

Adaptive model reduction and control for distributed parameter systems using DEIM/DAPOD combination

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Abstract—The output feedback control of distributed parameter systems (DPS) based on adaptive model reduction is explored in this paper. A significant computational hurdle when using model reduction for control is the numerical integration of integrals that appear in the reduced order model reducing their applicability when dealing with nonlinearities. The objective of this paper is to further reduce the computational cost in discrete adaptive proper orthogonal decomposition (DAPOD). It is addressed by using discrete empirical interpolation method (DEIM) in the observer and controller to reduce the computational cost associated with the computation of nonlinear functions. The proposed method is successfully applied in a tubular reactor with recycle.

I. INTRODUCTION

In recent years, control of distributed parameter systems (DPS) has been an important issue in many chemical industry process due to spatial variation as a result of diffusion, convection and chemical reaction. Some of the examples include plasma enhanced chemical vapor decomposition, polymerization catalysis and plug flow reactor. The controller design for DPS is nontrivial since the state varies in both time and space, which makes the problem infinite dimensional in functional space.

Most of these DPS are mathematically described by dissipative partial differential equation (PDE). A standard approach is to construct the reduced order model (ROM) using the method of weighted residuals (MWR)[5], [2], [3] which takes advantage of the property of dissipative PDEs that their behavior can be approximated by finite dimensional systems [4], [7]. It approximates the state variable by superposition of basis functions multiplied by time dependent coefficients. Basis functions are predetermined analytically by solving eigenfunction problem of spatial operator of the system.

Since solving the eigenproblem analytically is complex for systems with complex geometry and nonlinear systems, proper orthogonal decomposition (POD) [11] can be used to construct the basis functions using previous observation of the systems (snapshots). However, the quality of the basis functions depends on the quantity of the snapshots and how the snapshots are collected [9].

One promising approach to mitigate this situation is DAPOD [14], which updates the basis function when the controller and observer are implemented on-line. The basis

functions are updated using new snapshot collected from the process as it evolves. Since applying POD iteratively is computational intensive, DAPOD eliminates less important and less recent snapshots and determines new basis function size by estimating eigenvalues of new covariance matrix. Compared with the modified APOD [10], DAPOD reduces computational cost and improves the accuracy of the estimated eigenvalues.

Another issue with ROMs is that the resulting ordinary differential equations (ODEs) system may still be computationally expensive to evaluate since nonlinearities in the original partial differential equation(s) have to be numerically integrated in space in the resulting ROM. This can lead to delays in the computation of control action. To circumvent this issue, discrete empirical interpolation method (DEIM) [6] is adopted to reduce computational cost. In this method, the nonlinear term in governing equation in the whole domain is estimated by measurement at k points with k much less than the number of spatial grid points using nonlinear basis functions. Nonlinear basis functions are constructed off-line and the positions of those k points are determined based on DEIM algorithm.

In this paper, we propose a combination of DAPOD and DEIM to design nonlinear controllers of reduced computational requirements to force the closed-loop DPS evolution to a desired operating point. To illustrate the performance of the proposed control scheme and investigate the stability of it, a tubular reactor with recycle example is investigated.

II. PROBLEM FORMULATION

We consider a dissipative process described by the following PDE:

$$\frac{\partial x}{\partial t} = \mathcal{L}x + f(x) + b(x, z)u \quad (1)$$

$$y_i = \int_{\Omega_z} s_i(z)x dz \quad (i = 1, 2, \dots, k_1) \quad (2)$$

subject to the boundary condition

$$g_b(x, \frac{\partial x}{\partial z}) = 0 \text{ on } \Gamma \quad (3)$$

and the initial condition

$$x(z, 0) = x_0(z) \quad (4)$$

In this system, x is the state variable, t denotes time, $z \in \Omega_z \subset \mathbb{R}^3$ is the spatial coordinate; \mathcal{L} represents a linear operator, $f(x)$ is a nonlinear function. $u \in \mathbb{R}^{s_0 \times 1}$ denotes manipulated variables, where s_0 refers to the number of manipulated inputs. $b(z) \in \mathbb{R}^{1 \times s_0}$ describes how the

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manipulated variables u control the system spatially and is known. y_i denotes the i^{th} measured output, and $s_i(z)$ is a function of space. g is a function of x and its spatial derivative. Γ is the process boundary. The control objective is to maintain the state variable at a desired steady state.

