23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Stabilization of port-Hamiltonian systems by nonlinear boundary control in the presence of disturbances*

Jochen Schmid[1] and Hans Zwart[2,3]

***Abstract*— In this note, we are concerned with the stabilization of linear port-Hamiltonian systems on an interval (a, b) (for instance, vibrating strings or beams) in the presence of external disturbances. In order to achieve stabilization we couple the system to a nonlinear dynamic boundary controller whose output is allowed to be corrupted by an external disturbance before it is fed back into the system. We first establish the well-posedness of the resulting closed-loop system and then present two input-to-state stability results for the closed-loop system (with input being the external disturbance): for a special class of nonlinear controllers, we obtain uniform input-to-state stability and for a more general class of nonlinear controllers, we obtain weak input-to-stability. Also, in both cases we get convergence of all solutions to zero.**

***Index Terms*— Input-to-state stability, infinite-dimensional systems, port-Hamiltonian systems, nonlinear boundary control, actuator disturbances.**

I. INTRODUCTION

In this note, we consider a general linear port-Hamiltonian system $\mathfrak{S}$ on a 1-dimensional spatial domain (a, b). Such a system can be given, for example, by a vibrating string or a beam. What we are interested in is the stabilization of such systems in the presence of external disturbances and, for that purpose, we use nonlinear dynamic boundary controllers $\mathfrak{S}_c$, that is, dynamic controllers that act only via the boundary points a, b of the spatial domain (a, b) on which the system $\mathfrak{S}$ is defined. Since realistic controllers often exhibit nonlinear behavior (due to nonlinear potential energy or damping terms, for instance), we allow our controller to be nonlinear. Since, moreover, realistic controllers are typically affected by disturbances, we also allow external disturbances to corrupt the output of our controller before it is fed back into the system (actuator disturbances). We couple the controller to our system by standard feedback interconnection, that is,

$$y = u_c \qquad \text{and} \qquad -y_c + d = u,$$

where u, y and u_c, y_c are the input and output of $\mathfrak{S}$ and $\mathfrak{S}_c$ respectively and d is the external disturbance. So, in pictures the resulting closed-loop system $\tilde{\mathfrak{S}}$ with disturbance input d and output y looks as follows.

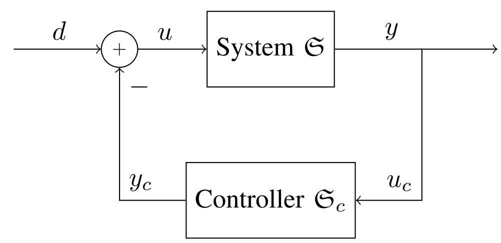

Fig. 1. Closed-loop system $\tilde{\mathfrak{S}}$

What we aim at in this note is the input-to-state stability of the closed-loop system $\tilde{\mathfrak{S}}$. And in order to achieve this goal, we will proceed as follows.

A. *Structure of this note*

Section II describes in detail the class of systems $\mathfrak{S}$ to be stabilized and the class of controllers $\mathfrak{S}_c$ used for that purpose. In Section III we present our solvability results for the closed-loop system $\tilde{\mathfrak{S}}$, namely solvability in the classical sense for classical (sufficiently regular) initial states and disturbance inputs (Section III-A) and solvability in the generalized sense for general initial states and locally square integrable disturbance inputs (Section III-B). In fact, we have well-posedness of the closed-loop system, that is, the generalized solutions and generalized outputs depend continuously on the initial states and disturbance inputs. In Section IV we present our input-to-state stability results for the closed-loop system $\tilde{\mathfrak{S}}$ w.r.t. square-integrable disturbance inputs. We establish uniform input-to-state stability for a special class of nonlinear dynamic boundary controllers (Section IV-A) and weak input-to-state stability for a more general class of nonlinear dynamic boundary controllers (Section IV-B). Additionally, we obtain convergence of all generalized solutions to zero.

B. *Some remarks on related works*

Similar settings have been considered in [10], [7], [15] [11], [1], but in these works no disturbances are allowed and, accordingly, the question of input-to-state stability does not arise there. In [13], disturbances are allowed, but in that paper they corrupt the input of the controller instead of its output (sensor disturbances). We do not strive for the utmost generality in this note, but rather concentrate on the simplest cases and on rough sketches of proof – for more general results, proofs, and applications we refer to the upcoming paper [12].

*This work was supported by DFG under the grant "Input-to-state stability and stabilization of distributed-parameter systems" (DA 767/7-1)

[1]J. Schmid is with the Institut für Mathematik, Universität Würzburg, 97074 Würzburg, Germany jochen.schmid@mathematik.uni-wuerzburg.de

[2] H. Zwart is with the Department of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, 7500 AE Enschede, The Netherlands h.j.zwart@utwente.nl

[3] H. Zwart is with the Department of Mechanical Engineering, Technische Universiteit Eindhoven, 5600 MB Eindhoven, The Netherlands h.j.zwart@tue.nl

C. Some notation used throughout this note

In the entire note, $\mathbb{R}_0^+ := [0, \infty)$ denotes the non-negative reals and $|\cdot|$ denotes the standard norm on $\mathbb{R}^k$ for every $k \in \mathbb{N}$. As usual, $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{L}$ denote the following classes of comparison functions:

$$\begin{aligned} \mathcal{K} &:= \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) : \gamma \text{ strictly increasing with } \gamma(0) = 0\} \\ \mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} : \gamma \text{ unbounded}\} \\ \mathcal{L} &:= \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) : \gamma \text{ strictly decreasing with } \lim_{r\to\infty} \gamma(r) = 0\}. \end{aligned}$$

