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Beurling-Lax representations for weighted Bergman shift-invariant subspaces: the free noncommutative setting

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***Abstract*— The Beurling-Lax-Halmos theorem tells us that any invariant subspace $\mathcal{M}$ for the shift operator $S\colon f(z) \mapsto zf(z)$ on the vectorial Hardy space over the unit disk $H^2_{\mathcal{Y}} = \{f(z) = \sum_{j=0}^{\infty} f_j z^j \colon \|f\|^2 = \sum_{j\ge 0} \|f_j\|^2 < \infty\}$ (the Reproducing Kernel Hilbert Space with reproducing kernel $K(z,w) = (1-z\overline{w})^{-1} I_{\mathcal{Y}}$) can be represented as $\mathcal{M} = M_\Theta H^2_{\mathcal{U}}$ where $M_\Theta\colon H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}}$ is an isometric multiplication operator $M_\Theta\colon u(z) \mapsto \Theta(z)u(z)$. One proof constructs Θ as the transfer-function of a discrete-time input-state-output linear system explicitly constructed from the shift-invariant subspace $\mathcal{M}$. We discuss analogues of this result and related constructions for the setting of multivariable weighted Bergman spaces in the free noncommutative setting.**

I. Introduction

For $\mathcal{X}$ and $\mathcal{Y}$ any pair of Hilbert spaces, we use the notation $\mathcal{L}(\mathcal{X},\mathcal{Y})$ to denote the space of bounded, linear operators from $\mathcal{X}$ to $\mathcal{Y}$, shortening the notation $\mathcal{L}(\mathcal{X},\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$. We start with the classical discrete-time linear system

$$\Sigma(\mathbf{U}): \quad \left\{ \begin{array}{ccc} \mathbf{x}(k+1) & = & A\mathbf{x}(k) + B\mathbf{u}(k) \\ \mathbf{y}(k) & = & C\mathbf{x}(k) + D\mathbf{u}(k) \end{array} \right. \tag{I.1}$$

with $\mathbf{x}(k)$ taking values in the *state space* $\mathcal{X}$, $\mathbf{u}(k)$ taking values in the *input space* $\mathcal{U}$ and $\mathbf{y}(k)$ taking values in the *output space* $\mathcal{Y}$, where $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{X}$ are given Hilbert spaces and where the *system matrix*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \tag{I.2}$$

is a given bounded linear operator. If we let the system evolve on the nonnegative integers $n \in \mathbb{Z}_+$, then the whole trajectory $\{\mathbf{u}(n), \mathbf{x}(n), \mathbf{y}(n)\}_{n\in\mathbb{Z}_+}$ is determined recursively from the input signal $\{\mathbf{u}(n)\}_{n\in\mathbb{Z}_+}$ and the initial state $\mathbf{x}(0) = x$. Application of the Z-transform

$$\{f(k)\}_{k\in\mathbb{Z}_+} \mapsto \widehat{f}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

to the system equations (I.1) eventually leads to

$$\begin{aligned} \widehat{\mathbf{x}}(\lambda) &= (I-\lambda A)^{-1}x + \lambda(I-\lambda A)^{-1}B\widehat{\mathbf{u}}(\lambda), \\ \widehat{\mathbf{y}}(\lambda) &= C(I-\lambda A)^{-1}x + [D + \lambda C(I-\lambda A)^{-1}B]\widehat{\mathbf{u}}(\lambda) \\ &= (\mathcal{O}_{C,A}x)(\lambda) + \Theta_{\mathbf{U}}(\lambda)\widehat{\mathbf{u}}(\lambda), \end{aligned} \tag{I.3}$$

where

$$\mathcal{O}_{C,A}\colon\ x \mapsto \sum_{k=0}^{\infty} (CA^k x)\,\lambda^k = C(I-\lambda A)^{-1}x \tag{I.4}$$

