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System Transformation Induced by Generalized Orthonormal Basis Functions Preserves Dissipativity

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***Abstract*— This paper discusses the dissipativity of transformed systems induced by generalized orthonormal basis functions. The dissipativity includes both passivity and the finite system gain property. We prove that the transformed system preserves the dissipativity and shares a common storage function when the original continuous-time system is dissipative. The results further illuminates the properties of this class of system transformation.**

Keywords. dissipativity, shift-invariant subspace, system transformation

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I. INTRODUCTION

A number of continuous-discrete conversion methods for linear systems are available. For example, the `c2d` command in MATLAB includes the zero-order hold, first-order hold, impulse-invariant mapping, Tustin approximation, and zero-pole matching equivalents methods [1]. Each method focuses on some properties of the input-output relation, and should be selected in accordance with applications.

The lifting technique [2] is proposed in the context of robust control for sampled-data systems. A discrete-time system is constructed from a continuous-time system where the input and output spaces are “lifted” to functional spaces and the input-output relation is preserved. The idea is extended in [3] using the orthogonal complement of the shift invariant space generated by an inner function to calculate Hankel singular values and vectors, which are useful for the H^∞ sensitivity minimization problem and the rational approximation problem for a class of infinite dimensional systems.

When the inner function is rational, the transformation derived in [3] is equivalent to the so called Hambo system transform (see [4]), when the orthogonal complement of the shift invariant space of the inner function is finite-dimensional and is equipped with an orthonormal basis. The basis of the orthogonal complement is extended to a basis of the whole signal space, consisting of so-called generalized orthonormal basis functions. The properties and applications of the Hambo system transform are discussed in [5], [6], [7], [8], and its relation with the lifting technique is discussed in [9].

A discretization technique is also investigated in the area of mechanics. The variational approach to discrete mechanics is summarized in a review paper [10], where the discretization of Lagrangian and Hamiltonian mechanics and integration algorithms are discussed. In [11], the technique is applied to the identification problem of mechanical systems, and it was shown that the mid-point rule applied to linear systems yields the Tustin transform, which preserves passivity.

The notion of dissipativity includes passivity as a special case and is discussed extensively in [12], [13], [14]. The notion of passivity is useful in analyzing the stability of feedback systems when the forward and the feedback paths are both passive. Standard methods for continuous-discrete conversion do not in general preserve passivity (see e.g. [15], [16], [17]), the question arises whether the system transformation in [3] and the Hambo transform in [4] preserve it instead.

In this paper, we shall show that the system transformation induced by the orthogonal complement of a shift invariant space and the Hambo system transform preserve dissipativity. Furthermore, we show that the continuous-time system and the transformed discrete-time system share a common storage function.

Here, $\mathbb{Z}_+$ denotes the set of nonnegative integers, and $\mathbb{R}$ the set of real numbers. For a complex number s, $\overline{s}$ denotes its complex conjugate. For a matrix or a vector, $^{\mathrm{T}}$ denotes the transposition.

II. PRELIMINARIES

A. Signal Spaces

The space of square integrable functions of time $0 < t < \infty$ is denoted as $L^2(0, \infty)$. The norm of $u \in L^2(0, \infty)$ is defined as

$$\|u\| := \sqrt{\int_0^\infty |u(t)|^2 \, dt} \; .$$

The Hardy space H^2 is the space of analytic functions on the right half place with the norm

$$\|\hat{u}\| := \sup_{\sigma>0} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(\sigma + j\omega)|^2 \, d\omega} \; .$$

The spaces $L^2(0, \infty)$ and H^2 can be identified with each other via the Fourier transform.

A function ϕ is called inner if it is analytic and bounded in the right half plane, and it satisfies $|h(j\omega)| = 1$ for almost all ω. The space ϕH^2 is called a shift invariant subspace, and its orthogonal complement $\mathcal{S} := H^2 \ominus \phi H^2$ is important because the following equation holds:

$$H^2 = \oplus_{k=0}^{\infty} \phi^k \mathcal{S} \; . \quad (1)$$

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It follows from (1) that every $\hat{u} \in H^2$ can be written as

$$\hat{u} = \sum_{k=0}^{\infty} \phi^k \hat{u}_k \ , \quad \hat{u}_k \in \mathcal{S} \ , \tag{2}$$

which means that H^2 is identified with $\ell^2_{\mathcal{S}}(\mathbb{Z}_+)$, the space of square summable sequences taking values in $\mathcal{S}$.

B. System Transformation

Consider a continuous-time state space realization of a transfer function $h(s)$,

$$\begin{cases} \dfrac{d}{dt}x = Ax + Bu, \\ \quad y = Cx + Du, \end{cases} \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ is assumed to be a stable matrix. To avoid clumsy notation, we assume that the system (3) is single-input single-output. When an input $u \in L^2(0,\infty)$ is applied to (3) with zero initial condition then the output y satisfies $y \in L^2(0,\infty)$. In the frequency domain, the equality $\hat{y} = h\hat{u}$ holds, where $\hat{u}$, $\hat{y} \in H^2$.