Also, $C_c^2(\mathbb{R}_0^+, \mathbb{R}^k)$ denotes the space of $C^2(\mathbb{R}_0^+, \mathbb{R}^k)$-functions d with compact support in $\mathbb{R}_0^+$ and for $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$ we will use the following short-hand notations:

$$\|d\|_2 := \|d\|_{L^2(\mathbb{R}_0^+, \mathbb{R}^k)}, \qquad \|d\|_{[0,t],2} := \left\|d|_{[0,t]}\right\|_{L^2([0,t], \mathbb{R}^k)}.$$

And finally, for a semigroup generator A and bounded operators B, C between appropriate spaces, the symbol $\mathfrak{S}(A, B, C)$ will stand for the state-linear system [2]

$$x' = Ax + Bu \quad \text{with} \quad y = Cx.$$

II. SETTING: DESCRIPTION OF THE SYSTEM AND THE CONTROLLER

A. Setting: the system to be stabilized

As has been pointed out above, the system $\mathfrak{S}$ to be stabilized is a linear first-order port-Hamiltonian system [5], [3] on a bounded interval (a, b) with control and observation at the boundary. Such a system evolves according to the following differential equation and boundary conditions:

$$x' = \mathcal{A}x = P_1 \partial_\zeta(\mathcal{H}x) + P_0 \mathcal{H}x \tag{1}$$

$$u(t) = \mathcal{B}x(t) \quad \text{and} \quad y(t) = \mathcal{C}x(t) \tag{2}$$

and the energy of such a system in the state x is given by

$$E(x) = \frac{1}{2} \int_a^b x(\zeta)^\top (\mathcal{H}x)(\zeta) \, \mathrm{d}\zeta. \tag{3}$$

In these equations, $\zeta \mapsto \mathcal{H}(\zeta) \in \mathbb{R}^{m\times m}$ is a measurable matrix-valued function (the energy density) such that for almost all $\zeta \in (a, b)$

$$0 < \underline{m} I \le \mathcal{H}(\zeta) \le \overline{m} I < \infty, \tag{4}$$

and $P_0, P_1 \in \mathbb{R}^{m\times m}$ are matrices such that $P_1 = P_1^\top$ is symmetric and invertible and $P_0 = -P_0^\top$ is skew-symmetric. As the state space of $\mathfrak{S}$ one chooses $X := L^2((a, b), \mathbb{R}^m)$ with norm $\|\cdot\|_X$ given by the system energy

$$\frac{1}{2} \|x\|_X^2 := E(x) = \frac{1}{2} \int_a^b x(\zeta)^\top (\mathcal{H}x)(\zeta) \, \mathrm{d}\zeta.$$

In view of (4) it is clear that the norm $\|\cdot\|_X$ is equivalent to the standard norm of $L^2((a, b), \mathbb{R}^m)$ and that it is induced by a scalar product which we denote by $\langle \cdot, \cdot\cdot \rangle_X$. Also, the domain of the linear differential operator $\mathcal{A}$ is

$$D(\mathcal{A}) := \{x \in X : \mathcal{H}x \in W^{1,2}((a, b), \mathbb{R}^m) \text{ and } W_{B,1} \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} = 0\}$$

where $W_{B,1} \in \mathbb{R}^{(m-k)\times 2m}$ with $k \in \{1, \dots, m\}$. Similarly, the boundary control and boundary observation operators $\mathcal{B}, \mathcal{C} : D(\mathcal{A}) \to \mathbb{R}^k$ are linear and of the form

$$\mathcal{B}x := W_{B,2} \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix} \quad \text{and} \quad \mathcal{C}x := W_C \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix}$$

where $W_{B,2}, W_C \in \mathbb{R}^{k\times 2m}$ are called the boundary control and boundary observation matrix, respectively.

Assumption II.1. *$\mathfrak{S}$ is impedance-passive, that is,*

$$\langle x, \mathcal{A}x \rangle_X \le (\mathcal{B}x)^\top \mathcal{C}x \qquad (x \in D(\mathcal{A})). \tag{5}$$

It follows that $A := \mathcal{A}|_{D(\mathcal{A})\cap \ker \mathcal{B}}$ is a contraction semigroup generator on X by [6] and that classical solutions of $\mathfrak{S}$ satisfy the following energy dissipation inequality:

$$E_x'(t) = \langle x(t), \mathcal{A}x(t) \rangle_X \le (\mathcal{B}x(t))^\top \mathcal{C}x(t) = u(t)^\top y(t).$$

B. Setting: the controller

As our controller $\mathfrak{S}_c$ we choose a finite-dimensional nonlinear system which evolves according to

$$v' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} K v_2 \\ -\nabla \mathcal{P}(v_1) - \mathcal{R}(K v_2) + B_c u_c \end{pmatrix} \tag{6}$$

$$y_c = B_c^\top K v_2 + S_c u_c \tag{7}$$

and whose energy in the state $v = (v_1, v_2)$ from the controller state space $V := \mathbb{R}^{2m_c}$ is given by

$$E_c(v) := \mathcal{P}(v_1) + \frac{1}{2} v_2^\top K v_2 \tag{8}$$

(potential energy plus kinetic energy). In these equations, $K \in \mathbb{R}^{m_c\times m_c}$, $B_c \in \mathbb{R}^{m_c\times k}$, $S_c \in \mathbb{R}^{k\times k}$ are such that

$$K > 0 \qquad \text{and} \qquad S_c > 0. \tag{9}$$

Additionally, the potential energy $\mathcal{P} : \mathbb{R}^{m_c} \to \mathbb{R}_0^+$ is differentiable such that $\nabla \mathcal{P}$ is locally Lipschitz continuous and $\mathcal{P}(0) = 0$ and the damping function $\mathcal{R} : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$ is locally Lipschitz continuous such that $\mathcal{R}(0) = 0$. As the norm on $V = \mathbb{R}^{2m_c}$ we choose $|\cdot|_V$ defined by

$$|v|_V^2 = |(v_1, v_2)|_V^2 := |v_1|^2 + v_2^\top K v_2$$

which is obviously equivalent to the standard norm on $\mathbb{R}^{2m_c}$ and is induced by a scalar product $\langle \cdot, \cdot\cdot \rangle_V$.