is the (frequency-domain) *observability operator* and where

$$\Theta_{\mathbf{U}}(\lambda) = D + \lambda C(I-\lambda A)^{-1}B \tag{I.5}$$

is the *transfer function* of the system Σ given by (I.1). In particular, if the input signal $\{\mathbf{u}(n)\}_{n\in\mathbb{Z}_+}$ is taken to be zero, the resulting output $\{\mathbf{y}(n)\}_{n\in\mathbb{Z}_+}$ is given by $\widehat{\mathbf{y}} = \mathcal{O}_{C,A}x(0)$. If $\mathcal{O}_{C,A}$ is injective, i.e., if (C,A) satisfies the so-called *observability condition*

$$\bigcap_{k=0}^{\infty} \operatorname{Ker} CA^k = \{0\}, \tag{I.6}$$

we say that the output pair (C,A) is *observable*. In case $\mathcal{O}_{C,A}$ is bounded as an operator from $\mathcal{X}$ into the standard vector-valued Hardy space of the unit disk

$$H^2_{\mathcal{Y}} = \left\{ f(\lambda) = \sum_{k=0}^{\infty} f_k \lambda^k \colon \sum_{k=0}^{\infty} \|f_k\|^2_{\mathcal{Y}} < \infty \right\},$$

we say that the pair (C,A) is *output-stable*.

The case where the system matrix $\mathbf{U}$ is unitary is of special interest. In system-theoretic terms this has the interpretation that the system $\Sigma(\mathbf{U})$ is *conservative* in the sense that the energy stored by the state at time k ($\|x(k+1)\|^2 - \|x(k)\|^2$) is exactly compensated by the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$), with a similar property for the adjoint system. From the operator- and function-theoretic points of view this case is interesting since the observability operator turns out to be contractive from $\mathcal{X}$ into $H^2_{\mathcal{Y}}$, while the transfer function $\Theta_{\mathbf{U}}$ turns out to be in the *Schur class* $\mathcal{S}(\mathcal{U},\mathcal{Y})$ (i.e., analytic on the open unit disk $\mathbb{D}$ and such that $\Theta(z)$ is a contraction in $\mathcal{L}(\mathcal{U},\mathcal{Y})$ for every $z \in \mathbb{D}$). A remarkable fact is that *any* function Θ in $\mathcal{S}(\mathcal{U},\mathcal{Y})$ can be realized as the transfer function of a conservative linear system of the form (I.1).

If in addition the state space operator A is *strongly stable* in the sense that $A^n x \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$, then the observability operator is a partial isometry (in fact an isometry if (C,A) is observable) and the transfer function is inner (the boundary values exist almost everywhere on the unit circle $\mathbb{T}$ and are isometric operators from $\mathcal{U}$ to $\mathcal{Y}$). In fact *any inner function arises in this way as the transfer function of a conservative system $\Sigma_{\mathbf{U}}$ with strongly stable state operator A*, as can be seen as a consequence of the Sz.-Nagy–Foias model theory (see [11]).

If we start with a shift-invariant subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}$ and we wish to construct an inner function Θ so that $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$, it suffices to find an appropriate unitary $\mathbf{U}$ so that $\Theta = \Theta_{\mathbf{U}}$ works. As a first step, take $\mathcal{X} = \mathcal{M}^\perp$, $A = S^*|_{\mathcal{M}^\perp}$,

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$C = \mathbf{ev}_0|_{\mathcal{M}^\perp}$ where $\mathbf{ev}_0$ is the evaluation-at-zero map $f \mapsto f(0)$. Then A is strongly stable and one can see that $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ is isometric with the additional property that $\mathcal{O}_{C,A} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ is isometric with range exactly equal to $\mathcal{M}^\perp$. If one then finds an injective solution $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] : \mathcal{U} \to \left[\begin{smallmatrix} \mathcal{M}^\perp \\ \mathcal{Y} \end{smallmatrix}\right]$ of the Cholesky factorization problem

$$\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] \left[\begin{smallmatrix} B^* & D^* \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] - \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] \left[\begin{smallmatrix} A^* & C^* \end{smallmatrix}\right]$$

and sets $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then $\Theta_{\mathbf{U}}$ is inner and a reproducing kernel computation shows that we recover $\mathcal{M}$ as $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$ and we have a constructive systems-theory proof of the Beurling-Lax theorem (see [5, Theorem 5.2] for this approach carried out in a multivariable context).

