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The infinite dimensional bounded real lemma for bicausal systems*

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***Abstract*— The Bounded Real Lemma, i.e., the state-space LMI characterization (referred to as Kalman-Yakubovich-Popov or KYP inequality) of when an input/state/output linear system satisfies a dissipation inequality, has recently been studied for infinite-dimensional discrete-time systems in a number of different settings. Here we focus on the Bounded Real Lemma in the context of infinite dimensional, discrete-time, bicausal systems. The transfer functions for such systems need not be stable, and as a result, one expects the solution to the KYP inequality to be only selfadjoint, and not (strictly) positive definite as in the classical case. This indeed turns out to be the case for the bicausal systems we consider. By a variation on Willems' storage-function approach, we prove variations on the standard and strict Bounded Real Lemma, leading to a minimal and maximal solution for the KYP inequality.**

I. THE CLASSICAL BOUNDED REAL LEMMA FOR DISCRETE-TIME LINEAR SYSTEMS

Consider the discrete-time linear system

$$\Sigma := \left\{ \begin{array}{rcl} \mathbf{x}(n+1) & = & A\mathbf{x}(n) + B\mathbf{u}(n), \\ \mathbf{y}(n) & = & C\mathbf{x}(n) + D\mathbf{u}(n), \end{array} \right. \quad (n \in \mathbb{Z}) \quad \text{(I.1)}$$

where $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are bounded linear Hilbert space operators i.e., $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and the *system matrix* associated with Σ takes the form

$$M = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] : \left[\begin{array}{c} \mathcal{X} \\ \mathcal{U} \end{array} \right] \to \left[\begin{array}{c} \mathcal{X} \\ \mathcal{Y} \end{array} \right]. \quad \text{(I.2)}$$

In this case an input sequence $\mathbf{u} = (\mathbf{u}(n))_{n\in\mathbb{Z}}$, with $\mathbf{u}(n) \in \mathcal{U}$, is mapped to an output sequence $\mathbf{y} = (\mathbf{y}(n))_{n\in\mathbb{Z}}$,with $\mathbf{y}(n) \in \mathcal{Y}$ through the state sequence $\mathbf{x} = (\mathbf{x}(n))_{n\in\mathbb{Z}}$, with $\mathbf{x}(n) \in \mathcal{X}$. With the system Σ we associate the *transfer function* given by

$$F_\Sigma(z) = D + zC(I - zA)^{-1}B. \quad \text{(I.3)}$$

Since A is bounded, F_Σ is defined, and analytic, on a neighborhood of 0 in $\mathbb{C}$.

The classical Bounded Real Lemma deals with the case where all spaces are finite dimensional, and asks for a characterization of the systems Σ for which F_Σ admits an analytic continuation to the open unit disk $\mathbb{D}$ such that the supremum norm $\|F_\Sigma\|_{\infty,\mathbb{D}} = \inf_{z\in\mathbb{D}} \|F_\Sigma(z)\|$ of F_Σ over $\mathbb{D}$ is at most one ($\|F_\Sigma\|_{\infty,\mathbb{D}} \leq 1$) in the case of the *Standard Bounded Real Lemma*, or strictly less than one ($\|F_\Sigma\|_{\infty,\mathbb{D}} < 1$) in the case of the *Strict Bounded Real Lemma*. The main result in the classical setting is the following characterization in terms of solutions to the Kalman-Yakubovich-Popov or KYP inequality.

Theorem 1.1: Suppose that Σ is a discrete-time linear system as in (I.1) with $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ finite dimensional.

(1) *Standard Bounded Real Lemma:* Assume that Σ is *minimal*, i.e., the output pair (C, A) is *observable* and the input pair (A, B) is *controllable*:

$$\bigcap_{k=0}^{n} \operatorname{Ker} CA^k = \{0\}, \quad \operatorname{Span}_{k=0}^{n-1} \operatorname{Im} A^k B = \mathcal{X}. \quad \text{(I.4)}$$

Then F_Σ is analytic on $\mathbb{D}$ and $\|F_\Sigma\|_{\infty,\mathbb{D}} \leq 1$ if and only if there is a positive definite matrix H satisfying the KYP inequality

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \preceq \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}. \quad \text{(I.5)}$$