Assumption II.2. (i) *$\mathcal{P}$ is positive definite and radially unbounded, that is, $\mathcal{P}(v_1) > 0$ for all $v_1 \in \mathbb{R}^{m_c} \setminus \{0\}$ and $\mathcal{P}(v_1) \longrightarrow \infty$ as $|v_1| \to \infty$*

(ii) *$\mathcal{R}$ is damping, that is, $v_2^\top \mathcal{R}(v_2) \ge 0$ for all $v_2 \in \mathbb{R}^{m_c}$.*

It follows (i) that $\underline{\psi}_c(|v|_V) \le E_c(v) \le \overline{\psi}_c(|v|_V)$ for some $\underline{\psi}_c, \overline{\psi}_c \in \mathcal{K}_\infty$ and (ii) that $\mathfrak{S}_c$ is passive (even strictly input-passive) w.r.t. E_c as storage function.

C. Setting: the closed-loop system

Coupling $\mathfrak{S}$ and $\mathfrak{S}_c$ by standard feedback interconnection

$$y(t) = u_c(t) \quad \text{and} \quad -y_c(t) + d(t) = u(t),$$

we obtain the closed-loop system $\tilde{\mathfrak{S}}$ described by the following evolution equation with boundary conditions:

$$\tilde{x}' = \tilde{\mathcal{A}}\tilde{x} + \tilde{f}(\tilde{x}) \tag{10}$$

$$d(t) = \tilde{\mathcal{B}}\tilde{x}(t) \qquad \text{and} \qquad y(t) = \tilde{\mathcal{C}}\tilde{x}(t). \tag{11}$$

Its state space is the Hilbert space $\tilde{X} := X \times V = X \times \mathbb{R}^{2m_c}$ with norm $\|\cdot\| = \|\cdot\|_{\tilde{X}}$ defined by

$$\|\tilde{x}\|^2 = \|(x,v)\|^2 := \|x\|_X^2 + |v|_V^2$$

and the energy of the closed-loop system in the state $\tilde{x} = (x,v) \in \tilde{X}$ is

$$\tilde{E}(\tilde{x}) := E(x) + E_c(v) = \frac{1}{2}\|x\|_X^2 + \mathcal{P}(v_1) + \frac{1}{2}v_2^\top K v_2.$$

In the equations above, the linear and nonlinear operators $\tilde{\mathcal{A}} : D(\tilde{\mathcal{A}}) \to \tilde{X}$ and $\tilde{f} : \tilde{X} \to \tilde{X}$ with $D(\tilde{\mathcal{A}}) := D(\mathcal{A}) \times \mathbb{R}^{2m_c}$ are given respectively by

$$\tilde{\mathcal{A}}\tilde{x} := \begin{pmatrix} \mathcal{A}x \\ Kv_2 \\ -v_1 + B_c\mathcal{C}x \end{pmatrix},$$

$$\tilde{f}(\tilde{x}) := \begin{pmatrix} 0 \\ 0 \\ v_1 - \nabla\mathcal{P}(v_1) - \mathcal{R}(Kv_2) \end{pmatrix}$$

and the linear boundary operators $\tilde{\mathcal{B}}, \tilde{\mathcal{C}} : D(\tilde{\mathcal{A}}) \to \mathbb{R}^k$ are given by

$$\tilde{\mathcal{B}}\tilde{x} := \mathcal{B}x + B_c^\top K v_2 + S_c\mathcal{C}x, \qquad \tilde{\mathcal{C}}\tilde{x} := \mathcal{C}x.$$

We record here for later use that the energy $\tilde{E}$ is equivalent to the norm $\|\cdot\|$ of $\tilde{X}$ in the following sense: there exist $\underline{\psi}, \overline{\psi} \in \mathcal{K}_\infty$ such that for all $\tilde{x} \in \tilde{X}$

$$\underline{\psi}(\|\tilde{x}\|) \le \tilde{E}(\tilde{x}) \le \overline{\psi}(\|\tilde{x}\|). \tag{12}$$

III. SOLVABILITY OF THE CLOSED-LOOP SYSTEM

We first have to settle the global solvability of the closed-loop equations before we can turn to stability investigations. In view of the structure of the right-hand side of (10) as a sum of a linear and a nonlinear part, we want to apply the standard theory [9] of semilinear evolution equations. Yet, this theory is not applicable directly here because first the linear part $\tilde{\mathcal{A}}$ of (10) is not a semigroup generator on $\tilde{X}$ and because second in addition to the mere differential equation (10) the boundary condition $d(t) = \tilde{\mathcal{B}}\tilde{x}(t)$ occurs. We therefore make the following additional assumption.

