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H_2 Optimization under Communication Interruptions

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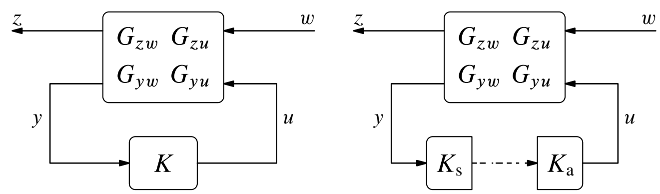
***Abstract*— The paper studies the problem of the H_2 (LQG) optimal control of LTI plants under irregular communication interruptions between the sensor- and actuator-side parts of the controller. The derived optimal solution is analytic, numerically simple, implementable, transparent, and requires no a priori information about the interruption intervals or their statistics. The result is generalized to incorporate a constant loop delay.**

***Index Terms*— Networked control, H_2 optimization.**



(a) Lumped controller (b) Networked controller

Fig. 1. The standard H_2 setup with generalized plant G

I. Introduction

Networked control raises numerous challenges for the analysis and design of feedback loops. One of them is expensive, and potentially unreliable, communication between sensors and actuators, see [1, 2] and the references therein. Communication problems may be accounted for as signal-to-noise ratio constraints [3], channel capacity constraints [4], loop delays [5], lost packets [6], etc. The setup studied in this paper is motivated by the latter approach.

We consider the standard continuous-time H_2 problem assuming that the information flow in the controller, from the measured plant output to the control input, may be interrupted during some time intervals. This assumption is similar to that made in [7, Ch. 4], where the setup is referred to as the "intermittent feedback" and periodic alternations of closed- and open-loop stretches are considered. In this paper, the communication is allowed to be interrupted irregularly. This results in intermittent alternations of closed- and open-loop stretches and may be thought of as a stripped-down version of the packets loss phenomenon in networked control. There are other motivations for this setup, see [7, Sec. 4.5] and the references therein. Another difference from [7, Ch. 4], which addressed only the stabilization problem, is that the present paper studies the closed-loop H_2 performance. From this viewpoint the setup is related to those in [6, 8][1].

The solution procedure proposed below is different from those in the above-mentioned references. Specifically, communication interruptions are treated not as time-varying parameters in the plant input and output channels, but rather as *causality constraints* imposed upon analog suboptimal controllers operating with no interruptions. The design procedure is then based on imposing appropriate causality constraints on the Youla parameter in the parametrization of all analog suboptimal controllers. This follows the logic of [9], although is applied to a different class of causality constraint. The result of this treatment is an analytic solution, which possesses two important properties. First, the solution is based on the very same gains as the standard, time-invariant, H_2 solution in the uninterrupted case. This is in contrast to the solutions in [6, 8], where the gains are time varying. Second, the optimal controller needs to know neither the interruption intervals in advance nor interruption statistics. The controller architecture does not depend on the interruption schedule and only switches one link on / off each time the communication becomes available / unavailable. This property might appear surprising, taking into account that the quadratic optimization has a two-point boundary value problem in its heart.

The approach is also extended to accommodate a constant loop delay. The presence of the delay requires a dead-time compensation element to be added to the sensor-side part of the controller. Otherwise, the properties remain unchanged.

Notation: The (complex-conjugate) transpose of a matrix M is denoted by M' and, if M is square, $\operatorname{tr}(M)$ stands for its trace. The Frobenius norm of a matrix $\|M\|_{\mathrm{F}} := \sqrt{\operatorname{tr}(M'M)}$. The notation $\|G\|_2$ is used for the H_2 norm of a linear system G, see Appendix for more details. The lower / upper linear-fractional transformations of Ω by Φ are

$$\mathcal{F}_{\mathrm{l}}(\Phi,\Omega) := \Phi_{11} + \Phi_{12}\Omega(I - \Phi_{22}\Omega)^{-1}\Phi_{21},$$
$$\mathcal{F}_{\mathrm{u}}(\Phi,\Omega) := \Phi_{22} + \Phi_{21}\Omega(I - \Phi_{11}\Omega)^{-1}\Phi_{12}$$

for appropriate partitions of Φ.

II. Problem Formulation and Solution

Consider the setup in Fig. 1(a) with a given generalized plant G and a controller K. The generalized plant is assumed to be LTI and given in terms of its state-space realization

$$G = \begin{bmatrix} G_{zw} & G_{zu} \\ G_{yw} & G_{yu} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]. \qquad (1)$$

We also assume that

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[1]Papers [6, 8] et alii consider discrete-time versions of the problem, which may be more natural in networked control. This difference, however, is minor. The continuous-time setup is adopted below solely to simplify the formulae, which streamlines the presentation of main ideas. Arguments of the paper extend to discrete-time systems mutatis mutandis.

$\mathcal{A}_1$: the triple (C_y, A, B_u) is stabilizable and detectable,

$\mathcal{A}_2$: the realization (A, B_u, C_z, D_{zu}) has no invariant zeros on the imaginary axis and $D'_{zu} D_{zu} > 0$,

$\mathcal{A}_3$: the realization (A, B_w, C_y, D_{yw}) has no invariant zeros on the imaginary axis and $D_{yw} D'_{yw} > 0$.