(2) *Strict Bounded Real Lemma:* Assume that A is *stable*, i.e., all eigenvalues of A are inside $\mathbb{D}$. Then F_Σ is analytic on $\mathbb{D}$ and $\|F_\Sigma\|_{\infty,\mathbb{D}} < 1$ if and only if there is a positive definite matrix H so that the strict KYP inequality holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{Y} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \prec \begin{bmatrix} H & 0 \\ 0 & I_\mathcal{U} \end{bmatrix}. \quad \text{(I.6)}$$

For the standard bounded real lemma we refer to [1], while the strict bounded real lemma can be found in [12]. In recent years there has been much attention for extensions of these results to infinite dimensional systems; we refer to [2] and [5], [6] for further details and references. See the papers [15] and [16] by Yakubovich for some early results. Other variations on the bounded real lemma exist for unbounded continuous-time linear systems [3] and non-stationary systems [9], to mention only a few.

Note that if $F = F_\Sigma$ is a transfer function of the form (I.3), then necessarily F is analytic at the origin. One approach to remove this restriction on the class of functions one can consider is to consider functions F that can be written as

$$\begin{aligned} F(z) = D + zC_+(I - zA_+)^{-1}B_+ + \\ + C_-(I - z^{-1}A_-)^{-1}B_-, \end{aligned} \quad \text{(I.7)}$$

where A_+ and A_- are stable matrices, or operators in the case of nonrational functions. All rational matrix functions without poles on the unit circle $\mathbb{T}$, can be represented in this way. Such functions appear as transfer functions of bicausal systems, which will be discussed in the next section. The objective in this case is no longer to determine when F has analytic continuation to the disc $\mathbb{D}$ with $\|F\|_{\infty,\mathbb{D}} \leq 1$ (or $\|F\|_{\infty,\mathbb{D}} < 1$ in the strict case), but rather F should be analytic on a neighborhood of the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$ and in the standard case:

$$\|F\|_{\infty,\mathbb{T}} := \sup_{z\in\mathbb{T}} \|F(z)\| \leq 1 \quad \text{(I.8)}$$

while in the strict case one requires:

$$\|F\|_{\infty,\mathbb{T}} < 1. \tag{I.9}$$

The main result presented in the sequel provides a variation on the KYP-type inequality solution criteria for the standard and strict bounded real lemma, see Theorem 4.2 below. Full details and proofs of the results presented here are given in [7].

II. Bicausal systems

A *bicausal system* Σ consists of a pair of input-state-output linear systems Σ_+ and Σ_- with Σ_+ running in forward time and Σ_- running in backward time

$$\Sigma_- : \begin{cases} \mathbf{x}_-(n) = A_-\mathbf{x}_-(n+1) + B_-\mathbf{u}(n), \\ \mathbf{y}_-(n) = C_-\mathbf{x}_-(n) \qquad\qquad (n \in \mathbb{Z}) \end{cases} \tag{II.1}$$

$$\Sigma_+ : \begin{cases} \mathbf{x}_+(n+1) = A_+\mathbf{x}_+(n) + B_+\mathbf{u}(n), \\ \mathbf{y}_+(n) = C_+\mathbf{x}_+(n) + D\mathbf{u}(n) \qquad (n \in \mathbb{Z}) \end{cases} \tag{II.2}$$

with Σ_- having state space $\mathcal{X}_-$ and state operator A_- on $\mathcal{X}_-$ exponentially stable (i.e., $\sigma(A_-) \subset \mathbb{D}$) and Σ_+ having state space $\mathcal{X}_+$ and A_+ on $\mathcal{X}_+$ exponentially stable ($\sigma(A_+) \subset \mathbb{D}$). The input and output spaces are again denoted by $\mathcal{U}$ and $\mathcal{Y}$, respectively. See [4] for the finite dimensional case and [8] for the infinite dimensional case considered here.

