller can be efficiently calculated via a closed form iterative expression.

II. SPARSE CONTROLLER FOR A LQ PROBLEM

We study two similar optimization problems that have the same Riccati equation. We construct a solution to this Riccati equation and show that it is optimal. The first problem is a standard infinite horizon linear quadratic problem, as given in (1).

The second problem is a discounted infinite horizon LQ problem. Let the discount factor α take values $0 < \alpha < 1$. The problem is formulated as

$$\begin{aligned} \min_u \quad & \mathbb{E}\left(\sum_{t=0}^{\infty} \alpha^t x[t]^T Q x[t]\right) \\ \text{Subject to} \quad & x[t+1] = A x[t] + B u[t] + w. \end{aligned} \tag{2}$$

Motivation for this type of problems is given later. It is easy to show for both problems that the corresponding difference Riccati equation is

$$X_{k+1} = \alpha A^T X_k A - \alpha A^T X_k B (B^T X_k B)^{-1} B^T X_k A + Q.$$

Any fix-point satisfies the algebraic Riccati equation,

$$\alpha A^T X A - X + Q = \alpha A^T X B (B^T X B)^{-1} B^T X A. \tag{3}$$

Let $x \in \mathbb{R}^{2N-1}$ and $u \in \mathbb{R}^{N-1}$.We consider

$$Q = \operatorname{diag}\{q_n, 0, q_{n-1}, \ldots, q_1\}, \tag{4}$$

and the following specific class of $A = A_N, B = B_N$ matrices,

$$A_k = \begin{bmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{e}_1 & A_{k-1} \end{bmatrix}, \quad B_k = \begin{bmatrix} -1 & \mathbf{0}^T \\ 1 & \mathbf{0}^T \\ \mathbf{0} & B_{k-1} \end{bmatrix}. \tag{5}$$

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With

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

We show that,for these matrices, there exist at least one solution of (3) by explicitly constructing it. The proposed solution is highly structured.

Theorem 1: Given A, B, Q as in (4-5), recursively define γ_k as

$$\gamma_{k+1} = \alpha \frac{q_{k+1}\gamma_k}{q_{k+1}+\gamma_k}, \quad \gamma_1 = \alpha q_1.$$

Then one solution to the Riccati equation (3) is given by

$$X = \frac{1}{1-\alpha}\gamma_N \mathbf{1}\mathbf{1}^T + Q + \tilde{X}_N, \tag{6}$$

where $\tilde{X}_N$ is defined by the recursion:

$$\tilde{X}_{k+1} = \begin{bmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & \gamma_k & \gamma_k \mathbf{1}^T \\ \mathbf{0} & \gamma_k \mathbf{1} & \gamma_k \mathbf{1}\mathbf{1}^T + \tilde{X}_k \end{bmatrix}, \quad \tilde{X}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma_1 & \gamma_1 \\ 0 & \gamma_1 & \gamma_1 \end{bmatrix}.$$

Corollary 1: The corresponding feedback matrix $K_N = -(B^T X B)^{-1} B^T X A$ for X in (6) is given by

$$\begin{aligned} K_{k+1} &= \begin{bmatrix} \frac{q_{k+1}}{q_{k+1}+\gamma_k} & -\frac{\gamma_k}{q_{k+1}+\gamma_k} & -\frac{\gamma_k}{q_{k+1}+\gamma_k}\mathbf{1}^T \\ 0 & K_k \mathbf{e}_1 & K_k \end{bmatrix}, \\ K_2 &= \begin{bmatrix} \frac{q_2}{q_2+\gamma_1} & -\frac{\gamma_1}{q_2+\gamma_1} & -\frac{\gamma_1}{q_2+\gamma_1} \end{bmatrix}. \end{aligned} \tag{7}$$