The performance is quantified by the H_2 norm of the system

$$T_{zw} = \mathcal{F}_l(G, K)$$

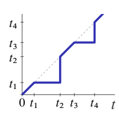
connecting the exogenous input w and the regulated output z (see Appendix for the definition of the H_2 norm and its interpretations).

A key assumption about the controller K is that the information flow in it, from the measured output y to the control input u, may be interrupted during some time intervals. A schematic representation of this is shown in Fig. 1(b), where K_s and K_a are controller parts located at its sensor and actuator sides, respectively. Formally, consider a sequence of time instances

$$0 = t_0 \le t_1 < t_2 \le t_3 < t_4 \le \cdots, \quad \lim_{i\to\infty} \frac{t_i}{i} > 0 \tag{2}$$

and define the function

$$\iota(t) := \begin{cases} t & \text{if } t \in [t_{2i}, t_{2i+1}] \\ t_{2i+1} & \text{if } t \in (t_{2i+1}, t_{2i+2}) \end{cases} = \tag{3}$$



for all $i \in \mathbb{Z}_+$. We assume that $K : y \mapsto u$ is such that

$\mathcal{A}_4$: $u(t)$ may depend only on $y(s)$ for $s \le \iota(t)$, $\forall t \in \mathbb{R}_+$.

Assumption $\mathcal{A}_4$ implies that the control loop is closed in "even" intervals $[t_{2i}, t_{2i+1}]$ and open in "odd" intervals, (t_{2i+1}, t_{2i+2}). Although this model is simplistic, it captures the essence of interrupted communication, the halt of the information flow from one end of the controller to the other. At the same time, it imposes no restriction on the way in which $y(t)$ is processed at the sensor side or on the waveform of $u(t)$ at the actuator side. This may be reasonable in the world where computational resources become more and more accessible. Likewise, there is no restriction on the split of the controller as $K = K_a K_s$ in Fig. 1(b).

Remark 2.1: It should be stressed that $\mathcal{A}_4$ does not imply any loss of information in $y(t)$. It merely says that the effect of measurements on the control signal might be occasionally delayed. If $t_{2i+1} = t_{2i}$ at some i, which is not ruled out by (2), we have a sampled-data information exchange, like that studied in [9]. If $t_{2i+1} > t_{2i}$, the controller K behaves as a standard analog causal system in $[t_{2i}, t_{2i+1}]$. ▽

We are now in a position to formulate the problem studied in this paper.

$\mathbf{OP}_{\{t_i\}}$: Given the generalized plant (1), satisfying $\mathcal{A}_{1\text{–}3}$, and the sequence $\{t_i\}$ as in (2), design a controller K satisfying $\mathcal{A}_4$, which internally stabilizes the system and minimizes $\|T_{zw}\|_2$.

The solution of $\mathbf{OP}_{\{t_i\}}$, much like of its conventional analog counterpart, is based on the stabilizing solutions to the following two algebraic Riccati equations:

$$A'X + XA + C'_z C_z - C'_u C_u = 0, \tag{4a}$$

where $C_u := (D'_{zu} D_{zu})^{-1/2}(B'_u X + D'_{zu} C_z)$, and

$$AY + YA' + B_w B'_w - B_y B'_y = 0, \tag{4b}$$

where $B_y := (YC'_y + B_w D'_{yw})(D_{yw} D'_{yw})^{-1/2}$. Solutions are said to be stabilizing if the matrices $A + B_u F$ and $A + LC_y$ are Hurwitz, where

$$F := -(D'_{zu} D_{zu})^{-1}(B'_u X + D'_{zu} C_z),$$
$$L := -(YC'_y + B_w D'_{yw})(D_{yw} D'_{yw})^{-1}.$$

Stabilizing solutions to (4) exist iff $\mathcal{A}_{1\text{–}3}$ hold and are unique and positive semi-definite then, see [10, Cor. 12.10].

The main result of the paper, whose proof is postponed to Section III, is then formulated as follows:

Theorem 2.1: The optimal performance in $\mathbf{OP}_{\{t_i\}}$ is

$$\|T_{zw}\|_2^2 = \gamma_0 + \lim_{i\to\infty} \frac{1}{t_{2i}} \sum_{j=1}^{i} \int_0^{h_j} \int_0^s \|C_u \mathrm{e}^{At} B_y\|_F^2 \, \mathrm{d}t \mathrm{d}s, \tag{5}$$

where $h_j := t_{2j} - t_{2j-1} > 0$ is the duration of the jth interruption and

$$\gamma_0 := \mathrm{tr}(B'_w X B_w) + \mathrm{tr}(C_z Y C'_z) + \mathrm{tr}(XAY + YA'X) \tag{6}$$

is the optimal performance in the conventional continuous-time H_2 problem. A (non-unique) controller attaining (5) is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_u u(t) - L(y(t) - C_y \hat{x}(t)) \tag{7a}$$
$$u(t) = F \mathrm{e}^{(A+B_u F)(t-\iota(t))} \hat{x}(\iota(t)) \tag{7b}$$

and it is stabilizing iff either A is Hurwitz or there is $\overline{\alpha} > 0$ such that $\sup_{i\in\mathbb{N}}(t_{2i} - t_{2i-1}) \le \overline{\alpha}$. ▽

