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A characteristics based curse-of-dimensionality-free approach for approximating control Lyapunov functions and feedback stabilization*

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Abstract— **This paper develops a curse-of-dimensionality-free numerical approach to construct control Lyapunov functions (CLFs) and stabilizing feedback strategies for deterministic control systems described by ODEs. An extension of the Zubov method is used to represent a CLF as the value function for an appropriate infinite-horizon optimal control problem. The infinite-horizon stabilization problem is approximated by an exit time problem, with target set given by a sufficiently small closed neighborhood of the origin in the state space. In order to compute the related value function and optimal feedback control law separately at different initial states and thereby to attenuate the curse of dimensionality, an extension of a recently developed characteristics based framework is proposed. Theoretical foundations of the developed approach are given together with practical discussions regarding its implementation, and numerical examples are also provided. In particular, it is pointed out that the curse of complexity may remain a significant issue even if the curse of dimensionality is avoided.**

I. INTRODUCTION

The area of designing state feedbacks for dynamic systems stabilization via control Lyapunov functions (or CLFs in short) has received wide interest in the last decade [1]–[3]. For deterministic control systems described by systems of ordinary differential equations without state constraints, the work [1] developed a general constructive approach of representing CLFs as value functions for appropriate infinite-horizon optimal control problems and, therefore, as unique viscosity solutions of certain Hamilton–Jacobi–Bellman (HJB) partial differential equations of first order. This was in fact an extension of the classical Zubov method for constructing Lyapunov functions [4] to the problem of weak asymptotic null-controllability. As was demonstrated in [3, Example 5.2], the framework of [1] can also be extended to some classes of state-constrained problems.

Many broadly used numerical methods for solving general Hamilton–Jacobi (HJ) equations and in particular HJB equations, such as semi-Lagrangian schemes [5], [6], finite difference schemes [8], [9] and level set techniques [10], [11], rely on sufficiently dense state space discretizations. The computational cost of such methods grows exponentially with the increase of the state space dimension n, and their implementation appears to be extremely difficult for $n \geqslant 4$. This leads to what was termed “the curse of dimensionality” by R. Bellman [12].

The paper [13] contains an overview of some existing studies on various approaches to mitigate the curse of dimensionality for certain classes of HJ equations. One group of methods exploits idempotent analysis tools and the max-plus algebra for problems where the Hamiltonian is represented as the maximum or minimum of a finite number of elementary Hamiltonians corresponding to linear-quadratic optimal control problems [14], [15]. Despite the attenuation of the curse of dimensionality, some practical issues may arise when implementing such methods, i. e., the curse of complexity may still be a formidable barrier.

Another promising direction is using the method of characteristics in order to reduce computation of a sought-after value function at selected states to finite-dimensional optimization problems [13], [16], [17]. Contrary to the aforementioned grid-based approaches, this allows the value function to be computed separately at different states. The curse of dimensionality can consequently be mitigated, and it is easy to arrange parallel computations. The curse of complexity may nevertheless remain when constructing global solution approximations. The sparse grid framework [18] may help in that effort if the state space dimension is not too high.

This work is aimed at combining the theoretical arguments of [1] with an extension of the characteristics based techniques of [13], [16], [17] and developing a curse-of-dimensionality-free approach for approximating CLFs and related state feedbacks.

The paper is organized as follows. Section II recalls the main results of [1] and contains additional theoretical discussions. The curse-of-dimensionality-free approach is described in Section III, and practical remarks on its implementation are also provided there. Section IV contains two numerical examples, while concluding remarks are given in Section V.