If instead we start with an inner function Θ, one can take the invariant subspace $\mathcal{M}$ to be $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$ and repeat the construction given in the previous paragraph. However the Cholesky factorization step can be done much more explicitly, the result being

$$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] = \begin{bmatrix} S^*|_{\mathcal{M}^\perp} & S^* M_\Theta|_{\mathcal{U}} \\ \mathbf{ev}_0|_{\mathcal{M}^\perp} & \Theta(0) \end{bmatrix} : \left[\begin{smallmatrix} \mathcal{M}^\perp \\ \mathcal{U} \end{smallmatrix}\right] \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{Y} \end{bmatrix}. \tag{I.7}$$

Then $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitary and we recover Θ as $\Theta = \Theta_{\mathbf{U}}$. This has been called the *functional-model colligation matrix* for the inner function Θ in the literature—see [1] for a general setting.

Versions of these constructions extend to more general settings. We mention (1) the setting of weighted Bergman spaces (see [8], [9] for background and [12], [13], [14], [2], [3] for the systems-theory approach in this setting), and more generally (2) multivariable weighted Bergman spaces in the free noncommutative setting (see [15], [4]). We focus here on extensions to setting (2).

II. Free noncommutative (nc) system theory

We let $\mathbb{Z}_d^+$ denote the unital free semigroup (i.e., monoid) generated by the set of d letters $\{1, \dots, d\}$. Elements of $\mathbb{Z}_d^+$ are words of the form $i_N \cdots i_1$ where $i_\ell \in \{1, \dots, d\}$ for each $\ell \in \{1, \dots, N\}$ with multiplication given by concatenation. The unit element of $\mathbb{Z}_d^+$ is the empty word denoted by $\emptyset$. For $\alpha = i_N i_{N-1} \cdots i_1 \in \mathcal{F}_d$, we let $|\alpha|$ denote the number N of letters in α and we let $\alpha^\top := i_1 \cdots i_{N-1} i_N$ denote the *transpose* of α. We propose to consider the following multidimensional system with evolution along the free semigroup $\mathbb{Z}_d^+$:

$$\left\{ \begin{array}{rcl} \mathbf{x}(1\alpha) & = & \frac{n+|\alpha|}{|\alpha|+1} A_1 \mathbf{x}(\alpha) + \binom{n+|\alpha|}{|\alpha|+1} B_{1,\alpha} \mathbf{u}(\alpha) \\ \vdots & \vdots & \vdots \\ \mathbf{x}(d\alpha) & = & \frac{n+|\alpha|}{|\alpha|+1} A_d \mathbf{x}(\alpha) + \binom{n+|\alpha|}{|\alpha|+1} B_{d,\alpha} \mathbf{u}(\alpha) \\ \mathbf{y}(\alpha) & = & C\mathbf{x}(\alpha) + \binom{n+|\alpha|-1}{|\alpha|} D_\alpha \mathbf{u}(\alpha) \end{array} \right. \tag{II.1}$$

with the d-tuple of state space operators $\mathbf{A} = (A_1, \dots, A_d)$ in $\mathcal{L}(\mathcal{X})^d$ and the state-output operator $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Here we have a family of system matrices and a family of input spaces indexed by $\alpha \in \mathbb{Z}_d^+$:

$$\mathbf{U}_\alpha = \left[\begin{smallmatrix} A & \widehat{B}_\alpha \\ C & D_\alpha \end{smallmatrix}\right] : \begin{bmatrix} \mathcal{X} \\ \mathcal{U}_\alpha \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix} \text{ where}$$
$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \ \widehat{B}_\alpha = \begin{bmatrix} B_{1,\alpha} \\ \vdots \\ B_{d,\alpha} \end{bmatrix}. \tag{II.2}$$