Some remarks are in order:

Remark 2.2 (implementation): To implement the optimal controller in (7) we need to run the Kalman filter (7a) at the sensor side K_s and two copies of the control signal generator (7b), one at the sensor side to produce u for (7a) and one at the actuator side K_a to generate the actual control signal. To that end, the sensor side needs to know interruption intervals in real time, which might not suit some protocols. ▽

Remark 2.3 (periodic interruptions): One may think of scenarios, where interruptions are periodic. For example, this would be the case if a communication channel is shared by several processes under round-robin scheduling. Periodic scenario corresponds to the choice

$$t_j = \begin{cases} \frac{j}{2} h_c + \frac{j}{2} h_i & \text{if } j \text{ is even,} \\ \frac{j+1}{2} h_c + \frac{j-1}{2} h_i & \text{if } j \text{ is odd,} \end{cases}$$

where $h_c > 0$ and $h_i > 0$ are the durations of the connection and interruption intervals, respectively. In this case, the optimal performance is

$$\|T_{zw}\|_2^2 = \gamma_0 + \frac{1}{h_c + h_i} \int_0^{h_i} \int_0^s \|C_u \mathrm{e}^{At} B_y\|_F^2 \, \mathrm{d}t \mathrm{d}s$$

and the optimal controller is unique (because $\|T_{zw}\|_2$ is then a norm, not a semi-norm). ▽

III. Proof of Theorem 2.1

To simplify the exposition, we assume throughout this section that $D'_{zu}D_{zu} = I$ and $D_{yw}D'_{yw} = I$. These assumptions are a matter of scaling the signals u and y in Fig. 1 and can thus be made without loss of generality. Under these assumptions $F = -C_u$ and $L = -B_y$.

A. All suboptimal analog controllers

Controllers satisfying $\boldsymbol{\mathcal{A}}_4$ constitute a subset of the set of causal continuous-time controllers $K : y \mapsto u$, i.e. such that $u(t)$ may depend only on $y(s)$ for $s \le t$. This suggests that controllers for $\mathbf{OP}_{\{t_i\}}$ can be designed by imposing additional (causality) constraints on a parametrization of suboptimal causal controllers. This line of reasonings is followed below.

Form the system

$$J = \left[\begin{array}{c|cc} A + B_u F + LC_y & -L & B_u \\ \hline F & 0 & I \\ -C_y & I & 0 \end{array}\right], \tag{8}$$

where F and L are the LQR and Kalman filter gains defined after (4). The following result parametrizes all suboptimal time-varying analog controllers:

Lemma 3.1: Let $\boldsymbol{\mathcal{A}}_{1\text{–}3}$ hold. Then all stabilizing continuous-time linear controllers for the setup in Fig. 1(a) can be parametrized as $\mathcal{F}_{\mathrm{l}}(J, Q)$, where Q is any stable, possibly time-varying, linear system. Moreover, if $Q \in H_2$, then

$$\|T_{zw}\|_2^2 = \gamma_0 + \|Q\|_2^2,$$

where γ_0 is as defined in Theorem 2.1. $\triangledown$

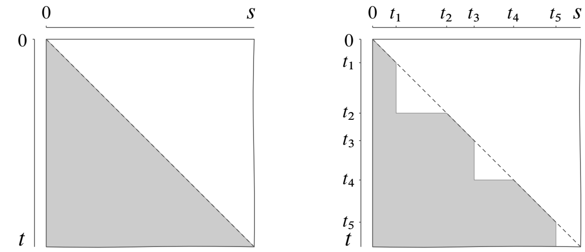
Proof (outline): It is essentially a version of the proof of [9, Lem. A.1]. There is one delicate point missed there though. Namely, the proof of the main result requires a switch from systems operating on the semi-axis $\mathbb{R}_+$ to those operating on the whole axis $\mathbb{R}$. This step is normally done silently and changes nothing in the LTI case. However, it is not obvious for time-varying Q's. Nonetheless, it can be worked out using the property that Hankel operators associated with systems of interest are always Hilbert-Schmidt operators and are thus in the null-space of the H_2 norm (as mentioned in the Appendix). Details, which are quite technical, will be reported elsewhere. ■

If no other constraints (apart from the standard causality) are imposed on K, Lemma 3.1 readily yields the solution to the problem of minimizing $\|T_{zw}\|_2$. Indeed, any choice verifying $\|Q\|_2 = 0$ yields an optimal causal K. In the LTI case, this condition is satisfied by the unique $Q = 0$. This choice is not unique if Q is allowed to be time varying. However, the resulting controllers under these choices do not satisfy $\boldsymbol{\mathcal{A}}_4$ in general. To end up with an admissible K, additional constraints on Q should be imposed.