A system trajectory for the bicausal system Σ consists of a triple $\{\mathbf{u}, \mathbf{x}, \mathbf{y}\} = \{\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n)\}_{n\in\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$ we have $\mathbf{u}(n) \in \mathcal{U}$, $\mathbf{x}(n) = \mathbf{x}_+(n) \oplus \mathbf{x}_-(n) \in \mathcal{X}_+ \oplus \mathcal{X}_- =: \mathcal{X}$ with $\mathbf{x}_\pm(n) \in \mathcal{X}_\pm$ and $\mathbf{y}(n) = \mathbf{y}_+(n) + \mathbf{y}_-(n)$ with $\mathbf{y}_\pm(n) \in \mathcal{Y}$, such that $(\mathbf{u}, \mathbf{x}_-, \mathbf{y}_-)$ is a system trajectory of Σ_- and $(\mathbf{u}, \mathbf{x}_+, \mathbf{y}_+)$ is a system trajectory of Σ_+. We say that the system trajectory $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ is ℓ^2*-admissible* if all system signals are in ℓ^2:

$$\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z}), \quad \mathbf{x}_+ \in \ell^2_{\mathcal{X}_+}(\mathbb{Z}), \quad \mathbf{x}_- \in \ell^2_{\mathcal{X}_-}(\mathbb{Z}), \quad \mathbf{y}_\pm \in \ell^2_{\mathcal{Y}}(\mathbb{Z}).$$

Due to the assumed exponential stability of A_+ and A_-, given $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$, there are uniquely determined $\mathbf{x}_+ \in \ell^2_{\mathcal{X}_+}(\mathbb{Z})$ $\mathbf{x}_- \in \ell^2_{\mathcal{X}_-}(\mathbb{Z})$ and $\mathbf{y}_\pm \in \ell^2_{\mathcal{Y}}(\mathbb{Z})$ so that $(\mathbf{u}, \mathbf{x}_+, \mathbf{y}_+)$ and $(\mathbf{u}, \mathbf{x}_-, \mathbf{y}_-)$ are ℓ^2-admissible system trajectories for Σ_+ and Σ_-, respectively, and hence $(\mathbf{u}, \mathbf{x}_+ \oplus \mathbf{x}_-, \mathbf{y}_+ + \mathbf{y}_-)$ is the unique ℓ^2-admissible system trajectory for $\Sigma = (\Sigma_+, \Sigma_-)$ with input sequence $\mathbf{u}$.

The transfer function $F = F_\Sigma$ associated with the bicausal system Σ is given by (I.7).

In order to state our standard bounded reel lemma for bicausal systems, we require an appropriate, and strong enough minimality notion. For this purpose we define the observability operator $\mathbf{W}_o$ and controllability operator $\mathbf{W}_c$ associated with the bicausal system Σ as

$$\begin{aligned} \mathbf{W}_c &= \begin{bmatrix} \mathbf{W}_c^+ & \mathbf{W}_c^- \end{bmatrix} : \begin{bmatrix} \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \end{bmatrix} \to \mathcal{X}, \\ \mathbf{W}_o &= \begin{bmatrix} \mathbf{W}_o^- \\ \mathbf{W}_o^+ \end{bmatrix} : \mathcal{X} \to \begin{bmatrix} \ell^2_{\mathcal{Y}}(\mathbb{Z}_-) \\ \ell^2_{\mathcal{Y}}(\mathbb{Z}_+) \end{bmatrix}. \end{aligned} \tag{II.3}$$

with the operators $\mathbf{W}_c^+$, $\mathbf{W}_c^-$, $\mathbf{W}_o^+$ and $\mathbf{W}_o^-$ are defined as

$$\begin{aligned} \mathbf{W}_c^+ &= \mathrm{row}_{j\in\mathbb{Z}_-}[A_+^{-j-1}B_+] \colon \ell^2_{\mathcal{U}}(\mathbb{Z}_-) \to \mathcal{X}_+, \\ \mathbf{W}_c^- &= \mathrm{row}_{j\in\mathbb{Z}_+}[A_-^{j}B_-] \colon \ell^2_{\mathcal{U}}(\mathbb{Z}_+) \to \mathcal{X}_-, \\ \mathbf{W}_o^+ &= \mathrm{col}_{i\in\mathbb{Z}_+}[C_+A_+^{i}] \colon \mathcal{X}_+ \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_+), \\ \mathbf{W}_o^- &= \mathrm{col}_{i\in\mathbb{Z}_-}[C_-eA_-^{-i}] \colon \mathcal{X}_- \to \ell^2_{\mathcal{Y}}(\mathbb{Z}_-). \end{aligned} \tag{II.4}$$

