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Reduction of Wave Linear Repetitive Processes to 2-D Singular State Space form

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***Abstract*— In this paper, a method is given that reduces a polynomial system matrix describing a discrete wave linear repetitive process to an equivalent 2-D singular Fornasini-Marchesini (SFM) model. It is shown that the transformation linking the original polynomial system matrix with its associated 2-D SFM form is zero coprime system equivalence. The exact nature of the resulting system matrix in singular form and the transformation involved are established.**

I. Introduction

An important aspect of the research in 2-D linear Systems theory has been the connection between various 2-D singular and nonsingular state space representations, e.g. [1] and the references therein. Recently, the 2-D state space models have been shown to represent linear repetitive processes and the motivation being that the theory already developed in the analysis and synthesis of the standard 2-D state space models may serve to tackle the problems related to these new classes of systems, see for example [2]. For example, in [3] it was shown that the way to obtain conditions for local controllability of discrete linear repetitive processes was to convert the repetitive process state-space model to that of a singular 2-D Roesser state-space model. Hence the motivation to establish the connection between the state space representations of the repetitive systems and those of the standard 2-D models, e.g. [4], [2]. Also, in [5] it was shown that a linear repetitive process is equivalent to a 2-D singular Roesser model. An elementary operations based method for transforming a polynomial matrix description of linear repetitive processes to a 2-D nonsingular Roesser model was proposed in [6]. This method was extended in [7] to reduce so-called wave repetitive processes to 2-D singular Roesser form. The reduction transformation involved is that of Input/Output equivalence. In this paper, a method is presented for the reduction of wave repetitive processes to 2-D Fornasini-Marchesini singular forms such that both the Input/Output properties of the system and the zero structure are preserved. Furthermore the exact equivalence transformation linking the original system with its associated singular form is established. The type of equivalence used has been the subject of considerable attention in the literature, e.g., [8], [9] and [10], [11].

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II. Discrete Linear 2-D Systems and Repetitive Processes

A singular version of the 2-D Fornasini-Marchesini state space model (SFM) model [12] is given by

$$\begin{array}{rcl} Ex(i+1,j+1) & = & A_1x(i+1,j)+A_2x(i,j+1) \\ & + & A_0x(i,j)+Bu(i,j), \\ y(i,j) & = & Cx(i,j)+Du(i,j), \end{array} \tag{1}$$

where $x(i,j)$ is the state vector, $u(i,j)$ is the input vector, $y(i,j)$ is the output vector, E, A_0, A_1, A_2, B, C and D are constant real matrices of appropriate dimensions and E may be singular.

Discrete linear repetitive processes evolve over the subset of the positive quadrant in the 2-D plane defined by $\{(p,k) : 0 \leq p \leq \alpha-1, k \geq 0\}$, and the most basic state-space model for their dynamics has the following form [2]

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p)+Bu_{k+1}(p)+B_0y_k(p), \\ y_{k+1}(p) &= Cx_{k+1}(p)+Du_{k+1}(p)+D_0y_k(p), \end{aligned} \tag{2}$$

where α denotes the number of samples along the pass. On pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^l$ is the vector of control inputs. The simplest form of boundary conditions are $x_{k+1}(0) = d_{k+1},\ k \geq 0$, where the $n \times 1$ vector d_{k+1} has known constant entries, and $y_0(p) = f(p)$, where $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

The 2-D systems structure of a repetitive process arises from the influence of the previous pass profile on the current pass state and pass profile vectors, i.e., due to the presence of the terms $B_0y_k(p)$ and $D_0y_k(p)$ in (2) respectively.

In the repetitive process model (2), the only previous pass (k) contribution to the dynamics at p on the current pass $(k+1)$ comes from the same instance. An alternative, more general, discrete linear repetitive process that also evolves over $\{(p,k) : 0 \leq p \leq \alpha-1, k \geq 0\}$, where the previous pass (k) contribution to the dynamics at the given sample p on the current pass $(k+1)$, comes from a pre-specified window of points and the state-space model is

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + \hat{B}u_{k+1}(p) \\ &\quad + \sum_{i=-w_L}^{w_H} B_i y_k(p+i), \\ y_{k+1}(p) &= Cx_{k+1}(p) + \hat{D}u_{k+1}(p) \\ &\quad + \sum_{i=-w_L}^{w_H} D_i y_k(p+i), \end{aligned} \tag{3}$$

where on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, $u_k(p) \in \mathbb{R}^l$ is the vector of control inputs and w_L, w_H are positive integers.