When H^2 is identified with $\ell^2_{\mathcal{S}}(\mathbb{Z}_+)$, the commutative diagram (4) defines the map h_D.

$$\begin{array}{ccc} H^2 & \xrightarrow{\ h\ } & H^2 \\ \downarrow & & \downarrow \\ \ell^2_{\mathcal{S}}(\mathbb{Z}_+) & \xrightarrow{\ h_D\ } & \ell^2_{\mathcal{S}}(\mathbb{Z}_+) \end{array} \tag{4}$$

We are interested in how the map h_D is characterized. The following result is in [9, Proposition 1 p. 523].

Proposition 1: Consider the map $h : H^2 \to H^2$ defined by the continuous-time state equation (3). Let ϕ be an inner function such that ϕ and its paraconjugate $\phi^{\sim}$ defined by $\phi^{\sim}(s) := \phi(-s)$ are analytic at the spectrum of A. Then the map $h_D : \ell^2_{\mathcal{S}}(\mathbb{Z}_+) \to \ell^2_{\mathcal{S}}(\mathbb{Z}_+)$ defined by the commutative diagram (4) coincides with the input-output relation of the discrete-time state equation

$$\begin{cases} \xi_{t+1} = \mathbf{A}\xi_t + \mathbf{B}\hat{u}_t, \\ \quad \hat{y}_t = \mathbf{C}\xi_t + \mathbf{D}\hat{u}_t, \end{cases} \tag{5}$$

where $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B} : \mathcal{S} \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to \mathcal{S}$, $\mathbf{D} : \mathcal{S} \to \mathcal{S}$ are defined by

$$\begin{aligned} \mathbf{A}\xi &:= \phi^{\sim}(A)\xi \ , \\ \mathbf{B}\hat{u} &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\Big(\phi^{\sim}(A)\,(j\omega I + A)^{-1} B} \\ &\qquad \overline{-\phi(j\omega)\,(j\omega I + A)^{-1} B}\Big)\hat{u}(j\omega)d\omega \ , \\ (\mathbf{C}\xi)(s) &:= \Big(C\,(sI - A)^{-1} \\ &\qquad -\phi(s)C\,(sI - A)^{-1}\,\phi^{\sim}(A)\Big)\,\xi \ , \\ (\mathbf{D}\hat{u})(s) &:= h(s)\hat{u}(s) - \phi(s)C\,(sI - A)^{-1}\,\mathbf{B}\hat{u} \ . \end{aligned}$$

C. Generalized Orthonormal Basis Functions

If an inner function $\phi(s)$ is rational of degree n_ϕ, then the space $\mathcal{S} = H^2 \ominus \phi H^2$ is n_ϕ dimensional. Let

$$\phi(s) = D_\phi + C_\phi\,(sI - A_\phi)^{-1} B_\phi, \tag{6}$$

where $A_\phi \in \mathbb{R}^{n_\phi \times n_\phi}$, be a balanced realization of $\phi(s)$. Then [4] showed that an orthonormal basis of the space $\mathcal{S}$ can be constructed from the elements of

$$\hat{v}(s) := \begin{bmatrix} \hat{v}_1(s) & \cdots & \hat{v}_{n_\phi}(s) \end{bmatrix} := C_\phi\,(sI - A_\phi)^{-1}. \tag{7}$$

If $\phi(s)$ is a first order rational inner function

$$\phi(s) := \frac{\lambda - s}{\lambda + s} \ , \quad \lambda > 0 \ , \tag{8}$$

then

$$\hat{v}(s) = \frac{\sqrt{2\lambda}}{\lambda + s} \tag{9}$$

is an orthonormal element in $\mathcal{S}$. An orthonormal basis of H^2 is given by $\{\hat{v}, \phi\hat{v}, \ldots, \phi^k\hat{v}, \ldots\}$, which is called the Laguerre basis.

When $n_\phi > 1$, an orthonormal basis of H^2 is constructed similarly from the basis of $\mathcal{S} = H^2 \ominus \phi H^2$ by multiplying $\phi(s)$ sequentially as follows:

$$\begin{aligned} \{\hat{v}_1, \ldots, \hat{v}_{n_\phi}, \phi\hat{v}_1, \ldots, \phi\hat{v}_{n_\phi}, \\ \ldots, \phi^k\hat{v}_1, \ldots, \phi^k\hat{v}_{n_\phi}, \ldots\} \ , \end{aligned} \tag{10}$$

which is called generalized orthonormal basis functions.

When the subspace $\mathcal{S} = H^2 \ominus \phi H^2$ is equipped with a basis, then $\mathcal{S}$ is identified with $\mathbb{R}^{n_\phi}$. Thus, the system $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ defined in Proposition 1 can be described by a discrete-time system whose input-output map $\tilde{h}_D$ satisfies the commutative diagram shown below:

$$\begin{array}{ccc} H^2 & \xrightarrow{\ h\ } & H^2 \\ \downarrow & & \downarrow \\ \ell^2_{\mathcal{S}}(\mathbb{Z}_+) & \xrightarrow{\ h_D\ } & \ell^2_{\mathcal{S}}(\mathbb{Z}_+) \\ \downarrow & & \downarrow \\ \ell^2_{\mathbb{R}^{n_\phi}}(\mathbb{Z}_+) & \xrightarrow{\ \tilde{h}_D\ } & \ell^2_{\mathbb{R}^{n_\phi}}(\mathbb{Z}_+) \end{array} \tag{11}$$