Assumption III.1. *$W := (W_B^\top, W_C^\top)^\top \in \mathbb{R}^{(m+k)\times 2m}$ has full rank $m+k$, where $W_B := (W_{B,1}^\top, W_{B,2}^\top)^\top$.*

It then easily follows that $\tilde{\mathcal{B}} : D(\tilde{\mathcal{A}}) \to \mathbb{R}^k$ is surjective and hence has a linear right-inverse $\tilde{R} : \mathbb{R}^k \to D(\tilde{\mathcal{A}})$ meaning that $\tilde{\mathcal{B}}\tilde{R}d = d$ for all $d \in \mathbb{R}^k$. We can thus define – following [4] – the new variable

$$\tilde{\xi}(t) = \tilde{x}(t) - \tilde{R}d(t)$$

and with this new variable the original closed-loop equations

$$\tilde{x}' = \tilde{\mathcal{A}}\tilde{x} + \tilde{f}(\tilde{x}) \quad \text{with} \quad d(t) = \tilde{\mathcal{B}}\tilde{x}(t) \tag{13}$$

become equivalent – for continuously differentiable disturbance signals d – to the evolution equation

$$\tilde{\xi}' = \tilde{A}\tilde{\xi} + \tilde{f}(\tilde{\xi} + \tilde{R}d(t)) + \tilde{\mathcal{A}}\tilde{R}d(t) - \tilde{R}d'(t) \tag{14}$$

where $\tilde{A} := \tilde{\mathcal{A}}|_{D(\tilde{\mathcal{A}})\cap\ker\tilde{\mathcal{B}}}$. What is important now is that (14), by the following proposition, is a truly semilinear evolution equation in the sense of [9] (with an explicitly time-dependent nonlinearity).

Proposition III.2. *With the above assumptions,*

(i) *$\tilde{A}$ is a contraction semigroup generator on $\tilde{X}$ with compact resolvent*
(ii) *$\tilde{f}$ is Lipschitz continuous on bounded subsets of $\tilde{X}$ and $\tilde{f}(0) = 0$.*

A. Solvability in the classical sense

With the above proposition at hand, we can now establish solvability in the classical sense for classical initial states $\tilde{x}_0$ and classical disturbances d. In the following, we abbreviate the set of these classical data as

$$\mathcal{D} := \big\{(\tilde{x}_0, d) \in D(\tilde{\mathcal{A}}) \times C_c^2(\mathbb{R}_0^+, \mathbb{R}^k) : d(0) = \tilde{\mathcal{B}}\tilde{x}_0\big\}.$$

Theorem III.3. *With the above assumptions, we have that*

(i) *$\tilde{\mathfrak{S}}$ has a unique global classical solution and a continuous classical output function*

$$\tilde{x}(\cdot, \tilde{x}_0, d) \in C^1(\mathbb{R}_0^+, \tilde{X}), \qquad y(\cdot, \tilde{x}_0, d) \in C(\mathbb{R}_0^+, \tilde{X})$$

for all $(\tilde{x}_0, d) \in \mathcal{D}$

(ii) *there exist $\sigma, \gamma \in \mathcal{K}$ such that*

$$\|\tilde{x}(t, \tilde{x}_0, d)\| \le \sigma(\|\tilde{x}_0\|) + \gamma(\|d\|_{[0,t],2}) \qquad (t \in \mathbb{R}_0^+)$$

for all $(\tilde{x}_0, d) \in \mathcal{D}$.

B. Solvability in the generalized sense

We can now establish solvability in some generalized sense also for general data $(\tilde{x}_0, d) \in \tilde{X} \times L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$. We do so by showing the following density and approximation result.

Theorem III.4. *With the above assumptions, we have for every $(\tilde{x}_0, d) \in \tilde{X} \times L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$:*

(i) *there exists a sequence $(\tilde{x}_{0n}, d_n)$ in $\mathcal{D}$ converging to $(\tilde{x}_0, d)$ in the locally convex topology of $\tilde{X} \times L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$*
(ii) *for every such sequence $(\tilde{x}_{0n}, d_n)$,*

$$\big(\tilde{x}(\cdot, \tilde{x}_{0n}, d_n)\big) \quad \text{and} \quad \big(y(\cdot, \tilde{x}_{0n}, d_n)\big)$$

are Cauchy sequences in the locally convex spaces $C(\mathbb{R}_0^+, \tilde{X})$ and $L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$, and the respective limits *are independent of the particular choice of the sequence* $(\tilde{x}_{0n}, d_n)$.

Assertion (i) rests on a simple density argument. Assertion (ii) ultimately rests on the following integral equation for classical solutions of (13) which follows from (14) by variation of constants:

$$\tilde{x}(t, \tilde{x}_0, d) = \mathrm{e}^{\tilde{A}t}\tilde{x}_0 + \int_0^t \mathrm{e}^{\tilde{A}(t-s)}\tilde{f}(\tilde{x}(s, \tilde{x}_0, d))\,\mathrm{d}s + \Phi_t(d)$$

for every $(\tilde{x}_0, d) \in \mathcal{D}$, where

$$\Phi_t(d) := -\mathrm{e}^{\tilde{A}t}\tilde{R}d(0) + \tilde{R}d(t) + \int_0^t \mathrm{e}^{\tilde{A}(t-s)}\big(\tilde{\mathcal{A}}\tilde{R}d(s) - \tilde{R}d'(s)\big)\,\mathrm{d}s.$$

A central ingredient in the proof of assertion (ii) is the following lemma which implies that the linearized boundary control system

$$\tilde{x}' = \tilde{\mathcal{A}}\tilde{x} \quad \text{with} \quad d(t) = \tilde{\mathcal{B}}\tilde{x}(t) \tag{15}$$

is input-admissible w.r.t. inputs $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$.