On each pass in the model (3), the previous pass (k) 'window' of samples $p - w_L \leq p \leq p + w_H$, that moves along the pass, contribute to the current pass $k+1$. This has led the term 'wave' repetitive process [13] to describe examples represented by this model. Also setting $w_L = 0$ and $w_H = 0$ recovers the previous state-space model.

The boundary conditions for a wave repetitive process are of the form

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0, \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1, \\ x_{k+1}(i) &= 0, \quad y_k(i) = 0, \\ i \in \{-w_L, \ldots, -1\} &\cup \{\alpha, \ldots, \alpha - 1 + w_H\}, \ k \geq 0, \end{aligned} \tag{4}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p. One area where a wave repetitive process state-space model arises is iterative learning control. This design method has been especially developed for the many applications where the same finite duration task is performed over and over again. Each repetition is termed a pass (or trial in some of the literature) and the duration of each trial is termed the trial length.

In the simplest form of operation the system resets to the starting location at the end of each pass and the next pass can begin either immediately resetting is complete or after a further period of time has elapsed. Once a pass is complete all information generated over the pass length is available for use in constructing the input for the next pass. The core task in iterative learning control design, therefore, is how to use this information to best effect in improving performance from pass-to-pass and the most common route is to construct the input for the next pass as the sum of the previous pass input and a correction term computed using previous trial data (or a finite number of previous trials).

Background references on iterative learning control can be found in the references cited in, e.g., [14] which reports the design and experimental verification of an iterative learning control law designed in the repetitive process setting. Let $e_k(p)$ denote the error between a supplied reference signal and the output on each pass. Also let $u_k(p)$ denote the input on pass k. Then one formula for computing the control input on pass $k+1$ is

$$u_{k+1}(p) = u_k(p) + Ke_k(p+1), \tag{5}$$

where K is a scalar gain (or matrix in the multi-input multi-output case). The term $e_k(p+1)$ is causal and hence implementable despite the shit in p because it can be computed for all p on the completion of pass k. This is the novel feature of iterative learning control and is termed 'phase-lead'.

In some applications it is beneficial to include in the control law further phase-lead terms, e.g., at $p+2$ and also 'phase-lag', e.g., at $p-1$ terms. Such a control law fits naturally within a wave repetitive process state-space model.

The similarity between wave repetitive processes and the Roesser model is much less obvious and hence there still remains the question: is it possible to convert a wave repetitive process to Roesser form?. This question is the subject of the future works with the emphasis on system matrix equivalence.

III. System Equivalence

Following the formulation of Rosenbrock [15], a general linear 2-D system can be represented by the following system of equations:

$$\begin{aligned} T(z_1, z_2)x &= U(z_1, z_2)u, \\ y &= V(z_1, z_2)x + W(z_1, z_2)u, \end{aligned} \tag{6}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^l$ is the input vector and $y \in \mathbb{R}^m$ is the output vector, T, U, V and W are polynomial matrices with elements in $\mathbb{R}[z_1, z_2]$ of dimensions $n \times n, n \times l, m \times n$ and $m \times l$ respectively. The operators z_1 and z_2 may have various meanings depending on the type of system. For example, in delay-differential systems z_1 may represent a differential operator and z_2 a delay-operator. In the case of 2-D discrete systems, z_1 and z_2 represent horizontal and vertical shift operators, respectively, but in this paper only the latter case is considered. The system (6) gives rise to the system matrix in the general form:

$$P(z_1, z_2) = \begin{bmatrix} T(z_1, z_2) & U(z_1, z_2) \\ -V(z_1, z_2) & W(z_1, z_2) \end{bmatrix}, \tag{7}$$

where

$$P(z_1, z_2) \begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}. \tag{8}$$

In the case when $T(z_1, z_2)$ invertible, the system matrix in (7) is said to be regular. The transfer-function matrix corresponding to the system matrix in (7) is given by:

$$G(z_1, z_2) = V(z_1, z_2)T^{-1}(z_1, z_2)U(z_1, z_2) + W(z_1, z_2). \tag{9}$$

The concept of coprimeness plays a crucial role in the theory of linear systems. In 2-D there are two distinct notions of coprimeness, namely factor coprimeness and zero coprimeness. The latter forms the basis of the definition of the transformation used in this paper and is defined as follows.