Via the commutative diagram (11), the transformed system (5) described by the operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ is now identified with the discrete-time linear system realizing $\tilde{h}_D$:

$$\begin{cases} \xi_{t+1} = \tilde{A}\xi_t + \tilde{B}\tilde{u}_t \ , \\ \quad \tilde{y}_t = \tilde{C}\xi_t + \tilde{D}\tilde{u}_t \ . \end{cases} \tag{12}$$

The system matrices $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are defined in the following proposition (see [9, Theorem 3 p. 525] for the proof). For this, define X and Y as the unique solutions to the Sylvester equations

$$AX + XA_\phi^{\mathrm{T}} + BB_\phi^{\mathrm{T}} = 0 \ , \tag{13}$$

$$A_\phi^{\mathrm{T}}Y + YA + C_\phi^{\mathrm{T}}C = 0 \ , \tag{14}$$

respectively.

Proposition 2: Consider the stable continuous-time system (3). Let (6) be a minimal balanced realization of a rational inner function $\phi(s)$. Let X and Y be defined by the Sylvester equations (13) and (14), respectively. Then the system matrices of (12) are given by

$$\begin{aligned}\tilde{A} &= \phi^{\sim}(A) \ , \\ \tilde{B} &= X \ , \\ \tilde{C} &= Y \ , \\ \tilde{D} &= h^{\sim}(A_{\phi}^{\mathrm{T}}) \ .\end{aligned}$$

Remark 1: When the inner function is first order and is given by (8), the matrices in Proposition 2 are as follows:

$$\begin{aligned}\tilde{A} &= (\lambda I + A)(\lambda I - A)^{-1} \ , \\ \tilde{B} &= \sqrt{2\lambda}\,(\lambda I - A)^{-1} B, \\ \tilde{C} &= \sqrt{2\lambda} C\,(\lambda I - A)^{-1} \ , \\ \tilde{D} &= h(\lambda) = D + C\,(\lambda I - A)^{-1} B \ .\end{aligned} \tag{15}$$

III. DISSIPATIVITY

In this section we summarize the relevant results on dissipative linear dynamical systems with quadratic supply rates. See [13], [14] for more detail.

Consider a continuous-time system described by a minimal realization

$$\begin{cases} \dfrac{d}{dt}x = Ax + Bu \ , \\ \quad y = Cx + Du \ , \end{cases} \tag{16}$$

and consider the following quadratic functional of the output and the input:

$$S(u,y) = \begin{bmatrix} u^{\mathrm{T}} & y^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R & S^{\mathrm{T}} \\ S & Q \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \ , \tag{17}$$

where $R = R^T$, S, $Q = Q^T$ are matrices of appropriate dimensions.

Definition 1: The system (16) is dissipative with respective to the supply rate (17) if there exists a storage function $V(x)$ such that for every (x, u, y) trajectory satisfying (16) it holds that

$$\dot{V}(x) \leq S(u,y), \tag{18}$$

where $\dot{V}(x)$ is the derivative along the trajectory of (16).
It can be shown (see [14, Theorem 8.4.1 p. 210]) that if V is a storage function, then there exists a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$ such that $V(x) = x^T P x$.

Definition (18) and $V(x) = x^T P x$ are equivalent with the following matrix inequality:

$$\begin{bmatrix} A^{\mathrm{T}}P + PA - C^{\mathrm{T}}QC & PB - C^{\mathrm{T}}(S + QD) \\ \left(PB - C^{\mathrm{T}}(S+QD)\right)^{\mathrm{T}} & -\left(D^{\mathrm{T}}QD + \left(D^{\mathrm{T}}S + S^{\mathrm{T}}D\right) + R\right) \end{bmatrix} \leq 0 \tag{19}$$

The transfer function of a system dissipative with respect to the supply rate (17) with $R = 0$, $Q = 0$, and $S = I$ is called positive-real; in that case it can be proved (see [14, Corollary 8.5.1]) that there exists a nonnegative storage function, i.e. $P \geq 0$. An analogous result (see [14, Corollary 8.5.2]) holds when a system has the finite system gain property, i.e. when it is dissipative with respect to the supply rate $R = I$, $Q = -I$, and $S = 0$. In such case its transfer function is called bounded-real.