Theorem 2: Take $x = [x_{2N-1}, \dots x_1]$ and $u = [u_{N-1}, \dots, u_1]$. Then the optimal u for (1) and (2), is given by

$$\begin{aligned} u_k &= \frac{q_{k+1}}{q_{k+1}+\gamma_k}(x_{2k+1}+x_{2k+2}) - \frac{\gamma_k}{q_{k+1}+\gamma_k}\sum_{i=1}^{2k} x_i, \\ u_{N-1} &= \frac{q_N}{q_N+\gamma_{N-1}} x_{2N-1} - \frac{\gamma_{N-1}}{q_N+\gamma_{N-1}}\sum_{i=1}^{2N-2} x_i. \end{aligned} \tag{8}$$

III. Graph Interpretation

Here we consider a directed line graph. The dynamics of transportation of goods with delays on such a graph can be described by the following dynamics,

$$\begin{aligned} g_k[t+1] &= g_k[t] - u_{k-1}[t] + r_k[t] \\ r_k[t+1] &= u_k[t]. \end{aligned} \tag{9}$$

Where g_k is the amount of goods in node k and r_k is the goods in transit, about to be received at node k. See Fig. 1 for an illustration of the three node case. Now, let

$$x = [g_N, r_{N-1}, g_{N-1}, r_{N-2}, \dots r_1, g_1]. \tag{10}$$

The dynamics of (9) on a line graph can then be described by $x[t+1] = Ax[t] + Bu[t]$, with A and B as in (5). If there were a decay of goods with decay rate p, we would instead take $x[t+1] = pAx[t] + Bu[t]$.

Note that the problem is not symmetric. The transportation for input k goes from node $k+1$ to node k. We say that the direction of increasing indices is upstream.

Remark 1: We do not restrict the input to be positive. Normally, sending a negative amount of goods with delay will results in the goods arriving immediately and being sent one time unit later. This is not physically possible. However, when working around a nominal flow, a negative input will correspond to sending less goods compared to the nominal flow.

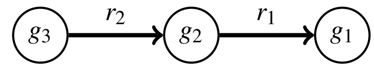

Fig. 1. Illustration of three node, delayed mass transfer. The delays are implemented using states at the links, r_i that corresponds to the mass in transit. g_k is the mass in each node.

A. Scalable Synthesis

It follows from the recursive nature of our optimal control law, that the solution for a problem of size N, can be used to construct the solution for a problem of size $N+1$. When adding a new node upstream there will not be any affect for the old nodes. The only calculations that needs to be done to implement the new feedback law is to calculate γ_N and the coefficients for the new input. Furthermore, the solution for size $N-1$ can be recovered from the solution for N. If the node furthest upstreams were to be removed, there would not be any effect on the other nodes. Hence, it is very computationally efficient to add and remove nodes upstream. In general when adding a node, only the nodes upstream of the new needs to be recalculated.

Calculating the initial feedback law for a system of N nodes is also efficient. The proposed method for solving the Riccati equation does so exactly, and its time-complexity is linear in the number of nodes.

B. Distributed Implementation

Now we revisit the feedback law in (8). We assume that node $k+1$ decides the value of u_k. In the variables in (10), u_k can be written as

$$u_k = \frac{q_k}{q_k+\gamma_{k-1}}(g_{k+1}+r_{k+1}) - \frac{\gamma_{k-1}}{q_k+\gamma_{k-1}}\sum_{i=1}^{k} g_k + r_k.$$

g_k and r_{k+1} is local information for node k. The sum $f_k = \sum_1^k g_k + x_k$, is the sum of goods downstream of node $k+1$. Thus each node need local information and the sum of goods downstream to calculate its output. This can efficiently be implemented by each node sending the sum of its own amount of goods and sum of goods downstream to the node upstream. The downside of this scheme is that node k can not send its information until it received information from node $k-1$. Thus, the latency of the communication is proportional to the number of nodes. However, the number of communication channels is also proportional to the number of nodes. If each node were to communicate with every other node, the number of communication channels are proportional to the number of nodes in square. The proposed scheme is also vulnerable to faulty communication channels as it becomes impossible to calculate the output for every node upstreams of the faulty communication channel.