Definition 1: Let $D = \mathbb{K}[z_1, z_2]$ be a polynomial ring in z_1 and z_2 and $\mathbb{K}$ an arbitrary but fixed field. Two polynomial matrices P_1 and S_1 of compatible dimensions, with elements in D, are said to be zero left coprime (ZLC) if the matrix

$$\begin{bmatrix} P_1 & S_1 \end{bmatrix} \tag{10}$$

admits a right inverse over D.
Similarly, two polynomial matrices P_2 and S_2, of compatible dimensions, with elements in D are said to be zero right coprime (ZRC) if the matrix

$$\begin{bmatrix} P_2 \\ S_2 \end{bmatrix} \tag{11}$$

admits a left inverse over D.
There are a number of equivalence concepts in n-D systems theory. Most of them are extensions from the 1-D to the n-D setting. More recently, a transformation using a module theoretic approach was used in [16]. A basic transformation proposed for the study of 2-D system matrices is zero coprime system equivalence as given in [8], [9]. This transformation may be regarded as an extension of Fuhrmann's strict system equivalence [17] from the 1-D to the 2-D case and is characterized by the following definition.

Definition 2: Let $\mathbb{P}(m, l)$ denote the class of $(n+m) \times (n+l)$ with polynomial system matrices in z_1 and z_2 with real coefficients.Two polynomial system matrices $P_1(z_1, z_2)$ and $P_2(z_1, z_2) \in \mathbb{P}(m, l)$, are said to be zero coprime system equivalent if they are related by the following

$$\underbrace{\begin{bmatrix} M & 0 \\ X & I_m \end{bmatrix}}_{S_1(z_1,z_2)} \underbrace{\begin{bmatrix} T_1 & U_1 \\ -V_1 & W_1 \end{bmatrix}}_{P_1(z_1,z_2)} = \underbrace{\begin{bmatrix} T_2 & U_2 \\ -V_2 & W_2 \end{bmatrix}}_{P_2(z_1,z_2)} \underbrace{\begin{bmatrix} N & Y \\ 0 & I_l \end{bmatrix}}_{S_2(z_1,z_2)} \tag{12}$$

where $P_1(z_1, z_2), S_2(z_1, z_2)$ are zero right coprime and $P_2(z_1, z_2), S_1(z_1, z_2)$ are zero left coprime and $M(z_1, z_2)$, $N(z_1, z_2)$, $X(z_1, z_2)$ and $Y(z_1, z_2)$ are polynomial matrices of compatible dimensions.
The concepts of controllability, observability and stability lie at the heart of the theory of linear systems and these concepts are characterized by the zero structure of their associated system matrices, see for example [18]. The transformation of zero coprime system equivalence plays a key role in many aspects of 2-D systems theory, see for example ([9], [8], [11] and [10]) as illustrated by the following lemma.

Lemma 1 (Johnson, [9]): The transformation of zero coprime system equivalence preserves the transfer-function matrix and, the zero structure of the matrices:

$$T_i(z_1, z_2),\ P_i(z_1, z_2),\ \begin{bmatrix} T_i(z_1, z_2) & U_i(z_1, z_2) \end{bmatrix}, \\ \begin{bmatrix} T_i(z_1, z_2) \\ -V_i(z_1, z_2) \end{bmatrix}, i = 1, 2.$$

IV. Polynomial System Matrix Descriptions

Consider the singular Fornasini-Marchesini model given in (1) and introduce the forward shift operators z_1 and z_2 in the horizontal and vertical directions, i.e.