The notion of dissipativity can be extended to systems written by discrete-time state equations (see e.g. [18] for the nonlinear case or [19]); consider a minimal realization of a linear discrete-time system:

$$\begin{cases} x(t+1) = \tilde{A}x(t) + \tilde{B}u(t), \\ \quad\ y(t) = \tilde{C}x(t) + \tilde{D}u(t). \end{cases} \tag{20}$$

Definition 2: The system (20) is dissipative with respective to the supply rate (17) if there exists a storage function $V(x)$ such that

$$V(x(t+1)) - V(x(t)) \leq S(u(t), y(t)). \tag{21}$$

Straightforward manipulations show that the system (16) is dissipative if and only if the following matrix inequality has a solution P:

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}P\tilde{A} - P - \tilde{C}^{\mathrm{T}}Q\tilde{C} & \tilde{A}^{\mathrm{T}}P\tilde{B} - \tilde{C}^{\mathrm{T}}\left(S + Q\tilde{D}\right) \\ \left(\tilde{A}^{\mathrm{T}}P\tilde{B} - \tilde{C}^{\mathrm{T}}\left(S + Q\tilde{D}\right)\right)^{\mathrm{T}} & -\left(\tilde{D}^{\mathrm{T}}Q\tilde{D} + \left(\tilde{D}^{\mathrm{T}}S + S^{\mathrm{T}}\tilde{D}\right) + R\right) + \tilde{B}^{\mathrm{T}}P\tilde{B} \end{bmatrix} \leq 0. \tag{22}$$

IV. DISSIPATIVITY OF TRANSFORMED SYSTEM

This section establishes the main result of this paper, namely that a continuous-time dissipative system and its transformed discrete-time system share a common storage function. The result is first proved in a stronger form for a first order inner function, and then it is proved for general rational inner functions.

A. First Order Inner Functions

When an inner function is given by (8), the transformed system (12) has the system matrices shown in equation (15) of Remark 1. Note that the transformed system is also single-input, single-output.

Theorem 1: Let the first order inner function ϕ be given by (8). The following conditions are equivalent:

1) The continuous-time system (3) is dissipative with respect to the supply rate (17), and $P \in \mathbb{R}^{n\times n}$ induces a storage function;
2) The discrete-time system defined by (15) is dissipative with respect to the supply rate (17), and $P \in \mathbb{R}^{n\times n}$ induces a storage function.

Proof: Denote the left hand sides of (19) and (22) by M and $\tilde{M}$, respectively. Then, by a direct calculation, we have

$$\tilde{M} = \begin{bmatrix} \sqrt{2\lambda}\left(\lambda I - A^{\mathrm{T}}\right)^{-1} & 0 \\ B^{\mathrm{T}}\left(\lambda I - A^{\mathrm{T}}\right)^{-1} & I \end{bmatrix} M$$

$$\begin{bmatrix} \sqrt{2\lambda}\,(\lambda I - A)^{-1} & (\lambda I - A)^{-1} B \\ 0 & I \end{bmatrix},$$

i.e. M and $\tilde{M}$ are congruent with each other. The equivalence of the two statements follows. ∎

From Theorem 1 we conclude that for $n_\phi = 1$, the dissipativity of the original and of the transformed system are equivalent. In the next section we prove a weaker statement, namely that if the continuous-time system is dissipative, then the transformed system is also dissipative (with respect to a suitably defined supply rate).

B. Higher-Order Rational Inner Functions

When the inner function $\phi(s)$ is rational of degree $n_\phi \geq 2$, the transformed system (12) has the system matrices defined in Proposition 2. The dimensions of the input and output spaces are n_ϕ times larger than those of the original continuous-time system (3). Hence, we need to consider a new supply rate for (12) induced by (17); this is defined by

$$\tilde{S}(\tilde{u},\tilde{y}) := \begin{bmatrix}\tilde{u}^{\mathrm{T}} & \tilde{y}^{\mathrm{T}}\end{bmatrix} \begin{bmatrix}\tilde{R} & \tilde{S}^{\mathrm{T}} \\ \tilde{S} & \tilde{Q}\end{bmatrix} \begin{bmatrix}\tilde{u} \\ \tilde{y}\end{bmatrix}, \tag{23}$$

$$\tilde{Q} := I_{n_\phi} \otimes Q,\ \tilde{R} := I_{n_\phi} \otimes R,\ \tilde{S} := I_{n_\phi} \otimes S, \tag{24}$$

where $\otimes$ denotes the Kronecker product and I_{n_ϕ} the identity matrix of size n_ϕ.

In this section we prove the following generalization of the necessary condition of Theorem 1 to the rational, higher-order case.

Theorem 2: Let the rational inner function ϕ have order n_ϕ, with a balanced realization (6). Assume that the continuous-time system (3) is dissipative with respect to the supply rate (17), and $P \in \mathbb{R}^{n\times n}$ induces a storage function. Then the discrete-time system (12) whose system matrices are given by Proposition 2 is dissipative with respect to the supply rate (23), and $P \in \mathbb{R}^{n\times n}$ induces a storage function.

The proof of Theorem 2 is deferred until we prove a couple of preliminary results.

First, we shall consider the dissipativity of periodic systems. Consider a set of discrete-time linear systems

$$\begin{cases} x(t+1) = \tilde{A}_i x(t) + \tilde{B}_i u(t), \\ y(t) = \tilde{C}_i x(t) + \tilde{D}_i u(t) \end{cases} \tag{25}$$

for $i = 1, 2, \ldots, n_\phi$. Now define a periodic linear, time-varying discrete-time system by the equations

$$\begin{cases} x(t+1) = \tilde{A}_i x(t) + \tilde{B}_i u(t), \\ y(t) = \tilde{C}_i x(t) + \tilde{D}_i u(t), \\ i = t - n_\phi \left\lfloor \dfrac{t}{n_\phi} \right\rfloor + 1 \,. \end{cases} \tag{26}$$