B. Communication interruptions as constraints on Q

A (sufficiently) broad class of linear controllers $K : y \mapsto u$ acting on the semi-axis $\mathbb{R}_+ := (0, \infty)$ can be described by the kernel representation

$$u(t) = \int_{\mathbb{R}_+} k(t, s) y(s) \mathrm{d}s \tag{9}$$



(a) Causal K (b) K satisfying $\boldsymbol{\mathcal{A}}_4$

Fig. 2. Impulse responses of analog systems in the time domain.

for an associated distribution $k(t, s)$ (impulse response). The class of causal analog controllers corresponds then to kernels that satisfy $k(t, s) = 0$ whenever $s > t$. The impulse response may be visualized as shown in Fig. 2(a), where the unshaded area represents zero values (the lower triangular structure in this case). It is readily seen that the kernels of controllers satisfying $\boldsymbol{\mathcal{A}}_4$ may be visualized as depicted in Fig. 2(b).

Our goal in this part is to apprehend the constraint, which $\boldsymbol{\mathcal{A}}_4$ imposes on Q. It is known [10, Lem. 10.4 (c)] that for J in (8) the mapping $Q \mapsto K$ is bijective, with

$$K = \mathcal{F}_{\mathrm{l}}(J, Q) \iff Q = \mathcal{F}_{\mathrm{u}}(J^{-1}, K) \tag{10}$$

for all causal Q's. Thus, we may be interested to understand how the "white triangles" in Fig. 2(b) are transformed by $\mathcal{F}_{\mathrm{u}}(J^{-1}, K)$. Arguably, a natural approach to do that is via considering the relation between Q and K in the lifted domain, in line with the reasonings in [9].

The lifting transformation associated with a sequence of time instances $\{t_i\}$ maps an analog signal $\xi(t)$ into an equivalent sequence of functions $\{\breve{\xi}[i]\}$ so that at each $i \in \mathbb{Z}_+$

$$\breve{\xi}[i](\theta) = \xi(t_i + \theta), \quad \forall \theta \in [0, t_{i+1} - t_i)$$

see Fig. 3, which visualizes the idea. Any continuous-time

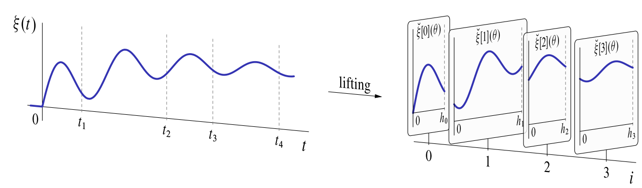
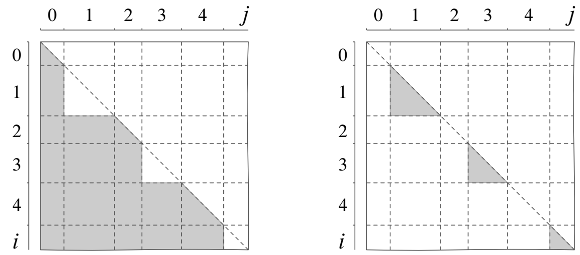


Fig. 3. Lifting transformation with nonuniform time axis partition.

system can then be lifted by lifting its input and output signals, resulting in a discrete-time system with infinite-dimensional i/o spaces. Specifically, controller (9) can be rewritten in the lifted domain as

$$\begin{aligned} \breve{u}[i](\theta) &= \sum_{j \in \mathbb{Z}_+} \int_{t_j}^{t_{j+1}} k(t_i + \theta, s) y(s) \mathrm{d}s \\ &= \sum_{j \in \mathbb{Z}_+} \int_0^{t_{j+1} - t_j} k(t_i + \theta, t_j + \sigma) \breve{y}[j](\sigma) \mathrm{d}\sigma \\ &=: \Big(\sum_{j \in \mathbb{Z}_+} \breve{k}[i, j]\, \breve{y}[j] \Big)(\theta). \end{aligned} \tag{11}$$



(a) $\breve{K}$ satisfying $\boldsymbol{\mathcal{A}_4}$ (b) Corresponding $\breve{Q}_{\text{stat}}$

Fig. 4. Impulse responses of analog systems in the lifted domain.

This relation describes a discrete linear system, denote it $\breve{K}$, whose kernel (impulse response) $\breve{k}[i,j]$ at each (i,j) is an integral operator mapping functions on $[0, t_{j+1}-t_j)$ to functions on $[0, t_{i+1}-t_i)$. In terms of the kernels in Fig. 2, this transformation may be viewed as merely chopping the t- and s-axes into pieces according to $\{t_i\}$. Then, the continuous-time kernel in Fig. 2(b) transforms to the lifted kernel as shown in Fig. 4(a). The latter can be thought of as a form of system matrix as in [11, Sec. 4.1]. The "diagonal" element $\breve{k}[i,i]$ of the lifted kernel, which is an integral operator on $[0, t_{i+1}-t_i)$, is referred to as the ith feedthrough term of $\breve{K}$.