Finally, let us indicate some notations that are used throughout this paper:

- given $j \in \mathbb{N}$, the origin in $\mathbb{R}^j$ is denoted by 0_j, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^j$ (we avoid any confusions when considering the norms of vectors with different dimensions together), the Euclidean distance between a point $\xi \in \mathbb{R}^j$ and a set $\Xi \subseteq \mathbb{R}^j$ is written as $\operatorname{dist}(\xi, \Xi)$, the open Euclidean ball with center $\xi \in \mathbb{R}^j$ and radius $R > 0$ is denoted by $B_R(\xi)$, while its closure is $\bar{B}_R(\xi)$;
- if a vector variable ξ consists of some arguments of a map $F = F(\ldots, \xi, \ldots)$, then $\mathrm{D}_\xi F$ denotes the

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standard (Fréchet) derivative of F with respect to ξ, and $\mathrm{D}F$ is the standard derivative with respect to the vector of all arguments (the exact definitions of the derivatives depend on the domain and range of F);

- given a function $F : \Xi_1 \to \mathbb{R}$, the set of its global minimizers on $\Xi \subseteq \Xi_1$ is denoted by $\operatorname{Arg\,min}_{\xi \in \Xi} F(\xi)$;
- $\mathcal{K}$ is the class of all strictly increasing continuous functions $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$;
- $\mathcal{K}_\infty$ is the class of all functions $\varphi(\cdot) \in \mathcal{K}$ satisfying $\lim_{\rho \to +\infty} \varphi(\rho) = +\infty$;
- $\mathcal{L}$ is the class of all nonincreasing continuous functions $\varphi : [0, +\infty) \to [0, +\infty)$ for which $\lim_{\rho \to +\infty} \varphi(\rho) = 0$;
- $\mathcal{KL}$ is the class of all continuous functions $\varphi : [0, +\infty)^2 \to [0, +\infty)$ such that $\varphi(\cdot, \rho) \in \mathcal{K}$ and $\varphi(\rho, \cdot) \in \mathcal{L}$ for every $\rho \geqslant 0$.

II. CHARACTERIZATION OF CONTROL LYAPUNOV FUNCTIONS VIA OPTIMAL CONTROL PROBLEMS

Let $G \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ be sets in the state and control spaces, respectively. Denote states by x and control inputs by u. Consider the autonomous control system

$$\begin{cases} \dot{x}(t) \;=\; f(x(t), u(t)), \quad t \geqslant 0, \quad x(0) \;=\; x_0 \;\in\; G, \\ u(\cdot) \;\in\; \mathcal{U} \;\stackrel{\text{def}}{=}\; L^\infty_{\text{loc}}([0, +\infty), U). \end{cases} \tag{1}$$

First, some basic conditions are imposed.

Assumption 2.1: The following properties hold:

1) U is a closed set in $\mathbb{R}^m$, and $0_m \in U$;
2) G and G_1 are open domains in $\mathbb{R}^n$, $0_n \in G$, and $\bar{G} \subset G_1$;
3) the function $G_1 \times U \ni (x, u) \longmapsto f(x, u) \in \mathbb{R}^n$ is continuous;
4) G is a strongly invariant domain in the state space for the control system (1), i. e., $x_0 \in G$ and $u(\cdot) \in \mathcal{U}$ imply that any corresponding state trajectory of (1) defined on some interval of the form $[0, T)$, $T \in (0, +\infty) \cup \{+\infty\}$ stays inside G and cannot reach the boundary ∂G ($G = \mathbb{R}^n$ is a trivial example of a strongly invariant domain);
5) there exists a function $\gamma_1(\cdot) \in \mathcal{K}_\infty$ such that, for any $R > 0$, one can choose $C_{1,R} > 0$ satisfying
$$\|f(x, u) \,-\, f(x', u)\| \;\leqslant\; (1 \,+\, \gamma_1(\|u\|))\, C_{1,R}\, \|x - x'\| \quad \forall\, x, x' \,\in\, \bar{B}_R(0_n) \cap \bar{G} \quad \forall u \in U;$$
6) the forward completeness property [19] holds here in the sense that there exist a continuously differentiable proper function $W : \bar{G} \to [0, +\infty)$ and a constant $C_2 > 0$ satisfying
$$\sup_{u \in U} \, \langle \mathrm{D}W(x),\, f(x, u) \rangle \;\leqslant\; C_2\, W(x) \quad \forall x \in \bar{G}.$$