For $i_1 \geq i_0$, we denote the matrix product of the A_i's by

$$\prod_{i=i_0}^{i_1} A_i := A_{i_1} \cdots A_{i_0+1} A_{i_0} \,.$$

Note that matrices with larger indices are multiplied from the left. If $i_0 > i_1$, then

$$\prod_{i=i_0}^{i_1} A_i := I_{n_\phi} \,.$$

Lemma 1: Assume that each discrete-time system in the set (25) is dissipative with respect to the supply rate (17), and that they share a common storage function $V(x) = x^{\mathrm{T}} P x$. Then the periodic linear, time-varying system (26) is dissipative with respect to the supply rate (17), with $\tilde{Q}$, $\tilde{R}$, and $\tilde{S}$ defined by (24). Define

$$\tilde{A} := \prod_{i=1}^{n_\phi} \tilde{A}_i,$$

$$\tilde{B} := \begin{bmatrix} \prod_{i=2}^{n_\phi} \tilde{A}_i \tilde{B}_1 & \cdots & \tilde{A}_{n_\phi} \tilde{B}_{n_\phi - 1} & \tilde{B}_{n_\phi} \end{bmatrix},$$

$$\tilde{C} := \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \tilde{A}_1 \\ \vdots \\ \tilde{C}_{n_\phi} \prod_{i=1}^{n_\phi - 1} A_i \end{bmatrix},$$

$$\tilde{D} := \begin{bmatrix} \tilde{D}_1 & 0 & \cdots & 0 \\ \tilde{C}_2 \tilde{B}_1 & \tilde{D}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_{n_\phi} \prod_{i=2}^{n_\phi - 1} \tilde{A}_i \tilde{B}_1 & \tilde{C}_{n_\phi} \prod_{i=3}^{n_\phi - 1} \tilde{A}_i \tilde{B}_2 & \cdots & \tilde{D}_{n_\phi} \end{bmatrix}.$$

Then P satisfies

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}} P \tilde{A} - P - \tilde{C}^{\mathrm{T}} \tilde{Q} \tilde{C} & \tilde{A}^{\mathrm{T}} P \tilde{B} - \tilde{C}^{\mathrm{T}} \left(\tilde{S} + \tilde{Q}\tilde{D}\right) \\ \left(\tilde{A}^{\mathrm{T}} P \tilde{B} - \tilde{C}^{\mathrm{T}} \left(\tilde{S} + \tilde{Q}\tilde{D}\right)\right)^{\mathrm{T}} & -\left(\tilde{D}^{\mathrm{T}} \tilde{Q} \tilde{D} + \left(\tilde{D}^{\mathrm{T}} \tilde{S} + \tilde{S}^{\mathrm{T}} \tilde{D}\right) + \tilde{R}\right) + \tilde{B}^{\mathrm{T}} P \tilde{B} \end{bmatrix} \leq 0 \,. \tag{27}$$

Proof: It is a matter of straightforward verification to check that the periodic system (26) satisfies

$$\begin{cases} x((t+1)n_\phi) = \tilde{A} x(t n_\phi) + \tilde{B} \begin{bmatrix} u(t n_\phi) \\ u(t n_\phi + 1) \\ \vdots \\ u(t n_\phi + n_\phi - 1) \end{bmatrix}, \\ \begin{bmatrix} y(t n_\phi) \\ y(t n_\phi + 1) \\ \vdots \\ y(t n_\phi + n_\phi - 1) \end{bmatrix} = \tilde{C} x(t n_\phi) + \tilde{D} \begin{bmatrix} u(t n_\phi) \\ u(t n_\phi + 1) \\ \vdots \\ u(t n_\phi + n_\phi - 1) \end{bmatrix}. \end{cases} \tag{28}$$

From the assumption of dissipativity of each system (25),

$$V(x(t+1)) - V(x(t)) - S\left(u(t), y(t)\right) \leq 0$$

holds for $t = 0, 1, 2, \ldots$. It follows that

$$V(x(n_\phi)) - V(x(0)) - \sum_{t=0}^{n_\phi - 1} S\left(u(t), y(t)\right) \leq 0,$$

which implies that the system (28) is dissipative with respect to the supply rate (23), and with the same storage function $x^T Px$. This means that P satisfies the inequality (27). ■

The following Lemma was proved in [3, Lemma 1] and is included here without proof.

Lemma 2: Let A be a square matrix and let ξ be a vector of compatible size. Suppose that ϕ is an inner function and that ϕ and $\phi^{\sim}$ are analytic at the eigenvalues of A. Define

$$g(s) := (sI - A)^{-1}\xi - \phi(s)\,(sI - A)^{-1}\phi^{\sim}(A)\xi\,.$$

Then $g \in H^2 \ominus \phi H^2$.

