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Uniform boundary stabilization of the Schrödinger equation with a nonlinear delay term in the boundary conditions*

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***Abstract*— We consider the Schrödinger equation in a bounded domain of $\mathbb{R}^n$ with a delay term in the nonlinear boundary feedback. Under suitable assumptions, we prove global existence of the solutions by the arguments of nonlinear semigroup theory. Moreover, we obtain uniform decay rates for the solutions by following an approach that is based on certain integral inequalities for the energy functional and a comparison theorem that relates the asmptotic behaviour of the energy and of the solutions to a dissipative ordinary differential equation.**

Keywords. Schrödinger equation, Time delays, Boundary stabilization, Nonlinear feedback law.
AMS subject classifications. 35B35, 93D15

I. Extended abstract

Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary Γ which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Let $\nu(.)$ denote the unit normal on Γ pointing towards the exterior of Ω.
In Ω, we consider the Schrödinger equation with a nonlinear delay term in the boundary conditions

$$\begin{cases} u_t(x,t) - \mathbf{i}\Delta u(x,t) = 0 \quad \text{in } \Omega \times (0,+\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \\ u(x,t) = 0 \text{ on } \Gamma_1 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = \mathbf{i}\alpha_1 f(u(x,t)) + \mathbf{i}\alpha_2 g(u(x,t-\tau)) \quad \text{on } \Gamma_2 \times (0,+\infty), \\ u(x,t-\tau) = f_0(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,\tau). \end{cases} \tag{1}$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative and $\tau > 0$ is the time delay. Moreover, α_1 and α_2 are positive constants, u_0 and f_0 are the initial data which belong to appropriate Hilbert spaces, f and g are real-valued functions of class $C(\mathbb{R})$.

In absence of delay ($\alpha_2 = 0$), stability problems for (1) have been considered by several authors, see for example [4], [2], [1]. In the presence of delay, Nicaise and Rebiai [5] established sufficient conditions that guarantee the exponential stability of solution of (1) when f and g are linear.

The main focus of the present paper is to study the asymptotic behaviour of the solutions of (1) in the case where both α_1 and α_2 are different from zero and f and g are nonlinear. To this aim, we need to make the following assumptions.

(**H1**) (i) f is a continuous strongly monotone function on $\mathbb{C}$, *i.e.*

$$Re(f(z)-f(v))(\bar{z}-\bar{v}) \geq \alpha |z-v|^2,$$
for all $z, v \in \mathbb{C}$ and fixed $\alpha > 0$.

(ii) $f(0) = 0$;
(iii) $Im(f(z)\bar{z}) = 0$ for all $z \in \mathbb{C}$;

(**H2**) There exists positive constant M such that

$$\begin{cases} |f(z)| \leq M|z|^r, \quad \text{for } |z| \geq 1, \\ \text{where: } r = 5 \quad \text{for } n = 2, \\ \qquad r = 3 \quad \text{for } n = 3. \end{cases}$$

(**H3**) (i) g is a Lipschitz continuous function on $\mathbb{R}$;
(ii) $g(0) = 0$;

(**H4**) $\alpha_1 > \alpha_2$

(**H5**) There exists a real vector field $l(.) \in (C^2(\overline{\Omega}))^n$ such that

$$l(x).\nu(x) \leq 0 \quad \text{on } \Gamma_1.$$

Regarding the well-posedness of the solutions to the problem (1), we have the following result which can be established by the arguments of nonlinear semigroup theory.

Theorem 1: Assume $(H1)$, $(H3)$, and $(H4)$. Then the following results hold true for problem (1):

(a) For any initial datum $(u_0, f_0) \in L^2(\Omega) \times L^2(\Gamma_2; L^2(0,1))$, problem (1) has a unique mild solution $u \in C([0,+\infty), L^2(\Omega))$.

(b) If $(u_0, f_0) \in W$ where

$$W = (z,v) \in H^2(\Omega) \times L^2(\Gamma_2; H^1(0,1)); z = 0$$
$$\text{on } \Gamma_1, \frac{\partial z}{\partial \nu} = \mathbf{i}\alpha_1 f(z) + \mathbf{i}\alpha_2 g(v(.,1)), z = v(.,0) \text{ on } \Gamma_2\}$$

then the corresponding solution u satisfies

$$\begin{cases} u \in C([0,+\infty); W); \\ u_t^+ \in C([0,+\infty), L^2(\Omega)); \\ u|_{\Gamma_2} \in C([0,+\infty), H^{1/2}(\Gamma_2)); \\ \frac{\partial u}{\partial \nu} \in L^2(0,+\infty; L^2(\Gamma_2)). \end{cases}$$

In order to state our stability result, we introduce as in [2] a real valued strictly increasing concave function $h(x)$ defined for $x \geq 0$ and satisfying

$$\begin{aligned} h(0) &= 0; \\ h(f(z)\bar{z}) &\geq |z| + |f(z)|^2 \quad \text{for } |z| \leq \delta; \end{aligned}$$

for some $\delta > 0, z \in \mathbb{C}$

and we define the following functions:

- $$\widetilde{h}(x) = h(\frac{x}{mes\ \Sigma_2}), x \geq 0,$$

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where $\Sigma_2 = \Gamma_2 \times (0,T)$, T is a given constant and *mes* Σ_2 is the mesure of Σ_2.

- $$p(x) = (cI + \widetilde{h})^{-1}Kx$$
  where c and K are positive constants. Then p is a positive, continuous, strictly increasing function with $p(0) = 0$.
- $$q(x) = x - (I+p)^{-1}(x), \ x > 0 \qquad (2)$$
  q is also a positive, continuous, strictly increasing function with $q(0) = 0$.

Let $E(t)$ be the energy function corresponding to the solution of (1) defined by

$$E(t) = \frac{1}{2}\int_{\Omega} |u(x,t)|^2 dx + \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 u(x,t-\rho\tau))d\rho d\Gamma$$

where the positive constant ξ is such that
(**H6**)

$$2\tau\alpha_2 < \xi < 2\tau(\alpha_1 - \alpha_2)$$

Theorem 2: Let $n = 2,3$. Assume hypotheses $(H1)-(H6)$. Then the energy corresponding to the solution of problem (1) satisfies the following decay rate

$$E(t) \leq S(\frac{t}{T_0} - 1)(E(0)) \quad \text{for all } t \geq T_0,$$

for some $T_0 >$, where the scalar function $S(t)$ is the solution of the nonlinear ordinary differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0), \qquad (3)$$

where the function $q(.)$ is defined by (2) in which the constants C and K depend on $E(0), r$ and *mes* Σ_2. Thus from (3) it follows that

$$E(t) \to 0 \ \text{ as } t \to +\infty, \text{ with rates specified by } S(t).$$

Theorem 2 follows from suitable integral estimates that can be proved by applying an energy estimate at the $L^2(\Omega)-$level for a fully Schrödinger equation with gradient and potentiel level terms obtained by Lasiecka et al [3].

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