$$z_1 x(i, j) = x(i+1, j),\ \ z_2 x(i, j) = x(i, j+1), \tag{13}$$

respectively. Then the process dynamics of (1) can be written in the polynomial matrix description form as:

$$P_{SFM}(z_1, z_2) \begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \tag{14}$$

where the polynomial matrix over $\mathbb{R}[z_1, z_2]$,

$$P_{SFM}(z_1, z_2) = \left[\begin{array}{c|c} z_1 z_2 E - z_1 A_1 - z_2 A_2 - A_0 & B \\ \hline -C & D \end{array}\right] \tag{15}$$

is the system matrix of (1). The transfer-function matrix is given by:

$$G_{SFM}(z_1, z_2) = C(z_1 z_2 E - z_1 A_1 - z_2 A_2 - A_0)^{-1} B + D. \tag{16}$$

Similarly, consider the wave linear repetitive process (3) and introduce the state vector

$$\nu(k, p) = [x_k^T(p)\ \ y_k^T(p)]^T. \tag{17}$$

Using the shift relations (13) adopted to the repetitive processes case, i.e.

$$z_1 w_k(p) = w_{k+1}(p),\ \ z_2 w_k(p) = w_k(p+1), \tag{18}$$

where a signal w stands for a state x, or output y, or an input u, the system matrix associated with (3) takes the form

$$P_{WR} = \left[\begin{array}{cc|c} z_1 z_2 I_n - z_1 A & -\sum_{i=-w_L}^{w_H} z_2^i B_i & z_1 \hat{B} \\ -z_1 C & z_1 I_m - \sum_{i=-w_L}^{w_H} z_2^i D_i & z_1 \hat{D} \\ \hline 0_{m,n} & -I_m & 0_{m,l} \end{array}\right] \tag{19}$$

with transfer-function matrix

$$G_{WR}(z_1, z_2) = \begin{bmatrix} 0 & I_m \end{bmatrix} \times \\ \begin{bmatrix} z_1 z_2 I_n - z_1 A & -\sum_{i=-w_L}^{w_H} z_2^i B_i \\ -z_1 C & z_1 I_m - \sum_{i=-w_L}^{w_H} z_2^i D_i \end{bmatrix}^{-1} \begin{bmatrix} z_1 \hat{B} \\ z_1 \hat{D} \end{bmatrix}, \tag{20}$$

provided the matrix inverse in (20) exists.

An alternative description of the system given in (3) is to multiply the system equations by the shift operator $z_2^{w_L}$. This corresponds to multiplying the system matrix P_{WR} in (19) on the left by the transfer function preserving matrix

$$\left[\begin{array}{cc|c} z_2^{w_L} I_n & 0 & 0 \\ 0 & z_2^{w_L} I_m & 0 \\ \hline 0 & 0 & I_l \end{array}\right].$$

$$\tilde{P}_{WR} \equiv \begin{bmatrix} \tilde{T}_{WR} & \tilde{U}_{WR} \\ -\tilde{V}_{WR} & 0_{m,l} \end{bmatrix} = \left[\begin{array}{cc|c} z_1 z_2^{w_L+1} I_n - z_1 z_2^{w_L} A & -\sum_{j=0}^{q} z_2^j B_{j-w_L} & z_1 z_2^{w_L} \hat{B} \\ -z_1 z_2^{w_L} C & z_1 z_2^{w_L} I_m - \sum_{j=0}^{q} z_2^j D_{j-w_L} & z_1 z_2^{w_L} \hat{D} \\ \hline 0_{m,n} & -I_m & 0_{m,l} \end{array}\right]. \tag{21}$$