We next wish to introduce the free nc Z-transform. Toward this end, we let $z = (z_1, \dots, z_d)$ to be a collection of d freely nc indeterminates and let $\mathcal{Y}\langle\langle z \rangle\rangle$ denote the set of nc formal power series $\sum_{\alpha \in \mathbb{Z}_d^+} f_\alpha z^\alpha$ where $f_\alpha \in \mathcal{Y}$ and where

$$z^\alpha = z_{i_N} z_{i_{N-1}} \cdots z_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \cdots i_1. \tag{II.3}$$

We extend the nc functional calculus (II.3) from nc indeterminates $z = (z_1, \dots, z_d)$ to a d-tuple of operators $\mathbf{A} = (A_1, \dots, A_d)$ by letting

$$\mathbf{A}^\alpha := A_{i_N} A_{i_{N-1}} \cdots A_{i_1} \quad \text{if} \quad \alpha = i_N i_{N-1} \cdots i_1 \in \mathbb{Z}_d^+, \tag{II.4}$$

where the multiplication is now operator composition. Letting

$$Z(z) = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix} \otimes I_{\mathcal{X}}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \tag{II.5}$$

we next observe that

$$(Z(z)A)^j = \left(\sum_{i=1}^d z_i A_i \right)^j = \sum_{\alpha \in \mathbb{Z}_d^+ : |\alpha| = j} \mathbf{A}^\alpha z^\alpha \tag{II.6}$$

for $j \geq 0$. We introduce the formal nc resolvent operator

$$R(Z(z)A) := (I - Z(z)A)^{-1} = \sum_{j=0}^\infty (Z(z)A)^j = \sum_{\alpha \in \mathbb{Z}_d^+} \mathbf{A}^\alpha z^\alpha$$

along with its n-th power

$$R_n(Z(z)A) := (I - Z(z)A)^{-n} = \sum_{\alpha \in \mathbb{Z}_d^+} \mu_{n,|\alpha|}^{-1} \mathbf{A}^\alpha z^\alpha$$

and shifted counterpart

$$R_{n,k}(Z(z)A) := \sum_{\alpha \in \mathbb{Z}_d^+} \mu_{n,|\alpha|+|\alpha|}^{-1} \mathbf{A}^\alpha z^\alpha,$$

where we have set $\mu_{n,k} = \frac{k\,(n-1)!}{(n+k-1)!}$. We define the formal nc Z-transform to be the map from functions on $\mathbb{Z}_+$ to nc formal power series given by

$$\{f_\alpha\}_{\alpha \in \mathbb{Z}_d^+} \mapsto \widehat{f}(z) = \sum_{\alpha \in \mathbb{Z}_d^+} f_\alpha z^\alpha. \tag{II.7}$$

Application of the nc Z-transform to the system equations (II.1) eventually leads to

$$\begin{aligned} \widehat{y}(z) &= C(I - Z(z)A)^{-n} x \\ &+ \sum_{\alpha \in \mathbb{Z}_d^+} \left(C R_{n,|\alpha|+1}(Z(z)A) Z(z) \widehat{B}_\alpha + \mu_{n,|\alpha|}^{-1} D_\alpha \right) z^\alpha u(\alpha) \\ &= \mathcal{O}_{n,C,\mathbf{A}} x + \sum_{\alpha \in \mathbb{Z}_d^+} \Theta_{n,\alpha}(z) z^\alpha \mathbf{u}(\alpha), \end{aligned} \tag{II.8}$$

where $x = \mathbf{x}(\emptyset)$. The first term on the right presents the n-observability operator

$$\mathcal{O}_{n,C,\mathbf{A}} x = C(I - Z(z)A)^{-n} x = \sum_{\alpha \in \mathbb{Z}_d^+} \mu_{n,|\alpha|}^{-1} (C\mathbf{A}^\alpha x) z^\alpha \tag{II.9}$$

associated with the state space d-tuple $\mathbf{A}$ and the state-output operator C and where

$$\Theta_{n,\alpha}(z) = \mu_{n,|\alpha|}^{-1} D_\alpha + C R_{n,|\alpha|+1}(Z(z)A) Z(z) \widehat{B}_\alpha \tag{II.10}$$

is the family of transfer functions indexed by $\alpha \in \mathbb{Z}_d^+$, in complete analogy with (I.3) with one exception: $\Theta_{n,\alpha}$ depends on α and hence we cannot set $\sum_{\alpha \in \mathbb{Z}_d^+} \Theta_{n,\alpha} z^\alpha \mathbf{u}(\alpha)$ equal to $\Theta_n(z) \cdot \widehat{\mathbf{u}}(z)$. Note that the dependence of $\Theta_{n,\alpha}$ on α is only through $|\alpha|$ as long as $\widehat{B}_\alpha$ depends on α only through $|\alpha|$.