Note that the above operators are well defined by the assumed exponential stability of A_+ and A_-. A bicausal system Σ as in (II.1)–(II.2) is said to be *minimal* if Σ is *controllable*, meaning that $\mathbf{W}_c$ has dense range in $\mathcal{X}$, as well as *observable*, meaning that $\operatorname{Ker}\mathbf{W}_o = \{0\}$. For our standard bounded real lemma, however, we need a stronger notion of minimality, namely, we say that Σ is ℓ^2*-exactly minimal* if Σ is ℓ^2*-exactly controllable*, meaning the range of $\mathbf{W}_c$ is equal to $\mathcal{X}$, and ℓ^2*-exactly observable*, meaning the range of $\mathbf{W}_o^*$ is equal to $\mathcal{X}$.

To conclude this section, we point out that in case the state operator A_- of the backward time system Σ_- is invertible, it is possible to rewrite the bicausal system $\Sigma = (\Sigma_+, \Sigma_-)$ in the form (I.1). Namely, in this case one puts $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ and

$$\begin{aligned} A &= \begin{bmatrix} A_+ & 0 \\ 0 & A_-^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} B_+ \\ -A_-^{-1}B_- \end{bmatrix} \\ C &= \begin{bmatrix} C_+ & C_- \end{bmatrix}. \end{aligned} \tag{II.5}$$

One can easily compute that for any input signal $\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z})$, both systems provide the same state and output signals.

Note that invertibility of A_- in (I.7) guarantees that there can be no pole at 0, as is the case in (I.3).

III. Storage functions for bicausal systems

The storage function approach to dissipative systems was introduced by Willems in [13] and [14]. In the context of bicausal systems $\Sigma = (\Sigma_+, \Sigma_-)$, as described in Section II, a *storage function* is a function $S : \mathcal{X} \to \mathbb{R}$, with $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$, that satisfies the following three conditions:

1) S is continuous at 0,
2) S satisfies the energy-balance relation: For each $n \in \mathbb{Z}$
$$S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) \le \|\mathbf{u}(n)\|^2_{\mathcal{U}} - \|\mathbf{y}(n)\|^2_{\mathcal{Y}} \tag{III.1}$$
along all ℓ^2-admissible system trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y})$,
3) S satisfies the normalization condition $S(0) = 0$.

We further say that S is a *strict storage function* for Σ if S is a storage function for Σ with condition (III.1) replaced by the stronger condition: *there is a* $\epsilon > 0$ *so that for all* $n \in \mathbb{Z}$:

$$\begin{aligned} S(\mathbf{x}(n+1)) - S(\mathbf{x}(n)) + \epsilon^2\|\mathbf{x}(n)\|^2 &\le \\ \le (1-\epsilon^2)\|\mathbf{u}(n)\|^2_{\mathcal{U}} &- \|\mathbf{y}(n)\|^2_{\mathcal{Y}} \end{aligned} \tag{III.2}$$

along all ℓ^2*-admissible system trajectories* $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ *of* Σ.

A storage function $S : \mathcal{X} \to \mathbb{R}$ is called quadratic in case there exists a bounded, selfadjoint operator $H = \left[\begin{smallmatrix} H_- & H_0 \\ H_0^* & H_+ \end{smallmatrix}\right]$ on $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$ which is invertible and such that

$$S(x) = S_H(x) = \langle Hx, x\rangle = \left\langle \left[\begin{smallmatrix} H_- & H_0 \\ H_0^* & H_+ \end{smallmatrix}\right] \left[\begin{smallmatrix} x_- \\ x_+ \end{smallmatrix}\right], \left[\begin{smallmatrix} x_- \\ x_+ \end{smallmatrix}\right] \right\rangle$$

for all $x = x_- \oplus x_+ \in \mathcal{X}$.