Lemma III.5. *Under the assumptions of the above theorem,* $\Phi_t : C_c^2(\mathbb{R}_0^+, \mathbb{R}^k) \to \tilde{X}$ *for every* $t \in \mathbb{R}_0^+$ *has a unique bounded extension* $\overline{\Phi}_t : L^2(\mathbb{R}_0^+, \mathbb{R}^k) \to \tilde{X}$ *and*

$$\sup_{t\in[0,t_0]} \left\|\overline{\Phi}_t\right\| \le \left\|\overline{\Phi}_{t_0}\right\|.$$

In view of the above theorem, we can define for $(\tilde{x}_0, d) \in \tilde{X} \times L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$ the functions

$$\tilde{x}(\cdot, \tilde{x}_0, d) := \lim_{n\to\infty} \tilde{x}(\cdot, \tilde{x}_{0n}, d_n) \in C(\mathbb{R}_0^+, \tilde{X})$$
$$y(\cdot, \tilde{x}_0, d) := \lim_{n\to\infty} y(\cdot, \tilde{x}_{0n}, d_n) \in L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k),$$

where $(\tilde{x}_{0n}, d_n)$ is any sequence in $\mathcal{D}$ converging to $(\tilde{x}_0, d)$ in the locally convex topology of $\tilde{X} \times L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$. We call these functions the generalized solution and the generalized output corresponding to $(\tilde{x}_0, d)$ because they coincide with the corresponding classical objects for classical data and because, by the next theorem, they share many important properties of classical solutions and classical outputs.

Theorem III.6. *With the above assumptions, we have:*

(i) *the generalized solution map* $(\tilde{x}_0, d) \mapsto \tilde{x}(\cdot, \tilde{x}_0, d)$ *satisfies the cocycle (or flow) property, that is,*

$$\tilde{x}(t+s, \tilde{x}_0, d) = \tilde{x}(t, \tilde{x}(s, \tilde{x}_0, d), d(s+\cdot))$$

for all $s, t \in \mathbb{R}_0^+$ *and all* $(\tilde{x}_0, d) \in \tilde{X} \times L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$

(ii) *the generalized solution map* $(\tilde{x}_0, d) \mapsto \tilde{x}(\cdot, \tilde{x}_0, d) \in C(\mathbb{R}_0^+, \tilde{X})$ *and output map* $(\tilde{x}_0, d) \mapsto y(\cdot, \tilde{x}_0, d) \in L^2_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^k)$ *both are continuous and causal.*

In particular, this theorem ensures that our closed-loop system $\tilde{\mathfrak{S}}$ falls within the general framework of [8] and that it is well-posed in the spirit of [14].

IV. INPUT-TO-STATE STABILITY OF THE CLOSED-LOOP SYSTEM

After having established the global solvability of the closed-loop system, we can now move on to stability. A first very simple result is the following uniform global stability [8] theorem. It is an immediate consequence of Theorem III.3 (ii) and Theorem III.4 (i).

Theorem IV.1. *With the above assumptions, there exist* $\sigma, \gamma \in \mathcal{K}$ *such that*

$$\|\tilde{x}(t, \tilde{x}_0, d)\| \le \sigma(\|\tilde{x}_0\|) + \gamma(\|d\|_2) \qquad (t \in \mathbb{R}_0^+)$$

for every $\tilde{x}_0 \in \tilde{X}$ *and every* $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$.

We are now going to improve this uniform global stability result to a (uniform) input-to-state stability result for a special class of nonlinear controllers and to a weak input-to-state stability result for a more general class of nonlinear controllers.

A. Input-to-state stability

Input-to-state stability of the closed-loop system means [8] that it is uniformly globally stable and of uniform asymptotic gain, where uniform asymptotic gain in turn means the following: there is a (so-called uniform gain) function $\gamma \in \mathcal{K} \cup \{0\}$ such that for every $\varepsilon, r > 0$ there is a time $\tau = \tau(\varepsilon, r)$ such that for every $\tilde{x}_0 \in \tilde{X}$ with $\|\tilde{x}_0\| \le r$ and every $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$

$$\|\tilde{x}(t, \tilde{x}_0, d)\| \le \varepsilon + \gamma(\|d\|_2)$$

for every $t \ge \tau$. With the help of the cocycle property and the fact that $\|d(t_0 + \cdot)\|_2 \longrightarrow 0$ as $t_0 \to \infty$, one easily obtains the following lemma.

Lemma IV.2. *If the assumptions of the well-posedness theorem (Assumption II.1, II.2, III.1) are satisfied and if* $\tilde{\mathfrak{S}}$ *is uniformly input-to-state stable w.r.t. inputs from* $L^2(\mathbb{R}_0^+, \mathbb{R}^k)$*, then for every* $(\tilde{x}_0, d) \in \tilde{X} \times L^2(\mathbb{R}_0^+, \mathbb{R}^k)$ *one has*

$$\tilde{x}(t, \tilde{x}_0, d) \longrightarrow 0 \qquad (t \to \infty). \tag{16}$$

In order to obtain input-to-state stability, we add the following assumptions to the assumptions from the well-posedness theorem (Theorem III.6).

Assumption IV.3. $\zeta \mapsto \mathcal{H}(\zeta)$ *is continuously differentiable and there is a constant* $\kappa > 0$ *such that*

$$|\mathcal{B}x|^2 + |\mathcal{C}x|^2 \ge \kappa |(\mathcal{H}x)(b)|^2 \qquad (x \in D(\mathcal{A})). \tag{17}$$

Assumption IV.4. (i) $\mathcal{P}$ *is quasi-quadratic, that is, for some constants* $\underline{c}_1, \overline{c}_1 > 0$

$$\overline{c}_1\, v_1^\top \nabla\mathcal{P}(v_1) \ge \mathcal{P}(v_1) \ge \underline{c}_1 |v_1|^2 \qquad (v_1 \in \mathbb{R}^{m_c})$$

(ii) $\mathcal{R}$ *is quasi-linear, that is, for some constants* $\underline{c}_2, \overline{c}_2 > 0$

$$\overline{c}_2\, v_2^\top \mathcal{R}(v_2) \ge |v_2|^2 \ge \underline{c}_2 |\mathcal{R}(v_2)|^2 \qquad (v_2 \in \mathbb{R}^{m_c}).$$

Theorem IV.5. *With the assumptions of the well-posedness theorem (Assumption II.1, II.2, III.1) and the assumptions* *just made (Assumption IV.3, IV.4), we have that the closed-loop system $\tilde{\mathfrak{S}}$ is input-to-state stable. In particular, the convergence* (16) *holds true.*

We now make some remarks on our strategy of proving the above theorem: first we record the most central ingredients in the form of two lemmas and then we roughly explain how and where these central ingredients come into play.