III. Free noncommutative (nc) weighted Bergman spaces

Given a positive integer n, the free semigroup $\mathbb{Z}_d^+$, and the coefficient Hilbert space $\mathcal{Y}$, we let $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ be the nc weighted Bergman space

$$\left\{ \sum_{\alpha \in \mathbb{Z}_d^+} f_\alpha z^\alpha \in \mathcal{Y}\langle\langle z \rangle\rangle \colon \sum_{\alpha \in \mathbb{Z}_d^+} \mu_{n,|\alpha|} \cdot \|f_\alpha\|_{\mathcal{Y}}^2 < \infty \right\}. \tag{III.1}$$

One can view $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ as a *formal nc reproducing kernel Hilbert space* in the sense of [7] with formal kernel

$$k_{\mathrm{nc},n}(z,w) \otimes I_{\mathcal{Y}} = \sum_{\alpha \in \mathbb{Z}_d^+} \mu_{n,|\alpha|}^{-1} I_{\mathcal{Y}}\, z^\alpha \overline{w}^{\alpha^\top}.$$

Alternatively, after substituting d-tuples of square matrices $(Z_1, \dots, Z_d)$ of arbitrary square size for the indeterminates $(z_1, \dots, z_d)$ and considering the space $\mathcal{A}_{n,\mathcal{Y}}$ as a space of nc functions in the sense of [10], one can consider $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ as a nc reproducing kernel Hilbert space in the sense of [6], but here it is convenient to restrict to the less general former point of view. The Hilbert space $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ is equipped with a d-tuple of shift operators $\mathbf{S} = (S_1, \dots, S_d)$ given by multiplication on the right by the j-th coordinate:

$$S_j \colon f(z) \mapsto f(z) \cdot z_j.$$

A subspace $\mathcal{M}$ of $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ is said to be shift-invariant if $S_j \cdot \mathcal{M} \subset \mathcal{M}$ for $j = 1, \dots, d$. We seek to describe such shift-invariant subspaces via a Beurling-Lax theorem for this setting.

We define a formal power series $\Theta(z) = \sum_{\alpha \in \mathbb{Z}_d^+} \Theta_\alpha z^\alpha$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ to be a *nc Bergman inner function* (actually here formal power series rather than function) if

(i) $M_\Theta \colon u \mapsto \Theta(z) \cdot u$ is isometric from the coefficient space $\mathcal{U}$ into $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$, and
(ii) $\Theta(z) \cdot u \perp \Theta(z) z^\alpha \cdot v$ in $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ for all $u, v \in \mathcal{U}$ and nonempty α in $\mathbb{Z}_d^+$.

We say that a collection $\{\Theta_\beta \colon \beta \in \mathbb{Z}_d^+\}$ where $\Theta_\beta \in \mathcal{L}(\mathcal{U}_\beta, \mathcal{Y})$ for a family of input spaces $\mathcal{U}_\beta$ ($\beta \in \mathbb{Z}_d^+$) is a *Bergman-inner family* if

(i) the operator $u_\beta \mapsto \Theta_\beta u_\beta z^\beta$ is isometric from $\mathcal{U}_\beta$ into $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$,
(ii) $\Theta_\beta z^\beta u_\beta \perp \Theta_\gamma(z) z^\gamma u_\gamma$ in $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ for all $u_\beta \in \mathcal{U}_\beta$, $u_\gamma \in \mathcal{U}_\gamma$ for all β and γ in $\mathbb{Z}_d^+$ with $\beta \neq \gamma$, and
(iii) for each $\alpha \in \mathbb{Z}_d^+$,

$$\mathbf{S}^{\alpha^\top} \left(\bigoplus_{\beta \in \mathbb{Z}_d^+} \Theta_\beta z^\beta \mathcal{U}_\beta \right) = \bigoplus_{\beta \in \mathbb{Z}_d^+} \Theta_{\beta\alpha} z^{\beta\alpha} \mathcal{U}_{\beta\alpha}.$$