IV. Application to Transportation

Consider inventory control for a set of stores. There is transportation of goods between the stores to keep optimal level of supply. We assume the topology of the stores and transportation takes the form of a directed line graph. This does not require that the stores are geographically distributed as a line.

Let the amount of goods in node k be denoted $\hat{g}_k$. There is a transportation of goods in the direction of the graph. The transportation has a delay of one time unit. Let the nodes be numbered in increasing order as we go upstream. We denote the goods in transit from node k as $\hat{r}_k$. Then the incoming goods to node k is $\hat{r}_{k-1}$. The amount of goods sent downstream in the graph by node k is denoted $\hat{u}_{k-1}$. There is also external influences $\hat{w}_k \in \mathcal{N}(\bar{w}_k, \sigma_k)$ for each node. See Fig. 2 for an illustration. The dynamics of edges and nodes are given by

$$\begin{cases} \hat{g}_k[t+1] = \hat{g}_k[t] + (\hat{r}_k[t] - \hat{u}_{k-1}[t]) + \hat{w}_k[t] \\ \hat{r}_k[t+1] = \hat{u}_k[t]. \end{cases} \tag{11}$$

Each node k have a utility function describing how much it values of having an inventory level of $\hat{g}_k$ goods,

$$U_k(\hat{g}_k) = q_k \hat{g}_k (a_k - \hat{g}_k). \tag{12}$$

The parameters q_k and a_k should both be positive. These utilities functions have the property that the benefit of having access to more goods is decreasing with the amount of goods, that is $\partial^2 U/\partial^2(\hat{g}) < 0$. Furthermore, when $\hat{g}_k > a_k/2$ we have that $\partial U/\partial(\hat{g}) < 0$. The intended working area is $0 < \hat{g} < a_k/2$.

We values higher inventory levels more the earlier we get them. Thus the following pay off function is chosen

$$\min_u \mathbb{E} \sum_{t=0}^{\infty} \alpha^t \sum_{k=1}^{N} U_k(\hat{g}[t])$$

Subject to dynamics in (11).

We assume that there is a underlying flow in the graph. Due to this the transportation is already payed for, and we do not put any penalty on the control signal u. Further, assume that the expected production and consumption are equal. The problem can be transformed to a problem of the form of (2) by controlling around the nominal flow. To do so, we must change variables so that the pay-off function is quadratic. We do this by letting $g = \hat{g} - a_k/2$. The new variable g can be interpreted as the negative demand for each node. Also, note that g is negative in the intended working area.

The input and flows will be controlled around the nominal flow $\bar{u} = \bar{r}$, so that $\hat{u} = \bar{u} + u$, $\hat{r} = \bar{r} + r$. Now we only require that $u > -\bar{u}$.

Remark 2: A positive disturbance in one node will give rise to a positive increase in all other nodes. The effect on transportation states upstream will be negative, while the effect on transportation states downstream will be positive. With proper selection of inputs and outputs, we will therefore get an externally positive system.

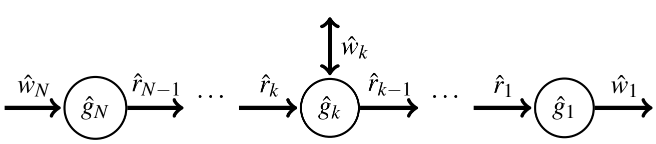

Fig. 2. Illustration of the inventory control problem. Each node k corresponds to a store with inventory level $\hat{g}_k$. Each store is affected by an external net production $\hat{w}_k$. To balance the inventory level over the stores there is transportation between the stores. $\hat{r}_k$ is the goods in transit from store $k+1$ to store k.

V. Conclusions

We have presented a recursive solution to a class of optimal control problems. This solution is easily extended as the system grows. The structure of the feedback law allows for an efficient implementation using a local communication scheme. We have showed that the optimal control problem can be used to solve an inventory control problem.

It is expected that the results presented here will generalize to tree graphs and periodic B matrices. This is subject to future work.

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