If the bicausal system is ℓ^2-exactly controllable, then the energy balance (III.1) can be replaced by the local condition

$$\begin{aligned} S(x_- \oplus (A_+x_+ + B_+u)) - S((A_-x_- + B_-u) \oplus x_+) \\ \leq \|u\|^2 - \|CA_-x_- + C_+x_+ + (C_-B_- + D)u\|^2, \end{aligned}$$

while the strict energy balance (III.2) can be replaced by

$$\begin{aligned} S(x_- \oplus (A_+x_+ + B_+u)) - S((A_-x_- + B_-u) \oplus x_+) \\ + \epsilon^2(\|A_-x_- + B_-u\|^2 + \|x_+\|^2 + \|u\|^2) \\ \leq \|u\|^2 - \|CA_-x_- + C_+x_+ + (C_-B_- + D)u\|^2, \end{aligned}$$

where in both inequalities $u \in \mathcal{U}$, $x_\pm \in \mathcal{X}_\pm$ are arbitrary.

As a result of these localized versions of the energy balance inequalities it follows that in the original inequalities (III.1) and (III.2), one may restrict to ℓ^2-admissible system trajectories.

Existence of a (strict) storage function for Σ is a sufficient condition for F_Σ to have the desired properties for the (strict) bounded real lemma.

Proposition 3.1: Suppose that S is a storage function for the bicausal system $\Sigma = (\Sigma_+, \Sigma_-)$ in (II.1)–(II.2), with $A_\pm$ exponentially stable. Then $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$. If S is a strict storage function for Σ, then $\|F_\Sigma\|_{\infty,\mathbb{T}} < 1$.

In analogy to the approach of Willems in [13], again assuming ℓ^2-exact controllability, one can define the available storage $S_a : \mathcal{X} \to \mathbb{R}$ and required supply $S_r : \mathcal{X} \to \mathbb{R}$ to be the functions given by

$$S_a(x) = \sup_{\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z}) \colon \mathbf{W}_c\mathbf{u} = x} \sum_{n=0}^{\infty} \left(\|\mathbf{y}(n)\|^2 - \|\mathbf{u}(n)\|^2\right) \quad \text{(III.3)}$$

$$S_r(x) = \inf_{\mathbf{u} \in \ell^2_{\mathcal{U}}(\mathbb{Z}) \colon \mathbf{W}_c\mathbf{u} = x} \sum_{n=-\infty}^{-1} \left(\|\mathbf{u}(n)\|^2 - \|\mathbf{y}(n)\|^2\right) \quad \text{(III.4)}$$

where $\mathbf{y}$ in both (III.3) and (III.4) is the output signal determined by the bicausal system Σ and the input $\mathbf{u}$.

Proposition 3.2: Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a ℓ^2-exactly minimal bicausal system as in (II.1)–(II.2), with $A_\pm$ exponentially stable and with transfer function $F = F_\Sigma$ as in (I.7). Assume $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$. Then:

1) S_a is a storage function for Σ.
2) S_r is a storage function for Σ.
3) If $\widetilde{S}$ is any storage function for Σ, then

$$S_a(x_0) \leq \widetilde{S}(x_0) \leq S_r(x_0) \text{ for all } x_0 \in \mathcal{X}.$$

Not only are S_a and S_r the smallest and largest storage functions, under the conditions of Proposition 3.2, they also turn out to be quadratic storage functions.

Theorem 3.3: Suppose that $\Sigma = (\Sigma_+, \Sigma_-)$ is a bicausal discrete-time linear system as in (II.1)–(II.2), with A_+ and A_- asymptotically stable. Assume Σ is ℓ^2-exactly minimal and $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$. Then the available storage S_a and required supply S_r are qudratic storage functions, i.e., there exist bounded, self-adjoint, invertible operators H_a and H_r on $\mathcal{X}$ such that for all $x \in \mathcal{X}$:

$$S_a(x) = \langle H_a x, x\rangle, \quad S_r(x) = \langle H_r x, x\rangle.$$

IV. Bounded Real Lemmas for bicausal systems

In the previous section we observed how the existence of a storage function implies that the objective of the bounded real lemma is met. Using quadratic storage functions, one can derive the KYP inequality in the present context.