Remark 2.2: For any $x_0 \in G$ and $u(\cdot) \in \mathcal{U}$, let

$$[0,\, T_{\text{ext}}(x_0, u(\cdot))) \;\ni\; t \;\longmapsto\; x(t; 0, x_0, u(\cdot)) \;\in\; G$$

be a solution of the Cauchy problem (1) defined on the maximum extendability interval with the right endpoint $T_{\text{ext}}(x_0, u(\cdot)) \in (0, +\infty) \cup \{+\infty\}$. It is in fact possible to use the simplified notation $x(\cdot; x_0, u(\cdot))$, because the system is time invariant and the initial time $t_0 = 0$ is fixed. Here the initial time is indicated to ensure consistency with the corresponding notation in [13]. The local existence and uniqueness of the solutions follow from Items 1–5 of Assumption 2.1, while Item 6 is added in order to guarantee their extendability to the whole time interval $[0, +\infty)$.

Suppose also that the origin 0_n is an equilibrium of (1) for $u \equiv 0_m$ and that a local asymptotic null-controllability property holds in a weak or strong form as follows [1, Section 2].

Assumption 2.3: $f(0_n, 0_m) = 0_n$, and one of the following two conditions hold (the second condition is a strengthened form of the first one):

1) there exist positive constants $r, \bar{u}$ and a function $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that $\bar{B}_r(0_n) \subseteq G$ and, for any $x_0 \in B_r(0_n)$, one can choose a control strategy $u_{x_0}(\cdot) \in L^\infty([0, +\infty), U)$ satisfying
$$\|u_{x_0}(\cdot)\|_{L^\infty([0,+\infty),\, U)} \;\leqslant\; \bar{u},$$
$$\|x\,(t;\, 0, x_0, u_{x_0}(\cdot))\| \;\leqslant\; \beta(\|x_0\|, t) \quad \forall t \geqslant 0;$$
2) there exists a constant $r > 0$ and a function $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that $\bar{B}_r(0_n) \subseteq G$ and, for any $x_0 \in B_r(0_n)$, one can choose a control strategy $u_{x_0}(\cdot) \in L^\infty([0, +\infty), U)$ satisfying
$$\|x\,(t;\, 0, x_0, u_{x_0}(\cdot))\| + \|u_{x_0}(t)\| \leqslant \beta(\|x_0\|, t) \quad \forall t \geqslant 0.$$

Remark 2.4: If $0_m \in \operatorname{int} U$, the function $f(\cdot, \cdot)$ is continuously differentiable, $f(0_n, 0_m) = 0_n$ and

$$\begin{gathered} \operatorname{rank}\, \left[B,\, AB,\, A^2 B,\, \ldots,\, A^{n-1} B\right] \;=\; n, \\ A \;\stackrel{\text{def}}{=}\; \mathrm{D}_x f(0_n, 0_m), \quad B \;\stackrel{\text{def}}{=}\; \mathrm{D}_u f(0_n, 0_m), \end{gathered} \tag{2}$$

then the well-known result [20, §6.1, Theorem 1] allows local null-controllability of (1) to be concluded (so that the origin in the state space can be exactly reached in finite time), and Item 2 of Assumption 2.3 obviously holds. If the condition (2) is not satisfied, verification of Assumption 2.3 for nonlinear systems may in general be a difficult task. For some classes of control-affine systems, the considerations of [21, §9.4] may be applied.

Remark 2.5: According to [22, Proposition 7], $\beta(\cdot, \cdot) \in \mathcal{KL}$ implies the existence of two functions $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ satisfying $\beta(\rho, t) \leqslant \alpha_2\left(\alpha_1(\rho)\, e^{-t}\right)$ for all $\rho, t \geqslant 0$. For example, if C_5, C_6 are positive constants, $\nu(\cdot) \in \mathcal{K}_\infty$ and $\beta(\rho, t) = C_5\, \nu(\rho)\, e^{-C_6\, t}$, then one can take $\alpha_1(\rho) = (\nu(\rho))^{1/C_6}$ and $\alpha_2(\rho) = C_5\, \rho^{C_6}$.