A. Method of Weighted Residuals

We use method of the weighted residuals (MWR) [13] to construct ROMs. Here we provide a brief introduction of MWR.

MWR approximates state variable, x , by a superposition of basis functions $\{\phi_q\}$.

$$x(z) = \sum_{q=1}^n c_q(t)\phi_q(z) \quad (5)$$

where c_q denotes time-dependent coefficients, called modes. They are the state variables in the reduced order model.

A set of ordinary differential equations is used to approximate the PDEs.

$$\begin{aligned} \sum_{q=1}^n \int_{\Omega_z} \varphi_j \dot{c}_q \phi_q dz &= \int_{\Omega_z} \varphi_j \left(\sum_{q=1}^n c_q \mathcal{L} \phi_q \right) dz \\ + \int_{\Omega_z} \varphi_j f \left(\sum_{q=1}^n c_q \phi_q \right) dz &+ \int_{\Omega_z} \varphi_j b(x, z) dz u \end{aligned} \quad (6)$$

$j = 1, 2, \dots, n$

When weighting functions $\{\varphi_j\}$ are constructed so that the residual r is orthogonal to basis functions $\{\phi_q\}$, this method is also called the Galerkin method [12]. In this manuscript, we will focus on the Galerkin method. Eq. 6 can be written in matrix form:

$$\dot{c} = Lc + F(c) + B(c)u \quad (7)$$

where $c = [c_1, c_2, \dots, c_n]^T$, $L_{ij} = \int_{\Omega_z} \varphi_j \mathcal{L} \phi_i dz$. Note that both L and B can be calculated off-line, while F is a function of c and has to be calculated by evaluating $f(x)$ in every point in space, which is computationally expensive.

B. Discrete Adaptive Proper Orthogonal Decomposition

We briefly review DAPOD for completeness. DAPOD can be summarized into the following steps:

- construct basis functions off-line
- incorporate new snapshots
- determine basis function size
- check the accuracy of ROM
- update basis functions
- eliminate old snapshot(s)

In the off-line initial basis function construction step, we can use singular value decomposition (SVD) to construct the basis function Φ , and initial basis function size is determined by energy captured by each basis function.

In on-line step, snapshot matrix A is updated when new snapshot becomes available. Both importance and "freshness" of snapshots are considered.

As new snapshots become available, the number of basis functions needed to construct ROM may increase or decrease.

We change the basis function size accordingly based on the energy captured by the basis functions truncated in previous steps.

After calculating energy captured by basis functions, we check the accuracy of basis functions if the basis function size is unchanged. When basis functions size changes or current basis functions are not accurate enough, basis functions are updated.

C. Discrete Empirical Interpolation Method

Discrete empirical interpolation method was proposed by Sorensen in 2010 [6]. The method seeks to reduce the computational cost associated with evaluating nonlinear terms in POD with Galerkin projection. DEIM applies to ordinary differential equations (ODEs) arising from discretization of PDEs. The nonlinear term $f(x)$ in Eq. 1 is discretized and approximated by a linear combination of nonlinear basis functions $U = [u_1, u_2, \dots, u_k] \in \mathbb{R}^{M \times k}$

$$\underbrace{f(t)}_{M \times 1} \approx U \tilde{c}(t) \quad (8)$$

where U is obtained using POD algorithm with nonlinear snapshots. Nonlinear snapshots $\{f(t_1), f(t_2), \dots, f(t_{\tilde{M}})\}$ are obtained from open loop process of the system (Eq. 1). The unknown coefficients \tilde{c} is determined using the value of f at k grid points. (Recalling that $k \ll M$, this problem is overdetermined for \tilde{c}). As a result, we obtain

$$\tilde{c}(t) = (P^T U)^{-1} P^T f(t) \quad (9)$$

where P is chosen to reduce the error of the approximation. The final approximation of nonlinear term is

$$f(t) \approx \underbrace{U(P^T U)^{-1}}_{M \times k} \underbrace{P^T f(t)}_{k \times 1} \quad (10)$$

Notices that $U(P^T U)^{-1}$ is predetermined and evaluating $P^T f(t)$ only requires the values of $f(t)$ at k spatial grid points.