V. Reduction of the WLRP to the SFM model

Let $t = n+m$ and let $\tilde{P}_{WR}(z_1,z_2)$ be a $(t+m)\times(t+l)$ be a polynomial system matrix given by (21). Then $\tilde{P}_{WR}(z_1,z_2)$ can be written as:

$$\tilde{P}_{WR}(z_1,z_2) = \sum_{i=0}^{1}\sum_{j=0}^{q} z_1^i z_2^j \tilde{P}_{i,j}, \quad (\tilde{P}_{1,0}=0), \tag{22}$$

where $\tilde{P}_{i,j}$ are $(t+m)\times(t+l)$ constant matrices. Now construct the matrices:

$$E = \begin{bmatrix} 0_{(q-1)(t+l),t+l} & 0_{(q-1)(t+l),t+l} & \cdots & 0_{(q-1)(t+l),t+l} \\ \tilde{P}_{1,q} & \tilde{P}_{1,q-1} & \cdots & \tilde{P}_{1,1} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} -I_{(q-1)(t+l)} & 0_{(q-1)(t+l),t+l} \\ 0_{t+m,(q-1)(t+l)} & -\tilde{P}_{0,0} \end{bmatrix}, A_1 = 0,$$

$$A_2 = \begin{bmatrix} 0_{(q-1)(t+l),t+l} & 0_{(q-1)(t+l),t+l} & \cdots & 0_{(q-1)(t+l),t+l} \\ -\tilde{P}_{0,q} & -\tilde{P}_{0,q-1} & \cdots & -\tilde{P}_{0,1} \end{bmatrix}$$

$$Y_m = \begin{bmatrix} 0_{(q-1)(t+l)+t,m} \\ I_m \end{bmatrix}, \; Z_l = \begin{bmatrix} 0_{l,q(t+l)-l} & I_l \end{bmatrix}. \tag{23}$$

The resulting system matrix:

$$P_{SFM}(z_1,z_2) \equiv \begin{bmatrix} T_{SFM} & U_{SFM} \\ -V_{SFM} & 0_{m,l} \end{bmatrix} = \left[\begin{array}{cc|c} z_1 z_2 E - z_2 A_2 - A_0 & Y_m & 0 \\ -Z_l & 0_{l,m} & I_l \\ \hline 0_{m,t+l} & -I_m & 0_{m,l} \end{array}\right] \tag{24}$$

is clearly in the singular Fornasini-Marchesini form (15) with $A_1 = 0$.

Theorem 1: Let $P_{SFM}(z_1,z_2)$ is constructed and given by (24). Then $P_{SFM}(z_1,z_2)$ is related to its corresponding system matrix $\tilde{P}_{WR}$ in (21) by the zero coprime system equivalence:

$$S_1 \tilde{P}_{WR} = P_{SFM} S_2, \tag{25}$$

where

$$S_1 = \left[\begin{array}{c|c} 0_{(q-1)(t+l),t} & 0_{(q-1)(t+l),m} \\ I_t & 0_{t,m} \\ 0_{m+l,t} & 0_{m+l,m} \\ \hline 0_{m,t} & I_m \end{array}\right],$$

$$S_2 = \left[\begin{array}{c|c} z_2^{q-1} I_t & 0_{t,l} \\ 0_{l,t} & z_2^{q-1} I_l \\ z_2^{q-2} I_t & 0_{t,l} \\ 0_{l,t} & z_2^{q-2} I_l \\ \vdots & \vdots \\ I_t & 0_{t,l} \\ 0_{l,t} & I_l \\ \tilde{V}_{WR} & 0_{m,l} \\ \hline 0_{l,t} & I_l \end{array}\right] \tag{26}$$

with the matrix $\tilde{V}_{WR}$ of (21).

Proof: The matrix P_{SFM} in (24) can be represented in the form

$$P_{SFM} = \left[\begin{array}{cccccc|c} I_{t+l} & -z_2 I_{t+l} & \cdots & 0 & 0 & 0 & 0 \\ 0 & I_{t+l} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{t+l} & -z_2 I_{t+l} & 0 & 0 \\ z_1 z_2 \tilde{P}_{1,q} + z_2 \tilde{P}_{0,q} & z_1 z_2 \tilde{P}_{1,q-1} + z_2 \tilde{P}_{0,q-1} & \cdots & z_1 z_2 \tilde{P}_{1,2} + z_2 \tilde{P}_{0,2} & z_1 z_2 \tilde{P}_{1,1} + z_2 \tilde{P}_{0,1} + \tilde{P}_{0,0} & \mathcal{F}_m^T & 0 \\ 0 & 0 & \cdots & 0 & -\mathcal{F}_l & 0 & I_l \\ \hline 0 & 0 & \cdots & 0 & 0 & -I_m & 0 \end{array}\right] \tag{27}$$