Definition 2.6: The global region of asymptotic null-controllability for the system (1) is

$$\mathcal{D}_0 \;\stackrel{\text{def}}{=}\; \left\{ x_0 \in G : \text{ there exists } u(\cdot) \,\in\, L^\infty([0, +\infty), U) \text{ such that } \lim_{t \to +\infty} \|x(t;\, 0, x_0, u(\cdot))\| \;=\; 0 \right\}.$$

Remark 2.7: Compared to [1], we consider the control system (1) not necessarily in the whole state space $\mathbb{R}^n$ but in a strongly invariant domain $G \subseteq \mathbb{R}^n$. That is why $\mathcal{D}_0 \subseteq G$

in Definition 2.6. If $G \neq \mathbb{R}^n$, the strong invariance of G still allows to obtain analogs of the key results of [1] (by using similar reasonings).

Proposition 2.8: [1, Proposition 2.3] Under Assumptions 2.1 and 2.3, $\mathcal{D}_0$ is an open domain containing $\bar{B}_r(0_n)$.

Assumption 2.9: $U \subseteq \mathbb{R}^m$ is a compact set.

Remark 2.10: The assumptions of [1, Section 2] do not exclude the situations when, for some initial states and control strategies, the related state trajectories of (1) explode and cannot be extended to the whole time interval $[0, +\infty)$. Item 6 of Assumption 2.1 excludes such situations. This and also Assumption 2.9 will be needed, in particular, to establish a general existence result for optimal control strategies in the next section (see Theorem 3.3).

Remark 2.11: In line with [1, Section 2], $L^\infty([0, +\infty), U)$ is selected as the control class in Assumption 2.3 and Definition 2.6. Under Assumption 2.9, this class obviously coincides with $\mathcal{U} = L^\infty_{\mathrm{loc}}([0, +\infty), U)$, as well as with the class of all Lebesgue measurable functions defined on $[0, +\infty)$ and taking values in U.

Now let us briefly describe the approach of [1, Section 3] for representing the domain of asymptotic null-controllability through the value function of an optimal control problem. Certain conditions should be imposed on the related running cost $\bar{G} \times U \ni (x, u) \longmapsto g(x, u) \in \mathbb{R}$.

Assumption 2.12: Let $\alpha_2^{-1}(\cdot)$ be the inverse of the function $\alpha_2(\cdot)$ introduced in Remark 2.5, and take the function $\gamma_1(\cdot)$ from Assumption 2.1 and the constants $r, \bar{u}$ from Assumption 2.3. The following properties hold:

1) the function $g(\cdot, \cdot)$ is continuous and nonnegative;
2) there exists a function $\gamma_2(\cdot) \in \mathcal{K}_\infty$ such that, for any $R > 0$, one can choose $C_{7,R} > 0$ satisfying
$$\begin{gathered} |g(x, u) - g(x', u)| \leqslant (1 + \gamma_2(\|u\|))\, C_{7,R}\, \|x - x'\| \\ \forall\, x, x' \in \bar{B}_R(0_n) \cap \bar{G} \quad \forall u \in U; \end{gathered}$$
3) for all $R > 0$, one has
$$g_R \overset{\mathrm{def}}{=} \inf \{ g(x, u) : (x, u) \in \bar{G} \times U, \ \|x\| \geqslant R \} > 0;$$
4) if Item 2 in Assumption 2.3 is not asserted, then there exist positive constants C_8, C_9 satisfying
$$\begin{gathered} g(x, u) \leqslant C_8 \left(\alpha_2^{-1}(\|x\|)\right)^{C_9} \\ \forall x \in \bar{B}_r(0_n) \quad \forall\, u \in \bar{B}_{\bar{u}}(0_m) \cap U; \end{gathered} \tag{3}$$
5) if Item 2 in Assumption 2.3 holds, then (3) is weakened to
$$\begin{gathered} g(x, u) \leqslant C_8 \left(\alpha_2^{-1}(\|x\| + \|u\|)\right)^{C_9} \\ \forall x \in \bar{B}_r(0_n) \quad \forall\, u \in \bar{B}_{\bar{u}}(0_m) \cap U, \end{gathered}$$
where C_8, C_9 are positive constants;
6) there exists a constant $C_{10} > 0$ such that
$$\begin{gathered} g(x, u) \geqslant C_{10} \left(\|f(x, u)\| + \max\{\gamma_1(\|u\|), \gamma_2(\|u\|)\}\right) \\ \forall\, (x, u) \in \{(x', u') \in \bar{G} \times U : \ \|x'\| \geqslant 2r \\ \text{or } \|u'\| \geqslant 2\bar{u}\}. \end{gathered} \tag{4}$$