The advantage of representing K in the lifted domain, with the lifted grid as in (2), stems from the fact that $\boldsymbol{\mathcal{A}_4}$ in the lifted domain reads as the condition that

$$\breve{k}[i,i] = 0 \text{ whenever } i \text{ is odd.} \tag{12}$$

This condition is easier to deal with. In particular, it fits well into the LFT framework in (10). Indeed, define

$$R := J^{-1} = \left[\begin{array}{c|cc} A & -B_u & L \\ \hline -C_y & 0 & I \\ F & I & 0 \end{array}\right].$$

Because the lifted $\breve{R}$ and $\breve{K}$ are causal, we have that

$$\breve{q}[i,i] = \mathcal{F}_{\mathrm{u}}(\breve{r}[i,i], \breve{k}[i,i]), \quad \forall i \in \mathbb{Z}_+.$$

Therefore, (12) holds true iff

$$\breve{q}[i,i] = \mathcal{F}_{\mathrm{u}}(\breve{r}[i,i], 0) = \breve{r}_{22}[i,i] \text{ whenever } i \text{ is odd.} \tag{13}$$

In other words, $\boldsymbol{\mathcal{A}_4}$ pre-specifies the odd feedthrough terms of the lifted version of Q and this is the only constraint imposed by it.

To formalize the constraint above, define the lifted system $\breve{Q}_{\text{stat}} : \breve{\epsilon} \mapsto \breve{\eta}$ as the system whose impulse response verifies

$$\breve{q}_{\text{stat}}[i,j] = \begin{cases} \breve{r}_{22}[i,i] & \text{if } j = i \text{ and is odd} \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

This kernel can be visualized as depicted in Fig. 4(b) (the nonzero diagonal blocks are triangular because R_{22} is causal, so the feedthrough terms of its lifting are causal, as operators on $[0, t_{i+1}-t_i)$, too). This $\breve{Q}_{\text{stat}}$ is a static system, in the sense that $\breve{\eta}[i]$ depends only on $\breve{\epsilon}[i]$. Them the following result summarizes the arguments above:

Lemma 3.2: $\breve{K} = \mathcal{F}_{\mathrm{l}}(\breve{J}, \breve{Q})$ is the lifting of a controller satisfying $\boldsymbol{\mathcal{A}_4}$ iff $\breve{Q} = \breve{Q}_{\text{stat}} + \breve{Q}_0$, where Q_{stat} is determined via (14) and $\breve{Q}_0$ is the lifting of a any causal system such that $\breve{q}_0[i,i] = 0$ whenever i is odd. ∇

C. The optimal Q, its H_2 norm and stability

The constraint on admissible Q's in Lemma 3.2 is given in the lifted domain and not readily usable. Transforming it back to the time domain will be discussed later on. But we start this part with one important property of the characterization of Lemma 3.2, which can be understood already in the lifted domain. Namely, it is readily seen that the kernels of $\breve{Q}_{\text{stat}}$ and $\breve{Q}_0$ do not overlap. Hence, the kernels of their time-domain counterparts, Q_{stat} and Q_0, do not overlap either. It then follows from (23) that Q_{stat} is orthogonal to any admissible Q_0. Thus, for every Q resulting in a controller, satisfying $\boldsymbol{\mathcal{A}_4}$, we have that

$$\|Q\|_2^2 = \|Q_{\text{stat}}\|_2^2 + \|Q_0\|_2^2. \tag{15}$$

This property leads us to the following result:

Lemma 3.3: Any $Q_0 \in H_2$ so that $\|Q_0\|_2 = 0$ is optimal for $\mathbf{OP}_{\{t_i\}}$, rendering $\|T_{zw}\|_2^2 = \gamma_0 + \|Q_{\text{opt}}\|_2^2$. ∇

Proof: Follows by combining Lemma 3.1 with (15). ■

The non-uniqueness of Q_0 in Lemma 3.3, which is the reason for the non-uniqueness of the optimal controller in Theorem 2.1, stems from the fact that the kernel of the H_2-norm is nontrivial in the time-varying case. For the sake of simplicity, in what follows we consider only the trivial choice, $Q_0 = 0$, which yields $Q = Q_{\text{stat}}$.

Our next step is to convert $\breve{Q}_{\text{stat}}$ defined by (14) back to the time domain. To this end, note that $R_{22}(s) = F(sI - A)^{-1}L$ and it is causal, so that its kernel

$$r_{22}(t,s) = F\,\mathrm{e}^{A(t-s)}L\,\mathbb{1}(t-s),$$

where $\mathbb{1}(t)$ is the Heaviside step function. It then follows from (11) that $\breve{r}_{22}[i,i] : \breve{\epsilon}[i] \mapsto \breve{\eta}[i]$ acts according to

$$\begin{aligned}\breve{\eta}[i](\theta) &= \int_0^{t_{i+1}-t_i} F\,\mathrm{e}^{A(\theta-\sigma)}L\,\mathbb{1}(\theta-\sigma)\breve{\epsilon}[i](\sigma)\,\mathrm{d}\sigma \\ &= F\int_0^{\theta} \mathrm{e}^{A(\theta-\sigma)}L\,\breve{\epsilon}[i](\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