III. OBSERVER & CONTROLLER DESIGN USING DEIM/APOD COMBINATION

In this section, we employ DEIM in APOD-based reduced order model to reduce the computational cost in controller and observer. Many controller design methods, such as feedback linearization and Lyapunov based control, and dynamic observer design methods require the evaluation of the nonlinear term. By using DEIM, evaluating nonlinear term at each grid point can be circumvented and the integration step can also be predetermined.

We consider a Luenberger-type dynamic observer based on the reduced order model (Eq. 7)

$$\frac{d\tilde{c}}{dt} = L\tilde{c} + F(\tilde{c}) + B(\tilde{c})u + G_m(y - \tilde{y}) \quad (11)$$

where \tilde{c} refers to the estimated state. The gain matrix G_m is determined using LQR theory[8].

Since the linear part $L\tilde{c}$ can be evaluated with lower cost than the nonlinear part $F(\tilde{c})$, we apply DEIM (Eq. 10) to $F(\tilde{c})$,

$$\begin{aligned} F(\tilde{c}) &= \int_{\Omega_z} \varphi_j \tilde{f} \left(\sum_{q=1}^n c_q \phi_q \right) dz \\ &\approx \Phi^T \tilde{f} \Delta z \\ &\approx \underbrace{\Phi^T U (P^T U)^{-1} \Delta z P^T}_{D} \tilde{f} \end{aligned} \quad (12)$$

where Φ denotes discretized basis functions. We assume the grid points are evenly spaced and Δz denotes the interval.

If $b(x, z)$ is a linear function of x , $B(\tilde{c})$ in Eq. 11 can be written as $\mathcal{B}\tilde{c}$, we still apply standard POD when evaluating $B(\tilde{c})$. If $b(x, z)$ is a nonlinear function of x , we generate another set of nonlinear basis functions to reduce the cost of evaluating $B(\tilde{c})$ like Eq. 12

$$\begin{aligned} B(\tilde{c}) &= \int_{\Omega_z} \varphi_j \tilde{b} \left(\sum_{q=1}^n c_q \phi_q \right) dz \\ &\approx \Phi^T \tilde{b} \Delta z \\ &\approx \underbrace{\Phi^T U_b (P_b^T U_b)^{-1} \Delta z P_b^T}_{D_b} \tilde{b} \end{aligned} \quad (13)$$

We obtain,

$$\frac{d\tilde{c}}{dt} = L\tilde{c} + DP^T \tilde{f}(t) + D_b P_b^T \tilde{b} u + G_m(y - \tilde{y}) \quad (14)$$

where

$$D = \Phi^T U (P^T U)^{-1} \Delta z$$

$$D_b = \Phi^T U_b (P_b^T U_b)^{-1} \Delta z$$

can be predetermined and updated when the basis functions are updated.

Similarly, we can use this approach in controller design methods.

IV. APPLICATION

V. TUBULAR REACTOR WITH RECYCLE

In this section, we consider a tubular reactor with recycle, where a first order reaction $A \rightarrow B$ takes place [1]. A cooling jacket is used to eliminate hot spot formulation.

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= -\frac{\partial x_1}{\partial z} + \frac{1}{Pe_1} \frac{\partial^2 x_1}{\partial z^2} + B_T B_C e^{-\frac{\gamma x_1}{1+x_1}} (1+x_2) \\ &\quad + \beta_T (u - x_1) + \delta(z-0)((1-r)x_{1f} + rx_1(1)) \\ \frac{\partial x_2}{\partial t} &= -\frac{\partial x_2}{\partial z} + \frac{1}{Pe_2} \frac{\partial^2 x_2}{\partial z^2} - B_C e^{-\frac{\gamma x_1}{1+x_1}} (1+x_2) \\ &\quad + \delta(z-0)((1-r)x_{2f} + rx_2(1)) \end{aligned} \quad (15)$$

with boundary condition:

TABLE I: Parameter values used in Eq. 15

Parameter	Values	Parameter	Values
Pe_1	1	r	0.1
Pe_2	1	x_{1f}	0
B_T	2.5	x_{2f}	1
B_C	0.1	γ	7
β_T	2		

$$\begin{aligned} z=0: & & z=L: \\ \frac{\partial x_1}{\partial z} &= Pe_1 x_1 & \frac{\partial x_1}{\partial z} &= 0 \\ \frac{\partial x_2}{\partial z} &= Pe_2 x_2 & \frac{\partial x_2}{\partial z} &= 0 \end{aligned}$$

where:

$$\begin{aligned} Pe_1 &= \frac{\rho c_p v L}{k} \\ Pe_2 &= \frac{v L}{D_A} \end{aligned} \quad (16)$$

The values of parameters used in Eq. 15 are reported in Table I. The objective of this problem is to force the temperature to the state $x_1(z, t) = 0$.