Define the functional
$$\begin{aligned} & G \times \mathcal{U} \ni (x_0, u(\cdot)) \longmapsto \\ & J(x_0, u(\cdot)) \overset{\mathrm{def}}{=} \int_0^\infty g(x(t; 0, x_0, u(\cdot)),\, u(t))\, dt \\ & \qquad \in [0, +\infty) \cup \{+\infty\}. \end{aligned} \tag{5}$$

The value function of the corresponding optimal control problem is specified as
$$G \ni x_0 \longmapsto V(x_0) \overset{\mathrm{def}}{=} \inf_{u(\cdot) \in \mathcal{U}} J(x_0, u(\cdot)). \tag{6}$$

In order to handle the infinite value $+\infty$, consider the Kruzhkov transformed function
$$G \ni x_0 \longmapsto v(x_0) \overset{\mathrm{def}}{=} 1 - e^{-V(x_0)} \in [0, 1] \tag{7}$$
with the convention $e^{-(+\infty)} \overset{\mathrm{def}}{=} e^{-\infty} \overset{\mathrm{def}}{=} 0$.

Remark 2.13: In [1, Section 3], the condition (4) is imposed with $C_{10} = 1$. However, it is clear that multiplying the running cost by an arbitrary positive constant leads to an equivalent optimization criterion.

The sought-after characterization of $\mathcal{D}_0$ is given in the next result.

Theorem 2.14: [1, Propositions 3.3, 3.5 and 3.6] Let Assumptions 2.1, 2.3, 2.9 and Items 1–5 of Assumption 2.12 hold. Then the domain of asymptotic null-controllability can be represented as
$$\mathcal{D}_0 = \{x_0 \in G \colon V(x_0) < +\infty\} = \{x_0 \in G \colon v(x_0) < 1\}.$$
If, moreover, Item 6 of Assumption 2.12 holds, then $V(\cdot)$ is continuous on $\mathcal{D}_0$, $v(\cdot)$ is continuous on G,
$$\{0_n\} = \{x_0 \in G \colon V(x_0) = 0\} = \{x_0 \in G \colon v(x_0) = 0\},$$
and, for any sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset G$ satisfying either $\lim_{k \to \infty} \operatorname{dist}\left(x^{(k)}, \partial\mathcal{D}_0\right) = 0$ or $\lim_{k \to \infty} \left\|x^{(k)}\right\| = +\infty$, one also has $\lim_{k \to \infty} V\left(x^{(k)}\right) = +\infty$ and $\lim_{k \to \infty} v\left(x^{(k)}\right) = 1$.

Remark 2.15: According to [1, Remark 4.2], $V(\cdot)$ is indeed a CLF on $\mathcal{D}_0$ (for a general definition of a CLF, see, e. g., [1, Introduction] and [2, Sections 2.5, 2.7]), and $v(\cdot)$ is a continuous viscosity supersolution on $\mathcal{D}_0$ of the partial differential inequality
$$\sup_{u \in U} \{-\langle \mathrm{D}v(x),\, f(x, u)\rangle\} \geqslant w(x)\, g_{\|x\|},$$
where $w \colon \mathcal{D}_0 \to \mathbb{R}$ is a continuous function satisfying
$$w(0_n) = 0, \quad 0 < w(x) < 1 - v(x) \quad \forall\, x \in \mathcal{D}_0 \setminus \{0_n\},$$
g_R is defined as in Item 3 of Assumption 2.12 if $R > 0$, and $g_0 \overset{\mathrm{def}}{=} 0$.

The next theorem can be obtained similarly to [1, Theorem 4.4]. Due to Assumption 2.9, one does not need to impose [1, Hypothesis (H6)].