Lemma IV.6. *Under the assumptions of the above theorem, there exists a function $c_0 \in \mathcal{L}$ and a $t_0 > 0$ such that for every $(\tilde{x}_0, d) \in \mathcal{D}$*

$$\tilde{E}(\tilde{x}(t, \tilde{x}_0, d)) \leq c_0(t)\bigg(\int_0^t |(\mathcal{H}x(s))(b)|^2 \,\mathrm{d}s + \int_0^t E_c(v(s)) \,\mathrm{d}s\bigg) + \int_0^t |d(s)||y(s)| \,\mathrm{d}s$$

for all $t \geq t_0$, where $(x, v) := \tilde{x}(\cdot, \tilde{x}_0, d)$ and $y := y(\cdot, \tilde{x}_0, d)$.

Lemma IV.7. *Under the assumptions of the above theorem, there exists a $C_0 > 0$ such that for every $(\tilde{x}_0, d) \in \mathcal{D}$*

$$\int_0^t E_c(v(s)) \,\mathrm{d}s \leq C_0\bigg(\tilde{E}(\tilde{x}_0) + \int_0^t |y(s)|^2 \,\mathrm{d}s\bigg)$$

for all $t \geq 0$, where $(x, v) := \tilde{x}(\cdot, \tilde{x}_0, d)$ and $y := y(\cdot, \tilde{x}_0, d)$.

A first step in the proof of the above theorem is to show that there exist $C_1, \tau_0 > 0$ such that for all $\tilde{x}_0 \in \tilde{X}$, $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$

$$\tilde{E}(\tilde{x}(\tau_0, \tilde{x}_0, d)) \leq \frac{1}{2}\,\tilde{E}(\tilde{x}_0) + C_1\,\|d\|^2_{[0,\tau_0],2}\,.$$

In order to get this, we start out from the following energy estimate:

$$\tilde{E}(\tilde{x}(t, \tilde{x}_0, d)) \leq \tilde{E}(\tilde{x}_0) + \int_0^t d(s)^\top y(s) \,\mathrm{d}s - \overline{c}_2^{-1}\int_0^t |Kv_2(s)|^2 \,\mathrm{d}s - \varsigma \int_0^t |y(s)|^2 \,\mathrm{d}s \quad (18)$$

for all $(\tilde{x}_0, d) \in \mathcal{D}$, where $\varsigma > 0$ is the smallest eigenvalue of $S_c > 0$. Assumption IV.3 yields a constant $\kappa > 0$ such that

$$|y(s)|^2 \geq \kappa |(\mathcal{H}x(s))(b)|^2 - |u(s)|^2 \quad (19)$$

for all $s \geq 0$. Also, by virtue of Lemma IV.6 and IV.7, there exist $c_0 \in \mathcal{L}$ and $t_0 > 0$ such that for all $(\tilde{x}_0, d) \in \mathcal{D}$

$$\tilde{E}(\tilde{x}(t, \tilde{x}_0, d)) \leq c_0(t)\bigg(\int_0^t |(\mathcal{H}x(s))(b)|^2 \,\mathrm{d}s + \tilde{E}(\tilde{x}_0) + \int_0^t |y(s)|^2 \,\mathrm{d}s\bigg) + \int_0^t |d(s)||y(s)| \,\mathrm{d}s \quad (20)$$

for all $t \geq t_0$. Inserting (19) and (20) into (18) and estimating terms appropriately, the first step follows. A second step in the proof of the above theorem is to show that there exists $C_2 > 0$ such that for all $\tilde{x}_0 \in \tilde{X}$, $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$

$$\tilde{E}(\tilde{x}(t, \tilde{x}_0, d)) \leq 2\left(\frac{1}{2}\right)^{t/\tau_0} \tilde{E}(\tilde{x}_0) + C_2\,\|d\|^2_{[0,t],2}$$

for all $t \geq 0$. With the help of induction and the cocycle property of generalized solutions, this follows from the first step. In view of (12) the second step yields the desired uniform asymptotic gain property.

B. Weak input-to-state stability

Weak input-to-state stability of the closed-loop system means, by definition, that it is uniformly globally stable and of weak asymptotic gain, where weak asymptotic gain in turn means the following: there is a (so-called weak gain) function $\gamma \in \mathcal{K} \cup \{0\}$ such that for every $\varepsilon > 0$ and every $\tilde{x}_0 \in \tilde{X}$, $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$ there is a time $\tau = \tau(\varepsilon, \tilde{x}_0, d)$ such that

$$\|\tilde{x}(t, \tilde{x}_0, d)\| \leq \varepsilon + \gamma(\|d\|_2)$$

for all $t \geq \tau$. So the only difference to the uniform asymptotic gain property is that the time τ is allowed to depend on the initial state $\tilde{x}_0$ (instead of only on the norm thereof) and on the disturbance d. In [8] the weak asymptotic gain property is called just asymptotic gain. With the help of the cocycle property and the fact that $\|d(t_0 + \cdot)\|_2 \longrightarrow 0$ as $t_0 \to \infty$, one easily obtains the following lemma.