It turns out that any nc Bergman inner function Θ can be embedded as $\Theta_\emptyset$ into a Bergman inner family $\{\Theta_\beta\}_{\beta \in \mathbb{Z}_d^+}$, and, in case $d = 1$, whenever $\Theta(z) z^\gamma$ is a nc Bergman inner function, then Θ can be embedded as $\Theta = \Theta_\gamma$ inside a Bergman inner family $\{\Theta_\beta \colon \beta \in \mathbb{Z}_d^+\}$. It is this notion of nc Bergman-inner family which leads to a compelling extension of the Beurling-Lax theorem to the nc weighted Bergman space setting, as demonstrated by the following result.

Theorem 3.1: Let $\mathcal{M}$ be a closed $\mathbf{S}_{n,R}$-invariant subspace of $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$. Define formal power series $\Theta_\beta \in \mathcal{L}(\mathcal{U}_\beta, \mathcal{Y})$ so that the map $u_\beta \mapsto \Theta_\beta u_\beta z^\beta$ is an isometry from $\mathcal{U}_\beta$ onto the space $\mathcal{M}_\beta = \mathbf{S}^{\beta^\top} \mathcal{M} \ominus \left(\bigoplus_{j=1}^d \mathbf{S}^{\beta^\top j} \mathcal{M} \right)$. Then $\Theta = \{\Theta_\beta\}_{\beta \in \mathbb{Z}_d^+}$ is a nc inner family giving rise to a Beurling-Lax representation for the shift-invariant subspace $\mathcal{M}$ in the following sense:

$$\mathcal{M} = M_\Theta H^2_{\{\mathcal{U}_\beta\}}(\mathbb{Z}_d^+) := \bigoplus_{\beta \in \mathbb{Z}_d^+} \Theta_\beta(z) z^\beta \cdot \mathcal{U}_\beta. \tag{III.2}$$

If $\Theta' = \{\Theta'_\beta\}_{\beta \in \mathbb{Z}_d^+}$ is another such nc inner family, then for each $\beta \in \mathbb{Z}_d^+$ there is a unitary operator $U_\beta \colon \mathcal{U}_\beta \to \mathcal{U}'_\beta$ so that $\Theta'_\beta(z) U_\beta = \Theta_\beta(z)$.

Conversely, if $\{\Theta_\alpha\}_{\alpha \in \mathbb{Z}_d^+}$ is a nc Bergman-inner family and we set $\mathcal{M}_\alpha = \Theta(z) z^\alpha \cdot \mathcal{U}_\alpha$, then $\mathcal{M} := \bigoplus \mathcal{M}_\alpha$ is a shift-invariant subspace for $\mathbf{S}$ in $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_+^d)$.

IV. Noncommutative transfer-function realization of nc Bergman-inner families

Theorem 3.1 shows how nc Bergman-inner families $\{\Theta_\alpha\}_{\alpha \in \mathbb{Z}_d^+}$ can be computed from a shift-invariant subspace $\mathcal{M}$ of the nc weighted Bergman space $\mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ via a nc analogue of the Halmos wandering-subspace construction. We now show how such nc Bergman-inner families can be constructed in transfer-function realization form for time-varying nc systems of the form (II.1) with system matrices $\mathbf{U}_\alpha = \left[\begin{smallmatrix} A & \widehat{B}_\alpha \\ C & D_\alpha \end{smallmatrix}\right]$ as in (II.2) satisfying some additional metric constraints as follows.