But this is the very response of the reset system

$$\begin{cases} \dot{x}_r(t) = Ax_r(t) + L\epsilon(t), & x_r(t_i) = 0 \\ \eta(t) = Fx_r(t), \end{cases} \tag{16}$$

in the time domain. Then, taking into account (14), Q_{stat} in the time domain is also described by (16), but only in the "odd" intervals, $[t_{2i+1}, t_{2i+2})$. In the "even" intervals, $[t_{2i}, t_{2i+1})$, the output of Q_{stat} should be $\eta(t) = 0$. A compact representation of these dynamics, which makes use of the function $\iota(t)$ defined by (3), is

$$\begin{cases} \dot{x}_{\text{stat}}(t) = Ax_{\text{stat}}(t) + L\epsilon(t), & x_{\text{stat}}(\iota(t)) = 0 \\ \eta(t) = Fx_{\text{stat}}(t) \end{cases} \tag{17}$$

with convention that the algebraic condition on x_{stat} overrules the differential equation at time intervals where they conflict.

Our next step is to calculate $\|Q_{\text{stat}}\|_2$. Using (24) and taking $T = t_{2i}$ we have:

$$\begin{aligned}\|Q_{\text{stat}}\|_2^2 &= \lim_{i\to\infty} \frac{1}{t_{2i}} \int_0^{t_{2i}} \int_0^{\infty} \|q_{\text{stat}}(t,s)\|_{\mathrm{F}}^2 \,\mathrm{d}t\mathrm{d}s \\ &= \lim_{i\to\infty} \frac{1}{t_{2i}} \sum_{j=1}^{2i} \int_{t_{j-1}}^{t_j} \int_0^{\infty} \|q_{\text{stat}}(t,s)\|_{\mathrm{F}}^2 \,\mathrm{d}t\mathrm{d}s \\ &= \lim_{i\to\infty} \frac{1}{t_{2i}} \sum_{j=1}^{i} \int_{t_{2j-1}}^{t_{2j}} \int_0^{t_{2j}} \|r_{22}(t,s)\|_{\mathrm{F}}^2 \,\mathrm{d}t\mathrm{d}s \\ &= \lim_{i\to\infty} \frac{1}{t_{2i}} \sum_{j=1}^{i} \int_{t_{2j-1}}^{t_{2j}} \int_s^{t_{2j}} \|F\mathrm{e}^{A(t-s)}L\|_{\mathrm{F}}^2 \,\mathrm{d}t\mathrm{d}s,\end{aligned}$$

whence (5) follows by straightforward variable changes.

Finally, Q_{stat} is a reset system (and zero in "even" intervals), so its stability is guaranteed if the "odd" intervals are uniformly bounded. This guarantees the stability of the closed-loop system and this is exactly what Theorem 2.1 claims. If A is Hurwitz, even this is not required.

D. The optimal K

To derive the resulting controller, we may plug (17) into the dynamics of $J : \left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ \epsilon \end{smallmatrix}\right]$. Denoting the state vector of the latter by x_J, we have that $K : y \mapsto u$ verifies

$$\begin{cases} \begin{bmatrix} \dot{x}_J \\ \dot{x}_{\text{stat}} \end{bmatrix} = \begin{bmatrix} A + B_u F + LC_y & B_u F \\ -LC_y & A \end{bmatrix} \begin{bmatrix} x_J \\ x_{\text{stat}} \end{bmatrix} + \begin{bmatrix} -L \\ L \end{bmatrix} y \\ \quad u = \begin{bmatrix} F & F \end{bmatrix} \begin{bmatrix} x_J \\ x_{\text{stat}} \end{bmatrix} \end{cases}$$

with the algebraic condition as in (17). Applying the similarity transform $\left[\begin{smallmatrix} I & 0 \\ I & I \end{smallmatrix}\right]$ and denoting $x_{\mathrm{a}} := x_J + x_{\text{stat}}$, we get

$$\begin{cases} \begin{bmatrix} \dot{x}_J \\ \dot{x}_{\mathrm{a}} \end{bmatrix} = \begin{bmatrix} A + LC_y & B_u F \\ 0 & A + B_u F \end{bmatrix} \begin{bmatrix} x_J \\ x_{\mathrm{a}} \end{bmatrix} - \begin{bmatrix} L \\ 0 \end{bmatrix} y \\ \quad u = \begin{bmatrix} 0 & F \end{bmatrix} \begin{bmatrix} x_J \\ x_{\mathrm{a}} \end{bmatrix} \end{cases}$$

and the algebraic condition $x_{\mathrm{a}}(\iota(t)) = x_J(\iota(t))$. It is readily seen that $x_J \equiv \hat{x}$, where the latter is the state of the Kalman filter (7a). Then, the control signal during the "even" intervals, where $\iota(t) = t$, is $u(t) = F\hat{x}(t)$. Because x_{a} is not controllable via y, during the "odd" intervals, where $\iota(t) = t_{2i-1}$, we have that

$$u(t) = F\mathrm{e}^{(A+B_uF)(t-t_{2i-1})}\hat{x}(t_{2i-1}), \quad \forall t \in [t_{2i-1}, t_{2i}).$$

Combining these intervals we end up with (7b).