Proposition 4.1: Suppose that $\Sigma = (\Sigma_+, \Sigma_-)$ is a bicausal system as in (II.1)–(II.2), with A_+ and A_- exponentially stable. Let H be a selfadjoint operator on $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_-$ with block matrix decomposition

$$H = \begin{bmatrix} H_- & H_0 \\ H_0^* & H_+ \end{bmatrix} \text{ on } \mathcal{X} = \begin{bmatrix} \mathcal{X}_- \\ \mathcal{X}_+ \end{bmatrix}. \quad \text{(IV.1)}$$

Then S_H is a quadratic storage function for Σ if and only if H is a solution of the bicausal KYP-inequality:

$$\begin{aligned} \begin{bmatrix} I & 0 & A_-^*C_-^* \\ 0 & A_+^* & C_+^* \\ 0 & B_+^* & B_-^*C_-^*+D^* \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & A_+ & B_+ \\ C_-A_- & C_+ & C_-B_-+D \end{bmatrix} \\ \preceq \begin{bmatrix} A_-^* & 0 & 0 \\ 0 & I & 0 \\ B_-^* & 0 & I \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_- & 0 & B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \quad \text{(IV.2)} \end{aligned}$$

Moreover, S_H is a strict storage function for Σ if and only if H is a solution of the strict bicausal KYP-inequality:

$$\begin{aligned} \begin{bmatrix} I & 0 & A_-^*C_-^* \\ 0 & A_+^* & C_+^* \\ 0 & B_+^* & B_-^*C_-^*+\widehat{D}^* \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & A_+ & B_+ \\ C_-A_- & C_- & C_-B_-+D \end{bmatrix} + \\ + \epsilon^2 \begin{bmatrix} A_-^*A_- & 0 & A_-^*B_- \\ 0 & I & 0 \\ B_-^*A_- & 0 & B_-^*B_-+I \end{bmatrix} \\ \preceq \begin{bmatrix} A_-^* & 0 & 0 \\ 0 & I & 0 \\ B_-^* & 0 & I \end{bmatrix} \begin{bmatrix} H_- & H_0 & 0 \\ H_0^* & H_+ & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_- & 0 & B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \quad \text{(IV.3)} \end{aligned}$$

for some $\epsilon > 0$.

In particular, under the assumptions of Theorem 3.3, the operators H_a and H_r associated with the available storage S_a and required supply S_r, respectively, are solutions to the KYP inequality IV.2.

By combining the above results along with additional arguments, it is possible to derive the following bicausal variations on the standard and strict bounded real lemma.

Theorem 4.2: Suppose that $\Sigma = (\Sigma_+, \Sigma_-)$ is a bicausal linear system as in (II.1) and (II.2),, with both A_+ and A_- exponentially stable and with associated transfer function F_Σ as in (I.7).

(1) *Standard Bounded Real Lemma:* Assume that Σ is ℓ^2-exactly minimal. Then $\|F_\Sigma\|_{\infty,\mathbb{T}} \leq 1$ if and only if there exists a bounded and boundedly invertible selfadjoint solution H as in (IV.1) of the bicausal KYP-inequality (IV.2).
(2) *Strict Bounded Real Lemma:* The strict inequality $\|F_\Sigma\|_{\infty,\mathbb{T}} < 1$ holds if and only if there is a bounded and boundedly invertible selfadjoint solution H as in (IV.1) of the strict bicausal KYP-inequality (IV.3).

Remark 4.3: Note that it is not directly obvious how the bicausal standard and strict KYP inequalities, (IV.2) and (IV.3), respectively, relate to the classical standard and strict KYP inequalities, (I.5) and (I.6), respectively. In case the operator A_- is invertible, one can transform the bicausal system (II.1)–(II.2) into a system of the form (I.1) via the transformation (II.5). It is then also possible to transfer the bicausal standard KYP inequality (IV.2) (respectively bicausal strict KYP inequality (IV.3)) into the classical standard KYP inequality (I.5) (respectively, classical strict KYP inequality (I.6)) by multiplying in (IV.2) on the right by T and on the left by T^*, where T is given by

$$T = \begin{bmatrix} A_-^{-1} & 0 & A_-^{-1}B_- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

and likewise by multiplying (IV.3) on the right by T and on the left by T^*.

In both cases one arrives at a selfadjoint, boundedly invertible solution H to (I.5) and (I.6), respectively, rather than strictly positive definite solutions.

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