Lemma IV.8. *If the assumptions of the well-posedness theorem (Assumption II.1, II.2, III.1) are satisfied and if $\tilde{\mathfrak{S}}$ is weakly input-to-state stable w.r.t. inputs from $L^2(\mathbb{R}_0^+, \mathbb{R}^k)$, then for every $(\tilde{x}_0, d) \in \tilde{X} \times L^2(\mathbb{R}_0^+, \mathbb{R}^k)$ one has*

$$\tilde{x}(t, \tilde{x}_0, d) \longrightarrow 0 \qquad (t \to \infty). \quad (21)$$

In order to obtain weak input-to-state stability, we add the following assumptions to the assumptions from the well-posedness theorem (Theorem III.6).

Assumption IV.9. *$\mathfrak{S}$ is even impedance-energy preserving (meaning that* (5) *holds with equality), $\zeta \mapsto \mathcal{H}(\zeta)$ is continuously differentiable, and there is a constant $\kappa > 0$ such that*

$$|\mathcal{B}x|^2 + |\mathcal{C}x|^2 \geq \kappa |(\mathcal{H}x)(b)|^2 \qquad (x \in D(\mathcal{A})). \quad (22)$$

Assumption IV.10. (i) *$\mathcal{R}$ is strictly damping, that is, for some constants $\underline{c}, \overline{c}, \delta > 0$,*

$$v_2^\top \mathcal{R}(v_2) \geq \underline{c}|v_2|^2 \qquad (|v_2| \leq \delta)$$
$$v_2^\top \mathcal{R}(v_2) \geq \overline{c} \qquad (|v_2| > \delta)$$

(ii) *B_c, the input operator of the controller, is injective and 0 is the only critical point of $\mathcal{P}$.*

Theorem IV.11. *With the assumptions of the well-posedness theorem (Assumption II.1, II.2, III.1) and the assumptions just made (Assumption IV.9, IV.10), we have that the closed-loop system $\tilde{\mathfrak{S}}$ is weakly input-to-state stable (with weak gain $\gamma = 0$). In particular, we have the convergence* (21).

We again make some remarks on our strategy of proving the above theorem. In doing so, it will be crucial to write the nonlinear part of (13) as

$$\tilde{f}(\tilde{x}) = \tilde{B}g(\tilde{B}^*\tilde{x}) + \tilde{B}h(\tilde{C}\tilde{x}),$$

where $\tilde{B} : \mathbb{R}^{m_c} \to \tilde{X}$, $\tilde{C} : \tilde{X} \to \mathbb{R}^{m_c}$ and $g, h : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$ are such that

$$\tilde{B}v_2 := (0, 0, v_2), \qquad \tilde{C}\tilde{x} := v_1,$$
$$g(v_2) := -\mathcal{R}(v_2) \quad \text{and} \quad h(v_1) := v_1 - \nabla\mathcal{P}(v_1).$$

In particular, $\tilde{B}^*\tilde{x} = Kv_2$ for every $\tilde{x} = (x, v_1, v_2) \in \tilde{X}$. We can thus rewrite the closed-loop equations (13) as

$$\tilde{x}' = \big(\tilde{\mathcal{A}} - \lambda\tilde{B}\tilde{B}^*\big)\tilde{x} + \tilde{B}\big(\lambda\tilde{B}^*\tilde{x} + g(\tilde{B}^*\tilde{x})\big) + \tilde{B}h(\tilde{C}\tilde{x})$$
$$d(t) = \tilde{\mathcal{B}}\tilde{x}(t) \tag{23}$$

that is, as a perturbation of the respective linearized system

$$\tilde{x}' = \big(\tilde{\mathcal{A}} - \lambda\tilde{B}\tilde{B}^*\big)\tilde{x} \quad \text{and} \quad d(t) = \tilde{\mathcal{B}}\tilde{x}(t), \tag{24}$$

where $\lambda > 0$. It follows from (23) by variation of constants that classical solutions of (13) satisfy the following integral equation:

$$\tilde{x}(t, \tilde{x}_0, d) = \mathrm{e}^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)t}\tilde{x}_0 + \int_0^t \mathrm{e}^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)(t-s)}$$
$$\tilde{B}\big(\lambda\tilde{B}^*\tilde{x}(s, \tilde{x}_0, d) + g(\tilde{B}^*\tilde{x}(s, \tilde{x}_0, d))\big)\,\mathrm{d}s \tag{25}$$
$$+ \int_0^t \mathrm{e}^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)(t-s)}\tilde{B}h(\tilde{C}\tilde{x}(s, \tilde{x}_0, d))\,\mathrm{d}s + \Phi_t^\lambda(d)$$

for every $(\tilde{x}_0, d) \in \mathcal{D}$, where

$$\Phi_t^\lambda(d) := -\mathrm{e}^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)t}\tilde{R}d(0) + \tilde{R}d(t)$$
$$+ \int_0^t \mathrm{e}^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)(t-s)}\big((\tilde{\mathcal{A}} - \lambda\tilde{B}\tilde{B}^*)\tilde{R}d(s) - \tilde{R}d'(s)\big)\,\mathrm{d}s.$$

A first important ingredient to the proof of the theorem is the following approximate observability result for the collocated linear system $\mathfrak{S}(\tilde{A}, \tilde{B}, \tilde{B}^*)$.

Lemma IV.12. *Under the assumptions of the above theorem, the linear system $\mathfrak{S}(\tilde{A}, \tilde{B}, \tilde{B}^*)$ is approximately observable in infinite time.*

A second important ingredient is the following stabilization result for the collocated linear system $\mathfrak{S}(\tilde{A}, \tilde{B}, \tilde{B}^*)$, which hinges on the approximate observability property just established and on the compactness of the resolvent of $\tilde{A}$ (Proposition III.2).