This completes the proof of Theorem 2.1. ■

IV. Incorporating Loop Delays

This section presents the extension of the result of Theorem 2.1 to the situation when the information transfer from u to y in the controller is delayed. Specifically, introduce the assumption

$\boldsymbol{\mathcal{A}}_5$: $u(t)$ may depend only on $y(s)$ for $s \le \iota(t) - \tau$ for a given $\tau > 0$

and define the problem $\mathbf{OP}^{\tau}_{\{t_i\}}$ as the version of $\mathbf{OP}_{\{t_i\}}$, where $\boldsymbol{\mathcal{A}}_4$ is replaced with $\boldsymbol{\mathcal{A}}_5$.

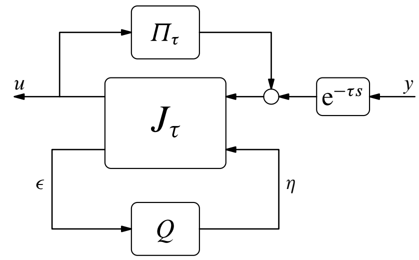


Fig. 5. All suboptimal dead-time controllers

The main result of this section, whose proof is outlined in §IV-A, is the following generalization of Theorem 2.1:

Theorem 4.1: The optimal performance in $\mathbf{OP}^{\tau}_{\{t_i\}}$ is

$$\begin{aligned}\|T_{zw}\|_2^2 = \gamma_0 &+ \int_0^{\tau} \|C_u\mathrm{e}^{At}B_y\|_{\mathrm{F}}^2 \,\mathrm{d}t \\ &+ \lim_{i\to\infty} \frac{1}{t_{2i}} \sum_{j=1}^{i} \int_0^{h_j} \int_0^{s} \|C_u\mathrm{e}^{A(t+\tau)}B_y\|_{\mathrm{F}}^2 \,\mathrm{d}t\mathrm{d}s, \quad (18)\end{aligned}$$

where $h_j := t_{2j} - t_{2j-1}$ is the duration of the jth interruption and γ_0 is given by (6), and this performance is an increasing function of τ. A (non-unique) controller attaining (18) is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \mathrm{e}^{-A\tau}B_u u(t+\tau) - L(y_\tau(t) - C_y\hat{x}(t)) \quad (19a)$$
$$u(t) = F\mathrm{e}^{(A+B_uF)(t-\iota(t))}\mathrm{e}^{A\tau}\hat{x}(\iota(t) - \tau), \quad (19b)$$

where the "compensated" output

$$y_\tau(t) = y(t) + C_y \int_{t-\tau}^{t} \mathrm{e}^{A(t-\tau-s)}B_u u(s+\tau)\mathrm{d}s. \quad (19c)$$

The control law (19) is stabilizing iff either A is Hurwitz or there is $\overline{\alpha} > 0$ such that $\sup_{i\in\mathbb{N}}(t_{2i} - t_{2i-1}) \le \overline{\alpha}$. ▽

Remark 4.1 (implementation): To implement the optimal controller (19) we need to run the Kalman filter (19a) and the "dead-time compensator" (19c) at the sensor side K_{s} and two copies of the control signal generator (19b), one at the sensor side to produce u for (19a) and (19c) and one at the actuator side K_{a} to generate the actual control signal. Both $u(t+\tau)$ and $y_\tau(t)$ in (19a) are based on the control signal in the time interval $[t, t+\tau]$. But u itself, see (19b), depends on the τ-delayed version of $\hat{x}$, so the implementation of all components of K_{s} is causal. Then the information transfer from K_{s} to K_{a} involves both the delay and interruptions. ▽

A. Proof of Theorem 4.1 (outline)

Assume again that $D'_{zu}D_{zu} = I$ and $D_{yw}D'_{yw} = I$. As in the delay-free case, the first step in solving $\mathbf{OP}^{\tau}_{\{t_i\}}$ is to parametrize all suboptimal solutions of an uninterrupted version of the controller. In this case, such a parametrization can be found in [12, IV-A], which considers the H_2 control of time-delay systems. While [12] studies only time-invariant controllers, its arguments extend to time-varying controllers seamlessly. All suboptimal solutions to the non-interrupted version of $\mathbf{OP}^{\tau}_{\{t_i\}}$ can be expressed in the form depicted in

Fig. 5, where

$$J_\tau = \left[\begin{array}{c|cc} A + B_u F + \mathrm{e}^{A\tau} L C_y \mathrm{e}^{-A\tau} & -\mathrm{e}^{A\tau} L & B_u \\ \hline F & 0 & I \\ -C_y \mathrm{e}^{-A\tau} & I & 0 \end{array}\right] \qquad (20)$$

and $\varPi_\tau$ is the LTI system having the transfer function

$$\varPi_\tau(s) = C_y \int_0^\tau \mathrm{e}^{A(t-\tau)} \mathrm{e}^{-st} \mathrm{d}t B_u. \qquad (21)$$