In this section, DEIM is exploited to reduce computational burden. Since both equations in Eq. 15 contain the same term $B_C e^{-\frac{\gamma x_1}{1+x_1}} (1+x_2)$, we use DEIM algorithm to reduce the computational cost in evaluating this term. To simplify the notation, we represent this term by f_0 . To apply DEIM, nonlinear snapshots of f_0 are collected from open loop process with initial condition $x_1(z) = 0.3$ and $x_2(z) = 0.3$. Process evolution based on DAPOD only and DAPOD & DEIM are compared in Fig. 1 and Fig. 2. The error of observer is displayed in Fig. 3. We observe that the error introduced by DEIM algorithm is negligible. The numbers of basis functions for x_1 and x_2 are provided in Fig. 4 The time used in simulating the processes is displayed in Table II

TABLE II: Simulation time

DAPOD	number of nonlinear basis functions	DEIM & DAPOD
	3	82s
83s	5	79s
	8	79s
	12	80s

VI. CONCLUSIONS

In this paper, we propose to use DAPOD/DEIM combination to reduce the computational cost in controlling dissipative distributed parameter system. By evaluating the performance of the proposed method in a tubular reactor with recycle example, we conclude that using DEIM can reduce the computational cost and has negligible impact on the performance of the controller and observer compared with systems using DAPOD only.

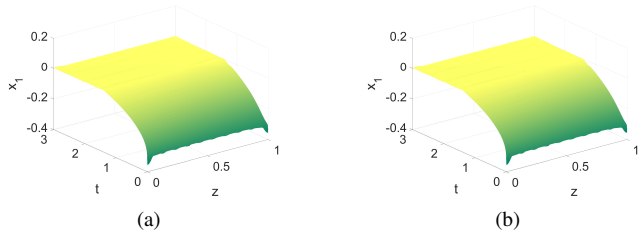


Fig. 1: The spatiotemporal profile of x_1 based on (a)DAPOD; (b)DAPOD & DEIM.

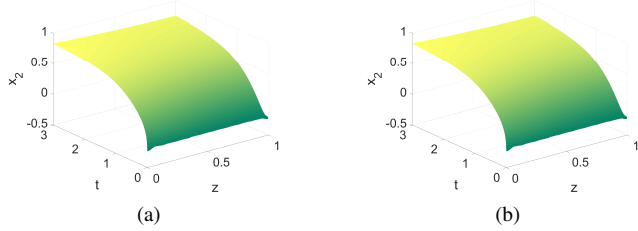


Fig. 2: The spatiotemporal profile of x_2 based on (a)DAPOD; (b)DAPOD & DEIM.

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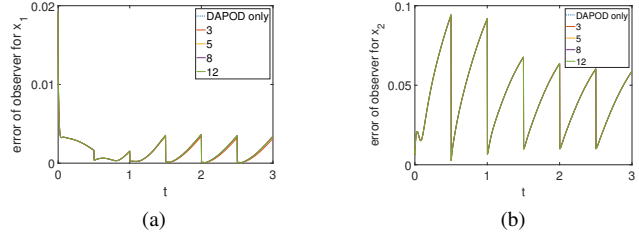


Fig. 3: The error of observer for (a) x_1 ; (b) x_2 .

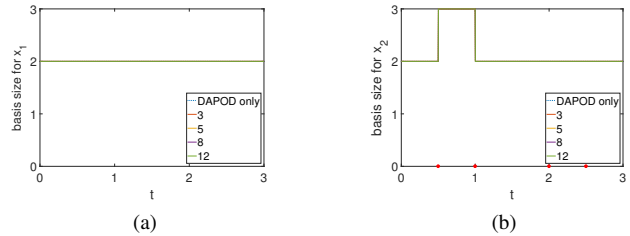


Fig. 4: The number of basis functions for (a) x_1 ; (b) x_2 .

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