For the sake of simplicity in the following we assume that the zeros and the poles of ϕ are real (the following results can be generalized in a straightforward way to the case of complex poles by taking complex conjugates when necessary). So $\lambda_i > 0$, $i = 1, 2, \ldots, n_\phi$. Define

$$\phi_i(s) := \frac{(\lambda_i - s)}{(\lambda_i + s)}\ ,\ i = 1, \ldots, n_\phi\ , \tag{29}$$

and note that

$$\phi(s) = \frac{(\lambda_1 - s)(\lambda_2 - s)\cdots(\lambda_{n_\phi} - s)}{(\lambda_1 + s)(\lambda_2 + s)\cdots(\lambda_{n_\phi} + s)} = \prod_{i=1}^{n_\phi}\phi_i(s)\ . \tag{30}$$

A balanced realization of $\phi_i(s)$ is $\left(-\lambda_i, \sqrt{2\lambda_i}, \sqrt{2\lambda_i}, -1\right)$, $i = 1, \ldots, n_\phi$, and a balanced realization of $\phi(s)$ is constructed as follows (this result can be also found in [8, Proposition 2.4]).

Lemma 3: Suppose $\phi(s)$ is an inner function given by (30), and define

$$A_\phi = \begin{bmatrix} -\lambda_1 & 2\sqrt{\lambda_1\lambda_2} & \cdots & (-1)^{n_\phi}2\sqrt{\lambda_1\lambda_{n_\phi}} \\ 0 & \lambda_2 & \cdots & (-1)^{n_\phi-1}2\sqrt{\lambda_2\lambda_{n_\phi}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda_{n_\phi} \end{bmatrix},$$

$$B_\phi = \begin{bmatrix} (-1)^{n_\phi-1}\sqrt{2\lambda_1} \\ (-1)^{n_\phi}\sqrt{2\lambda_2} \\ \vdots \\ \sqrt{2\lambda_{n_\phi}} \end{bmatrix},$$

$$C_\phi = \begin{bmatrix} \sqrt{2\lambda_1} & -\sqrt{2\lambda_2} & \cdots & (-1)^{n_\phi-1}\sqrt{2\lambda_{n_\phi}} \end{bmatrix},$$

$$D_\phi = (-1)^{n_\phi}.$$

Then $(A_\phi, B_\phi, C_\phi, D_\phi)$ is a balanced realization of $\phi(s)$. Furthermore, the following equations hold:

$$(sI - A_\phi)^{-1}B_\phi = \begin{bmatrix} \prod_{i=2}^{n_\phi}\phi_i(s)\frac{\sqrt{2\lambda_1}}{\lambda_1+s} \\ \vdots \\ \phi_{n_\phi}(s)\frac{\sqrt{2\lambda_{n_\phi-1}}}{\lambda_{n_\phi-1}+s} \\ \frac{\sqrt{2\lambda_{n_\phi}}}{\lambda_{n_\phi}+s} \end{bmatrix}, \tag{31}$$

$$C_\phi\,(sI - A_\phi)^{-1} \tag{32}$$

$$= \begin{bmatrix} \frac{\sqrt{2\lambda_1}}{\lambda_1+s} & \phi_1(s)\frac{\sqrt{2\lambda_2}}{\lambda_2+s} & \cdots & \prod_{i=1}^{n_\phi-1}\phi_i(s)\frac{\sqrt{2\lambda_{n_\phi}}}{\lambda_{n_\phi}+s} \end{bmatrix}.$$

Proof: Consider the series connection shown in Fig. 1, where each block is a first order inner function having a balanced realization $\left(-\lambda_i, \sqrt{2\lambda_i}, \sqrt{2\lambda_i}, -1\right)$. If the state vector of a realization of $\phi(s)$ is constructed by stacking the states of $\phi_i(s)$, $i = 1, \ldots, n_\phi$, then we have the realization $(A_\phi, B_\phi, C_\phi, D_\phi)$.

From Fig. 1 it follows that the transfer function from the state to the output is given by (32), and that from the input to the state is given by (31). Denote by $\langle\cdot,\cdot\rangle$ the inner product in H^2. If $i \le j$, then

$$\left\langle \prod_{k=1}^{i-1}\phi_k\frac{\sqrt{2\lambda_i}}{\lambda_i + s}, \prod_{k=1}^{j-1}\phi_k\frac{\sqrt{2\lambda_j}}{\lambda_j + s} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i + s}, \prod_{k=i}^{j-1}\phi_k\frac{\sqrt{2\lambda_j}}{\lambda_j + s} \right\rangle = \begin{cases} 0, & \text{if } i < j, \\ 1, & \text{if } i = j\ . \end{cases}$$

The claim follows from the fact that if $i < j$, then the left side in the last bracket belongs to $H^2 \ominus \phi_i H^2$, while the right side is in $\phi_i H^2$. The equality for $i = j$ follows from the fact that basis elements have unit norm.

Such equalities imply that the realization is output normal. The realization is also input normal by the same argument. Hence the realization is a balanced realization of $\phi(s)$. ■

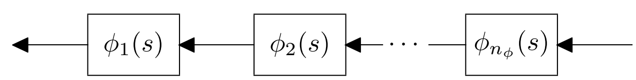

Fig. 1 Series connection.