The latter is an entire function of s, bounded in $\operatorname{Re} s > 0$, so it is stable. The impulse response of $\varPi_\tau$ has support in $[0, \tau]$. The performance attainable by any such controller is

$$\|T_{zw}\|_2^2 = \gamma_0 + \int_0^\tau \|C_u \mathrm{e}^{At} B_y\|_{\mathrm{F}}^2 \, \mathrm{d}t + \|Q\|_2^2.$$

The next step is to express $\boldsymbol{\mathcal{A}_5}$ in terms of Q. To this end, denote by $\tilde{K}$ the delay-free part of the controller in Fig. 5, so that $K = \tilde{K} \mathrm{e}^{-\tau s}$. Then the mapping $Q \mapsto \tilde{K}$ is bijective, with the inverse $Q = \mathcal{F}_{\mathrm{u}}(R_\tau, \tilde{K})$, where

$$R_\tau := J_\tau^{-1} + \left[\begin{smallmatrix} \varPi_\tau & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

Repeating the arguments of §§III-B and III-C, the optimal choice of Q under communication interruptions is built on the $(2, 2)$ sub-block of this R_τ, which is that of J_τ^{-1}. Hence, the optimal $Q = Q_{\mathrm{stat}}$, where Q_{stat} is described by

$$\begin{cases} \dot{x}_{\mathrm{stat}}(t) = A x_{\mathrm{stat}}(t) + \mathrm{e}^{A\tau} L \epsilon(t), \quad x_{\mathrm{stat}}(\iota(t)) = 0 \\ \quad \eta(t) = F x_{\mathrm{stat}}(t) \end{cases} \qquad (22)$$

These dynamics differ from (17) only by the factor $\mathrm{e}^{A\tau}$, so its norm is exactly the last term in the right-hand side of (18). The fact that the resulting $\|T_{zw}\|_2$ is an increasing function of τ follows by the very fact that as τ increases, the restriction imposed by $\boldsymbol{\mathcal{A}_5}$ on the controller becomes more severe.

The optimal controller (19) is then derived by the arguments of §III-D. Just note that the second term in the right-hand side of (19c) is exactly the output of $\mathrm{e}^{\tau s} \varPi_\tau u$. Finally, the stability conditions for (22) are not different from those of (17). This completes the proof of Theorem 4.1. ■

V. Concluding Remarks

The paper has studied the standard continuous-time H_2 problem under irregular and unknown a priori communication interruptions in the controller. A closed-form solution has been derived and the optimal attainable performance has been characterized. The solution is transparent and computationally simple. In particular, it is based on the same Riccati equations and optimal gains as the uninterrupted version of the problem. It has also been shown how to incorporate a constant loop delay. The presence of such a delay adds a predictor block to the sensor-side part of the controller, but does not alter computational procedures.

Immediate extensions include the H_∞ performance, which can also be treated in line with the approach in [9], and the coordination of homogeneous agents, like in the setup of [13]. A discrete version of the problem, which is more practical in networked systems, can be treated similarly.

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Appendix

H_2 Space for Time-Varying Linear Systems

This Appendix collects properties of linear time-varying causal systems with square integrable impulse responses, relevant for the developments in this paper. Some definitions can be found in [14], others are adopted from their better studied time-invariant counterparts.

Let $G : w \mapsto z$ be a linear causal system on the semi-axis $\mathbb{R}_+$ described by its kernel representation

$$z(t) = \int_0^\infty g(t, s) w(s) \mathrm{d}s,$$

where the distribution $g(t, \tau)$ is the impulse response (kernel) of G. In the causal case $g(t, s) = 0$ whenever $s > t$. With some abuse of notation, we say that $G \in H_2$ if the Frobenius norm of its kernel $\|g(\cdot, s)\|_{\mathrm{F}} \in L_2(\mathbb{R}_+)$, $\forall s \in \mathbb{R}_+$. The set of all $G \in H_2$ is a vector space, referred to as H_2. It can be endowed with the degenerate inner product

$$\langle G_1, G_2 \rangle_2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^\infty \operatorname{tr}(g_2'(t, s) g_1(t, s)) \, \mathrm{d}t \mathrm{d}s, \quad (23)$$

rendering H_2 a degenerate Hilbert space with the (semi) norm

$$\|G\|_2 = \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^\infty \|g(t, s)\|_{\mathrm{F}}^2 \, \mathrm{d}t \mathrm{d}s \right)^{1/2}. \qquad (24)$$

This is a semi-norm as $\|G\|_2 = 0 \;\not\Longrightarrow\; G = 0$. In particular, it can be shown that all bounded Hilbert-Schmidt operators, like the Hankel operators associated with BIBO-stable LTI systems, have zero H_2 norm.