where $\mathcal{F}_k = \begin{bmatrix} 0_{k,t} & I_k \end{bmatrix}$, from which it can be verified that (25) is satisfied, i.e.,

$$S_1 \tilde{P}_{WR} = P_{SFM} S_2 = \begin{bmatrix} 0_{(q-1)(t+l),t} & 0_{(q-1)(t+l),l} \\ \tilde{T}_{WR} & \tilde{U}_{WR} \\ 0_{m+l,t} & 0_{m+l,l} \\ -\tilde{V}_{WR} & 0_{m,l} \end{bmatrix} \tag{28}$$

with $\tilde{V}_{WR}$, $\tilde{T}_{WR}$ and $\tilde{U}_{WR}$ given in (21). Now it remains to show the zero coprimeness of the matrices. The zero right coprimeness of the matrices $\tilde{P}_{WR}$ and S_2 follows from the fact that the matrix

$$\left[\begin{array}{c} \tilde{P}_{WR} \\ S_2 \end{array}\right] = \left[\begin{array}{cc} \tilde{T}_{WR} & \tilde{U}_{WR} \\ -\tilde{V}_{WR} & 0_{m,l} \\ \hline z_2^{q-1} I_t & 0_{l,l} \\ 0 & z_2^{q-1} I_t \\ z_2^{q-2} I_t & 0_{l,l} \\ 0_{l,t} & z_2^{q-2} I_l \\ \vdots & \vdots \\ I_t & 0_{t,l} \\ 0_{l,t} & I_l \\ \tilde{V}_{WR} & 0_{m,l} \\ 0_{l,t} & I_l \end{array}\right] \tag{29}$$

contains a highest order minor which is equal to 1 obtained from the two block rows before the last two block rows of (29). Similarly the zero left coprimeness of the matrices P_{SFM} and S_1 follows from the fact that the matrix $\left[\begin{array}{cc} P_{SFM} & S_1 \end{array}\right]$ given by:

$$\left[\begin{array}{ccccccc|cc} I_{t+l} & -z_2 I_{t+l} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{t+l} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{t+l} & -z_2 I_{t+l} & 0 & 0 & I_{t+l} & 0 \\ z_1 z_2 \tilde{P}_{1,q} + z_2 \tilde{P}_{0,q} & z_1 z_2 \tilde{P}_{1,q-1} + z_2 \tilde{P}_{0,q-1} & \cdots & z_1 z_2 \tilde{P}_{1,2} + z_2 \tilde{P}_{0,2} & z_1 z_2 \tilde{P}_{1,1} + z_2 \tilde{P}_{0,1} + \tilde{P}_{0,0} & \mathcal{F}_m^T & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\mathcal{F}_l & 0 & I_l & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & -I_m & 0 & 0 & I_m \end{array}\right] \tag{30}$$

has a highest order minor equal to 1 obtained by deleting the columns $(q-1)(t+l)+1, \ldots, (t+l)q$ from the matrix in (30). ■

Example 1: Consider the system matrix $\tilde{P}_{WR}$ in (21) with $w_L = 1, w_H = 1$ and $q = 2$, i.e.,

$$\tilde{P}_{WR} = \left[\begin{array}{cc|c} z_1 z_2^2 I_n - z_1 z_2 A & -B_{-1} - B_0 z_2 - B_1 z_2^2 & z_1 z_2 \hat{B} \\ -z_1 z_2 C & z_1 z_2 I_m - D_{-1} - D_0 z_2 - D_1 z_2^2 & z_1 z_2 \hat{D} \\ \hline 0_{m,n} & -I_m & 0_{m,l} \end{array}\right] \tag{31}$$

