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Stable polynomials in matrix variables, their properties and representations

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A multivariate polynomial is *stable* if it has no zeros in the positive orthant in $\mathbb{C}^n$. Stable polynomials are a classical topic in control theory and in areas of mathematics ranging from complex analysis to real algebraic geometry. Of course all of this classical theory concerns polynomials (in commuting variables) acting on $\mathbb{C}^n$.

The talk describes a new class of results on noncommutative polynomials f acting on g-tuples of matrix variables X. Let $\mathbb{H}^g$ denote the *matricial positive orthant*, the set of all tuples of square complex matrices $X = (X_1, \dots, X_g)$ such that $\operatorname{im} X_j := \frac{1}{2i}(X_j - X_j^*)$ is a positive definite matrix for every $j = 1, \dots, g$. A complex noncommutative polynomial f is *stable* if for each $X \in \mathbb{H}^g$ the matrix $f(X)$ is invertible. The talk will describe considerable recent progress on nc stable polynomials.

Now to some classical context. Within the mathematics community, in recent years, there has been an increased interest in multivariate stable polynomials and closely related families (e.g. hyperbolic polynomials or rational inner functions). One of the most prominent new problems is the existence of distinguished determinantal representations for stable polynomials, which is related to the generalized Lax conjecture, the multivariate Pólya-Schur program, and the celebrated Kadison–Singer problem. Such a determinantal representation is a structural feature that naturally extends from scalar variables to matrix variables, which leads to questions about stable linear matrix pencils, and stable noncommutative polynomials and rational functions. The talk will address this noncommutative aspect of stability inspired by free analysis and free real algebraic geometry.

Next we give an idea of our results on nc stable polynomials. A linear matrix pencil $L = A_0 + A_1 x_1 + \cdots + A_g x_g$ with coefficients $A_j \in \mathrm{M}_d(\mathbb{C})$ is said to be *stable* if

$$L(X) := A_0 \otimes I + A_1 \otimes X_1 + \cdots + A_g \otimes X_g \qquad (1)$$

is invertible for every $X \in \mathbb{H}^g$. For example, if

$$L = (H + iP_0) + P_1 x_1 + \cdots + P_g x_g \qquad (2)$$

where H is hermitian and P_j are positive semidefinite matrices without common nullspace, then it is easy to check that L is stable. One of the main results presented in the talk states that every stable linear matrix pencil is essentially composed of blocks of the form (1).

Using state space realization theory and its relation to factorization of noncommutative polynomials one proves that if f is a stable nc polynomial, then

$$\det f(X) = \det L(X) \qquad (3)$$

for some stable L of the distinguished form (2). The relation (3) can be seen as a determinantal representation of f.

Lastly, hermitian polynomials are the noncommutative analogs of real commutative polynomials. Contrary to the classical commutative theory, stable hermitian polynomials have an extremely rigid structure. For example, irreducible stable hermitian polynomials are necessarily affine. However, by extending the scope of stability to hermitian noncommutative rational functions one obtains a much richer class, indicating that determinantal representations are related to noncommutative lifts of stable commutative polynomials to stable noncommutative rational functions.

No paper will be submitted to the proceedings.

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