The inner function ϕ_i in (29) defines the system transformation (12) whose matrices are described by

$$\begin{aligned} \tilde{A}_i &= (\lambda_i I + A)(\lambda_i I - A)^{-1}, \\ \tilde{B}_i &= \sqrt{2\lambda_i}\,(\lambda_i I - A)^{-1}B, \\ \tilde{C}_i &= \sqrt{2\lambda_i}C\,(\lambda_i I - A)^{-1}, \\ \tilde{D}_i &= h(\lambda_i) = D + C\,(\lambda_i I - A)^{-1}B. \end{aligned} \tag{33}$$

From Theorem 1, the system (12) is dissipative if and only if the continuous-time system (3) is dissipative. Furthermore, $P = P^T$ induces a storage function for the one system if and only if it induces a storage function for the other.

We proceed to show that the system matrices given in Proposition 2 is exactly the system (28) when an orthonormal basis of the subspace $\mathcal{S} = H^2 \ominus \phi H^2$ is given by (32). In order to do this, we first prove the following Lemma stating that the unique solutions of the Sylvester equations (13) and (14) respectively coincide with the input and output matrices in (33).

Lemma 4: Let $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{C}_i$, and $\tilde{D}_i$ be defined by (33). Then the solutions to the Sylvester equations (13), (14) are given by

$$X = \begin{bmatrix} \prod_{i=2}^{n_\phi}\tilde{A}_i\tilde{B}_1 & \prod_{i=3}^{n_\phi}\tilde{A}_i\tilde{B}_2 & \cdots & \tilde{B}_{n_\phi} \end{bmatrix},$$

$$Y = \begin{bmatrix} \tilde{C}_1 \\ \vdots \\ \tilde{C}_{n_\phi-1}\prod_{i=1}^{n_\phi-2}\tilde{A}_i \\ \tilde{C}_{n_\phi}\prod_{i=1}^{n_\phi-1}\tilde{A}_i \end{bmatrix}.$$

Proof: From Lemma 2 and the definition of inner product in H^2, we have

$$\begin{aligned}&\left\langle \prod_{k=1}^{i-1}\phi_k(s)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}, C\left(sI-A\right)^{-1}\xi\right\rangle\\&=\left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i+s}, C\left(sI-A\right)^{-1}\prod_{k=1}^{i-1}\phi_k^{\sim}(A)\xi\right\rangle\\&=\sqrt{2\lambda_i}C\left(\lambda_i I-A\right)^{-1}\prod_{k=1}^{i-1}\phi_k^{\sim}(A)\xi\\&=\tilde{C}_i\prod_{k=1}^{i-1}\tilde{A}_k\xi.\end{aligned}$$

Observe that the solution Y of (14) satisfies

$$e_i^{\mathrm{T}}Y\xi=\int_0^\infty e_i^{\mathrm{T}}e^{A_\phi^{\mathrm{T}}t}C_\phi^{\mathrm{T}}Ce^{At}\xi dt,$$

where e_i is the ith unit vector. The right hand side is the inner product of $C_\phi e^{A_\phi t}e_i$ and $Ce^{At}\xi$, and hence from Lemma 3 the ith row of Y is equal to $\tilde{C}_i\prod_{k=1}^{i-1}\tilde{A}_k$. The proof for X is similar. ■

We now show that the direct feedthrough matrix $\tilde{D}$ in Proposition 2 coincides with that defined in Lemma 1.

Lemma 5: When the orthonormal basis of $\mathcal{S}=H^2\ominus\phi H^2$ is given by $C_\phi\left(sI-A_\phi\right)^{-1}$ in (32), the direct through term $\tilde{D}$ in Proposition 2 satisfies

$$\begin{aligned}&\tilde{D}=h^{\sim}(A_\phi^{\mathrm{T}})\\&=\begin{bmatrix}\tilde{D}_1 & 0 & \cdots & 0\\ \tilde{C}_2\tilde{B}_1 & \tilde{D}_2 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{C}_{n_\phi}\prod_{i=2}^{n_\phi-1}\tilde{A}_i\tilde{B}_1 & \tilde{C}_{n_\phi}\prod_{i=3}^{n_\phi-1}\tilde{A}_i\tilde{B}_2 & \cdots & \tilde{D}_m\end{bmatrix}.\end{aligned}$$

Proof: The (i,j) th element of $\tilde{D}$ is given by

$$\tilde{d}_{ij}=\left\langle \prod_{k=1}^{i-1}\phi_k(s)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}, h(s)\prod_{k=1}^{j-1}\phi_k(s)\frac{\sqrt{2\lambda_j}}{\lambda_j+s}\right\rangle.$$

If $i<j$, then

$$\tilde{d}_{ij}=\left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i+s}, \prod_{k=i}^{j-1}\phi_k(s)h(s)\frac{\sqrt{2\lambda_j}}{\lambda_j+s}\right\rangle=0$$

because the left side in the bracket is in $H^2\ominus\phi_i H^2$ and the right side is in $\phi_i H^2$. If $i=j$, then

$$\begin{aligned}\tilde{d}_{ii}&=\left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i+s}, h(s)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}\right\rangle\\&=\left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i+s}, h(\lambda_i)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}\right\rangle\\&\quad+\left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i+s}, \left(h(s)-h(\lambda_i)\right)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}\right\rangle\\&=h(\lambda_i)=\tilde{D}_i,\end{aligned}$$

where the second term of the second line is zero because the left side of the bracket is in $H^2\ominus\phi_i H^2$ and the right side