Lemma IV.13. *Under the assumptions of the above theorem,*

(i) $e^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)\cdot}$ *is strongly stable and* $(\tilde{A} - \lambda\tilde{B}\tilde{B}^*)^{-1}$ *is compact*

(ii) *for every* $u \in L^2(\mathbb{R}_0^+, \mathbb{R}^{m_c})$,
$$\int_0^t e^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)(t-s)}\tilde{B}u(s)\,ds \longrightarrow 0 \qquad (t \to \infty)$$

(iii) *for every bounded* $u \in AC_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^{m_c})$ *with* $u' \in L^2(\mathbb{R}_0^+, \mathbb{R}^{m_c})$ *and every sequence* (t_n) *with* $t_n \longrightarrow \infty$, *there is a subsequence* (t_{n_l}) *such that*
$$\lim_{l\to\infty}\int_0^{t_{n_l}} e^{(\tilde{A}-\lambda\tilde{B}\tilde{B}^*)(t_{n_l}-s)}\tilde{B}u(s)\,ds$$
$$= -\lim_{l\to\infty}(\tilde{A} - \lambda\tilde{B}\tilde{B}^*)^{-1}\tilde{B}u(t_{n_l}).$$

Lemma IV.14. *Under the assumptions of the above theorem, we have for every $\tilde{x}_0 \in \tilde{X}$ and $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$ that*

(i) $\tilde{B}^*\tilde{x}(\cdot, \tilde{x}_0, d), g \circ (\tilde{B}^*\tilde{x}(\cdot, \tilde{x}_0, d)) \in L^2(\mathbb{R}_0^+, \mathbb{R}^{m_c})$

(ii) $h \circ (\tilde{C}\tilde{x}(\cdot, \tilde{x}_0, d)) \in AC_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{R}^{m_c})$ *is bounded with derivative* $(h \circ (\tilde{C}\tilde{x}(\cdot, \tilde{x}_0, d)))' \in L^2(\mathbb{R}_0^+, \mathbb{R}^{m_c})$.

A third important ingredient is the following lemma which implies that the linearized boundary control system

$$\tilde{x}' = \big(\tilde{\mathcal{A}} - \lambda\tilde{B}\tilde{B}^*\big)\tilde{x} \quad \text{and} \quad d(t) = \tilde{\mathcal{B}}\tilde{x}(t)$$

for $\lambda > 0$ is infinite-time input-admissible w.r.t. inputs $d \in L^2(\mathbb{R}_0^+, \mathbb{R}^k)$.

Lemma IV.15. *Under the assumptions of the above theorem, $\Phi_t^\lambda : C_c^2(\mathbb{R}_0^+, \mathbb{R}^k) \to \tilde{X}$ for every $t \in \mathbb{R}_0^+$ and $\lambda > 0$ has a unique bounded extension $\overline{\Phi}_t^\lambda : L^2(\mathbb{R}_0^+, \mathbb{R}^k) \to \tilde{X}$ and*

$$\sup_{t\in[0,\infty)} \left\|\overline{\Phi}_t^\lambda\right\| < \infty.$$

With the help of Lemma IV.15 it follows that the above integral equation (25) for classical solutions extends to generalized solutions. Applying the last three lemmas (Lemma IV.13, IV.14, IV.15) to this extended integral equation, we can finally show that every generalized solution converges to zero as desired.

References

[1] B. Augner: *Well-posedness and stability of linear port-Hamiltonian systems with nonlinear boundary feedback.* arXiv:1506.02921 (2015)

[2] R. Curtain, H. Zwart: An introduction to infinite-dimensional linear systems theory. Springer (1995)

[3] V. Duindam, A. Macchelli, S. Stramigioli, H. Bruyninckx (editors): Modeling and control of complex physical systems – the port-Hamiltonian approach. Springer (2009)

[4] H. O. Fattorini: *Boundary control systems.* SIAM J. Contr. **6** (1968), 349-388

[5] B. Jacob, H. Zwart: Linear port-Hamiltonian systems on infinite-dimensional spaces. Birkäuser (2012)

[6] B. Jacob, K. Morris, H. Zwart: *C_0-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain.* J. Evol. Equ. **15** (2015), 493-502

[7] M. Miletic, D. Stürzer, A. Arnold, A. Kugi: *Stability of an Euler-Bernoulli beam with a nonlinear dynamic feedback system.* IEEE Trans. Autom. Contr. **61** (2016), 2782-2795

[8] A. Mironchenko, F. Wirth: *Characterizations of input-to-state stability for infinite-dimensional systems.* arXiv:1701.08952 (2017)

[9] A. Pazy: Semigroups of linear operators and applications to partial differential equations. Springer (1983)

[10] H. Ramirez, Y. Le Gorrec, A. Macchelli, H. Zwart: *Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback.* IEEE Trans. Autom. Contr. **59** (2014), 2849-2855

[11] H. Ramirez, H. Zwart, Y. Le Gorrec: *Stabilization of infinite-dimensional port-Hamiltonian systems by nonlinear dynamic boundary control.* Automatica **85** (2017), 61-69

[12] J. Schmid, H. Zwart: *Stabilization of port-Hamiltonian systems by nonlinear boundary control in the presence of disturbances.* arXiv:1804.10598 (2018)

[13] A. Tanwani, C. Prieur, S. Tarbouriech: *Disturbance-to-state stabilization and quantized control for linear hyperbolic systems.* arXiv:1703.00302 (2017)

[14] M. Tucsnak, G. Weiss: *Well-posed systems – the LTI case and beyond.* Automatica **50** (2014), 1757-1779

[15] H. Zwart, H. Ramirez, Y. Le Gorrec: *Asymptotic stability for a class of boundary control systems with non-linear damping.* In: The second IFAC workshop on control of systems governed by partial differential equations (CPDE). Bertinoro, Italy (2016)