It follows that

$$\tilde{P}_{WR} = \tilde{P}_{0,0} + \tilde{P}_{0,1} z_2 + z_1 z_2 \tilde{P}_{1,1} + z_2^2 \tilde{P}_{0,2} + z_1 z_2^2 \tilde{P}_{1,2}, \tag{32}$$

where

$$\tilde{P}_{0,0} = \left[\begin{array}{cc|c} 0 & -B_{-1} & 0 \\ 0 & -D_{-1} & 0 \\ \hline 0 & -I_m & 0 \end{array}\right], \ \tilde{P}_{0,1} = \left[\begin{array}{cc|c} 0 & -B_0 & 0 \\ 0 & -D_0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$
$$\tilde{P}_{0,2} = \left[\begin{array}{cc|c} 0 & -B_1 & 0 \\ 0 & -D_1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

$$\tilde{P}_{1,1} = \left[\begin{array}{cc|c} -A & 0 & \hat{B} \\ -C & I_m & \hat{D} \\ \hline 0 & 0 & 0 \end{array}\right], \ \tilde{P}_{1,2} = \left[\begin{array}{cc|c} I_n & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]. \tag{33}$$

Then the matrices in (23) are given by:

$$E = \left[\begin{array}{cc} 0 & 0 \\ \tilde{P}_{1,2} & \tilde{P}_{1,1} \end{array}\right], \ A_0 = \left[\begin{array}{cc} -I_{t+m} & 0 \\ 0 & -\tilde{P}_{0,0} \end{array}\right],$$
$$A_2 = \left[\begin{array}{cc} 0 & 0 \\ -\tilde{P}_{0,2} & -\tilde{P}_{0,1} \end{array}\right],$$

$$Y_m = \left[\begin{array}{c} 0_{2(t+l)-m,m} \\ I_m \end{array}\right], \ Z_l = \left[\begin{array}{cc} 0_{l,2(t+l)-n} & I_l \end{array}\right]. \tag{34}$$

The resulting system matrix $P_{SFM}(z_1, z_2)$ has the form

$$P_{SFM} = \left[\begin{array}{ccccccc|c} I_n & 0 & 0 & -z_2 I_n & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & -z_2 I_m & 0 & 0 & 0 \\ 0 & 0 & I_l & 0 & 0 & -z_2 I_l & 0 & 0 \\ z_1 z_2 I_n & -z_2 B_1 & 0 & -z_1 z_2 A & -B_{-1} - z_2 B_0 & z_1 z_2 \hat{B} & 0 & 0 \\ 0 & -z_2 D_1 & 0 & -z_1 z_2 C & z_1 z_2 I_m - D_{-1} - z_2 D_0 & z_1 z_2 \hat{D} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_m & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_l & 0 & I_l \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -I_m & 0 \end{array}\right], \tag{35}$$

where the transformation matrices S_1 and S_2 are given by:

$$S_1 = \left[\begin{array}{c|c} 0_{(t+l),t} & 0_{(t+l),m} \\ I_t & 0_{t,m} \\ 0_{m+l,t} & 0_{m+l,m} \\ \hline 0_{m,t} & I_m \end{array}\right], \quad S_2 = \left[\begin{array}{c|c} z_2 I_t & 0_{t,l} \\ 0_{l,t} & z_2 I_l \\ I_t & 0_{t,l} \\ 0_{l,t} & I_l \\ \tilde{V}_{WR} & 0_{m,l} \\ \hline 0_{l,t} & I_l \end{array}\right] \tag{36}$$

Furthermore it can be confirmed that

$$G_{SFM}(z_1, z_2) = \tilde{G}_{WR}(z_1, z_2) = G_{WR}(z_1, z_2).$$

CONCLUSIONS

In this paper, an equivalent representation is obtained in 2-D Fornasini-Marchesini singular form for a given system matrix arising from a discrete linear wave repetitive process. The exact connections between the original system matrix with its corresponding 2-D singular forms have been developed and shown to be zero coprime system equivalence. The zero structures of the original polynomial system matrix are preserved making it possible to analyze the polynomial system matrix in terms of its associated 2-D singular form. It is worth mentioning that in order to preserve the zero structure, the resulting system matrices become quite large. It will be an interesting problem to find a way to reduce the size of these matrices while not sacrificing their zero structure.

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