Given $A, \widehat{B}_\alpha, C, D_\alpha$ as in (II.2), introduce shifted weighted observability gramians $\mathfrak{G}_{n,k,C,\mathbf{A}}$ (for fixed n equal to the index for the weight sequence $\{\mu_{n,j}\}_{j \geq 0}$, $k \in \mathbb{N}$, C and $\mathbf{A} = (A_1, \dots, A_d)$ as in (II.2)) by

$$\mathfrak{G}_{n,k,C,\mathbf{A}} = \sum_{\alpha \in \mathbb{Z}_d^+} \mu_{n,|\alpha|+k}^{-1} \mathbf{A}^{*\alpha^\top} C^* C \mathbf{A}^\alpha.$$

We assume that the series defining $\mathfrak{G}_{n,k,C,\mathbf{A}}$ is strongly convergent and that $\mathfrak{G}_{n,k,C,\mathbf{A}}$ has a bounded inverse for $k = 1, 2, \dots$. The additional metric constraints which we shall impose on $\mathbf{U}_\alpha$ are

$$\begin{bmatrix} A^* & C^* \\ \widehat{B}_\beta^* & D_\beta^* \end{bmatrix} \begin{bmatrix} \mathfrak{G}_{n,|\beta|+1,C,\mathbf{A}} \otimes I_d & 0 \\ 0 & \mu_{n,|\beta|}^{-1} \cdot I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & \widehat{B}_\beta \\ C & D_\beta \end{bmatrix} = \begin{bmatrix} \mathfrak{G}_{n,|\beta|,C,\mathbf{A}} & 0 \\ 0 & I_{\mathcal{U}_\beta} \end{bmatrix}, \tag{IV.1}$$

$$\begin{bmatrix} A & \widehat{B}_\beta \\ C & D_\beta \end{bmatrix} \begin{bmatrix} \mathfrak{G}_{n,|\beta|,C,\mathbf{A}}^{-1} & 0 \\ 0 & I_{\mathcal{U}_\beta} \end{bmatrix} \begin{bmatrix} A^* & C^* \\ \widehat{B}_\beta^* & D_\beta^* \end{bmatrix} = \begin{bmatrix} \mathfrak{G}_{n,|\beta|+1,C,\mathbf{A}}^{-1} \otimes I_d & 0 \\ 0 & \mu_{n,|\beta|} I_{\mathcal{Y}} \end{bmatrix}. \tag{IV.2}$$

We then have the following result, the nc multivariable weighted-Bergman analogue of the result in Section I that the transfer function $\Theta_{\mathbf{U}}(\lambda)$ is inner whenever $\mathbf{U}$ is unitary.

Theorem 4.1: Suppose that the collection of system matrices $\{\mathbf{U}\}_{\alpha \in \mathbb{Z}_d^+}$ (II.2) *gives rise to bounded and boundedly invertible shifted weighted observability gramian operators* $\mathfrak{G}_{n,k,C,\mathbf{A}}$ *for which the metric constraints* (IV.1)-(IV.2) *are satisfied. Define* $\Theta_{n,\alpha}$ *by* (II.10). *Then* $\{\Theta\}_{\alpha \in \mathbb{Z}_d^+}$ *is a nc Bergman-inner family.*

A canonical way to achieve all these objectives is to start with $\mathbf{A} = (A_1, \dots, A_d)$ equal to a *n-hypercontractive d-tuple of operators on $\mathcal{X}$*. To define this notion, we first define an operator $B_{\mathbf{A}} \in \mathcal{L}(\mathcal{L}(\mathcal{X}))$ by $B_{\mathbf{A}}(X) = \sum_{j=1}^d A_j^* X A_j$ and then set $\Gamma_{k,\mathbf{A}} = (I - B_{\mathbf{A}})^k$ for $k = 1, 2, \dots,$ We say that the operator d-tuple $\mathbf{A} = (A_1, \dots, A_d)$ is *n-hypercontractive* if $\Gamma_{k,\mathbf{A}}(I) \succeq 0$ for $1 \le k \le n$ (or equivalently as it turns out, for $k = 1$ and $k = n$). We say that $\mathbf{A}$ is *strongly stable* if $\lim_{N \to \infty} \sum_{\alpha \in \mathbb{Z}_d^+ \colon |\alpha| = N} \|\mathbf{A}^\alpha x\|^2 = 0$ for each $x \in \mathcal{X}$. Then we have the following algorithm:

1) Choose $\mathbf{A}$ to be any strongly stable n-hypercontractive operator d-tuple,.
2) Choose $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ so that $C^* C = \Gamma_{n,\mathbf{A}}(I)$. Then the shifted weighted observability gramians $\mathfrak{G}_{n,k,C,\mathbf{A}}$ ($k = 1, 2, \dots$) turn out all to be well defined and invertible.
3) Choose $\begin{bmatrix} \widehat{B}_\beta \\ D_\beta \end{bmatrix}$ to be an injective solution of the Cholesky factorization problem

$$\begin{bmatrix} \widehat{B}_\beta \\ D_\beta \end{bmatrix} \mathfrak{G}_{n,|\beta|,C,\mathbf{A}}^{-1} \begin{bmatrix} \widehat{B}_\beta^* & D_\beta^* \end{bmatrix} = \begin{bmatrix} \mathfrak{G}_{n,|\beta|+1,C,\mathbf{A}}^{-1} \otimes I_d & 0 \\ 0 & \mu_{n,|\beta|} I \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \mathfrak{G}_{n,|\beta|,C,\mathbf{A}}^{-1} \begin{bmatrix} A^* & C^* \end{bmatrix}.$$

Then the family of system matrices $\mathbf{U}_\beta = \begin{bmatrix} A & \widehat{B}_\beta \\ C & D_\beta \end{bmatrix}$ meets all the hypotheses of Theorem 4.1, and hence $\{\Theta_\beta = D_\beta + C R_{n,|\beta|+1}(Z(z)A) Z(z) \widehat{B}_\beta\}_{\beta \in \mathbb{Z}_d^+}$ is a nc Bergman-inner family.

If we start with a shift-invariant subspace $\mathcal{M} \subset \mathcal{A}_{n,\mathcal{Y}}(\mathbb{Z}_d^+)$ and seek a nc Bergman-inner family $\{\Theta_\beta\}_{\beta \in \mathbb{Z}_d^+}$ giving the nc Beurling-Lax representation (III.2), we need only modify Steps 1 and 2 in the above algorithm: ($1'$) $A_j = S_j^*|_{\mathcal{M}^\perp =: \mathcal{X}}$ and ($2'$) $C = \mathbf{ev}_\emptyset \colon f(z) = \sum_{\alpha \in \mathbb{Z}_d^+} f_\alpha z^\alpha \mapsto f_\alpha$ for $f \in \mathcal{M}^\perp$. Continue with Steps (3) and (4) of the algorithm to arrive at the nc Bergman-inner family $\{\Theta_{\mathbf{U}_\beta}\}$ giving the nc Beurling-Lax representation (III.2) for the subspace $\mathcal{M}$.

If we start with a nc Bergman-inner family $\{\Theta_\beta\}$, we can perform Steps ($1'$) and ($2'$) with $\mathcal{M} = \bigoplus_{\alpha \in \mathbb{Z}_d^+} \Theta_\alpha \mathcal{U}_\alpha$. But then Step 3 can be done more explicitly: simply take $\mathbf{U}_\beta$ equal to

$$\begin{bmatrix} S^*|_{\mathcal{M}^\perp} & S^* \mathbf{S}_{n,R}^{*\beta} \mathbf{S}^{\beta^\top} M_{\Theta_\beta}|_{\mathcal{U}_\beta} \\ E|_{\mathcal{M}^\perp} & \mu_{n,|\beta|} [\Theta_\beta]_\emptyset \end{bmatrix} \colon \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{U}_\beta \end{bmatrix} \to \begin{bmatrix} (\mathcal{M}^\perp)^d \\ \mathcal{Y} \end{bmatrix} \tag{IV.3}$$

where here $S^* = \begin{bmatrix} S_1^* \\ \vdots \\ S_d^* \end{bmatrix}$. We have thus arrived at a compelling nc weighted Bergman-space analogues of the realization formulas presented for the single-variable case at the end of Section I. Complete details will appear in [4].

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