$$\left(h(s)-h(\lambda_i)\right)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}=\phi_i(s)\frac{\sqrt{2\lambda_i}\left(h(s)-h(\lambda_i)\right)}{\lambda_i-s}$$

is in $\phi_i H^2$. Using the resolvent equation $\left(sI-A\right)^{-1}-\left(rI-A\right)^{-1}=\left(r-s\right)\left(sI-A\right)^{-1}\left(rI-A\right)^{-1}$, we have

$$\begin{aligned}&\tilde{d}_{(j+1)j}\\&=\left\langle \phi_j(s)\frac{\sqrt{2\lambda_{j+1}}}{\lambda_{j+1}+s}, h(s)\frac{\sqrt{2\lambda_j}}{\lambda_j+s}\right\rangle\\&=\left\langle \phi_j(s)\frac{\sqrt{2\lambda_{j+1}}}{\lambda_{j+1}+s}, D\frac{\sqrt{2\lambda_j}}{\lambda_j+s}\right\rangle\\&\quad+\left\langle \frac{\sqrt{2\lambda_{j+1}}}{\lambda_{j+1}+s}, C\left(sI-A\right)^{-1}B\phi_j^{\sim}(s)\frac{\sqrt{2\lambda_j}}{\lambda_j+s}\right\rangle\\&=\left\langle \frac{\sqrt{2\lambda_{j+1}}}{\lambda_{j+1}+s}, C\left\{\left(sI-A\right)^{-1}\right.\right.\\&\qquad\left.\left.-\left(\lambda_j I-A\right)^{-1}+\left(\lambda_j I-A\right)^{-1}\right\}B\frac{\sqrt{2\lambda_j}}{\lambda_j-s}\right\rangle\\&=\left\langle \frac{\sqrt{2\lambda_{j+1}}}{\lambda_{j+1}+s}, C\left(sI-A\right)^{-1}\left(\lambda_j I-A\right)^{-1}B\sqrt{2\lambda_j}\right\rangle\\&\quad+\left\langle \frac{\sqrt{2\lambda_{j+1}}}{\lambda_{j+1}+s}, C\left(\lambda_j I-A\right)^{-1}B\frac{\sqrt{2\lambda_j}}{\lambda_j-s}\right\rangle\\&=\sqrt{2\lambda_{j+1}}\sqrt{2\lambda_j}C\left(\lambda_{j+1}I-A\right)^{-1}\left(\lambda_j I-A\right)^{-1}B\\&=\tilde{C}_{j+1}\tilde{B}_j.\end{aligned}$$

Similarly, if $i>j+1$, then

$$\begin{aligned}&\tilde{d}_{ij}\\&=\left\langle \prod_{k=j}^{i-1}\phi_k(s)\frac{\sqrt{2\lambda_i}}{\lambda_i+s}, h(s)\frac{\sqrt{2\lambda_j}}{\lambda_j+s}\right\rangle\\&=\left\langle \prod_{k=j+1}^{i-1}\phi_k(s)\frac{\sqrt{2\lambda_i}}{\lambda_i+s},\right.\\&\qquad\left.C\left(sI-A\right)^{-1}\left(\lambda_j I-A\right)^{-1}B\sqrt{2\lambda_j}\right\rangle\\&=\left\langle \frac{\sqrt{2\lambda_i}}{\lambda_i+s}, C\left(sI-A\right)^{-1}\right.\\&\qquad\left.\prod_{k=j+1}^{i-1}\phi_k^{\sim}(A)\left(\lambda_j I-A\right)^{-1}B\sqrt{2\lambda_j}\right\rangle\\&=\tilde{C}_i\prod_{k=j+1}^{i-1}\tilde{A}_k\tilde{B}_j.\end{aligned}$$

This completes the proof. ■

Now, we are ready to prove the main result of this section.

Proof of Theorem 2: If the continuous-time system is dissipative with respect to the supply rate (17) and has a storage function $V(x)=x^{\mathrm{T}}Px$, then Theorem 1 implies that the discrete time system $\left(\tilde{A}_i,\tilde{B}_i,\tilde{C}_i,\tilde{D}_i\right)$ defined by

(33) for $i = 1, 2, \ldots, n_\phi$ are dissipative for the supply rate (17) with the common storage function induced by P. Then the periodic system defined by (26) is also dissipative. From Lemma 1, the lifted system (28) is also dissipative with the common storage function. When we use the balanced realization in Lemma 3 and the orthonormal basis (32) for the subspace $\mathcal{S} = H^2 \ominus \phi H^2$, Lemmas 4 and 5 imply that the lifted system (28) has the system matrices as in Proposition 2. This concludes the proof. ■

V. CONCLUSIONS

In this paper we studied whether the system transformation induced by generalized orthonormal basis functions preserves dissipativity. The system transformation in this paper is based on the orthogonal complement of the shift invariant subspace defined by a rational inner function. This includes as special case the bilinear transformation, which is induced by the Laguerre basis. The transformation is also called Hambo transform.

It was shown that dissipative continuous-time systems are transformed to dissipative discrete-time systems. Furthermore, a storage function of the original continuous-system is also a storage function of the transformed system.

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