Theorem 2.16: Under Assumptions 2.1, 2.3, 2.9 and 2.12, the transformed value function $v(\cdot)$ is the unique bounded viscosity solution of the following boundary value problem for the HJB equation:

$$\begin{cases} \sup\limits_{u \in U} \{- \langle \mathrm{D}v(x),\, f(x,u) \rangle \\ \qquad -\, (1 - v(x))\, g(x,u)\} \;=\; 0, \quad x \in G, \\ v(0_n) \,=\, 0. \end{cases} \tag{8}$$

Remark 2.17: Theorems 2.14 and 2.16 extend the classical Zubov method for constructing Lyapunov functions [4] to the problem of weak asymptotic null-controllability [1].

Remark 2.18: A well-known numerical approach for solving a stationary first-order HJB equation is to implement an appropriate semi-Lagrangian scheme [5], [6]. Since the problem (8) has a singularity at the origin $x = 0_n$, the fixed point argument of [5, Appendix A, §1] fails here, and the convergence of the related semi-Lagrangian scheme is not guaranteed (see the discussion in [7, Section 4] that is relevant for (8), although it was originally provided for the somewhat different problem of strong asymptotic null-controllability). In order to ensure the convergence, one can consider the regularized problem

$$\begin{cases} \sup\limits_{u \in U} \{- \langle \mathrm{D}v_\varepsilon(x),\, f(x,u) \rangle \,-\, g(x,u) \\ \qquad +\, v_\varepsilon(x)\, \hat{g}_\varepsilon(x,u)\} \;=\; 0, \quad x \in G, \\ v(0_n) \,=\, 0, \end{cases} \tag{9}$$

with $\hat{g}_\varepsilon(x,u) = \max\{g(x,u), \varepsilon\}$ and a sufficiently small $\varepsilon > 0$. Similarly to [7, Section 4], one can verify that there exists a unique bounded viscosity solution of (9) and that it uniformly converges to the solution of (8) on G as $\varepsilon \to +0$. However, semi-Lagrangian schemes usually require dense state space discretizations, as well as many other grid-based numerical methods for solving HJ equations, such as finite difference schemes [8], [9] and level set techniques [10], [11]. The computational cost of such methods grows exponentially with the increase of the state space dimension n, and their application in general becomes extremely difficult for $n \geqslant 4$, even when using supercomputers. The corresponding circumstances were referred to as the curse of dimensionality by R. Bellman [12]. Moreover, the practical dilemma of selecting a suitable bounded computational region in the state space (so as to reduce boundary cutoff errors in a relevant subregion) often arises when implementing grid-based methods.

With this in mind, the aim of the current work is to extend the framework of [13], [16], [17] in order to develop a characteristics based curse-of-dimensionality-free approach for approximating CLFs and related feedback strategies. Such an extension becomes possible due to the aforementioned representation of CLFs via optimal control problems.

III. A CHARACTERISTICS BASED CURSE-OF-DIMENSIONALITY-FREE APPROACH FOR APPROXIMATING CONTROL LYAPUNOV FUNCTIONS AND FEEDBACK STABILIZATION

For computational purposes, it is reasonable to approximate the infinite-horizon optimal control problem (6) by an exit time problem, stated with respect to a closed ball $\bar{B}_\delta(0_n)$ with center $x = 0_n$ and sufficiently small radius $\delta \in (0, r]$ (the constant r is taken from Assumption 2.3). Define the associated value function by

$$\begin{gathered} V_\delta(x_0) \;\stackrel{\text{def}}{=}\; \inf \left\{ \int\limits_0^T g(x(t;\, 0, x_0, u(\cdot)),\, u(t))\, dt \;:\; u(\cdot) \in \mathcal{U}, \right. \\ \left. x(T;\, 0, x_0, u(\cdot)) \;\in\; \bar{B}_\delta(0_n) \;\text{ at some }\; T \in [0, +\infty) \right\} \\ \forall x_0 \in G \quad \forall \delta \in (0, r]. \end{gathered} \tag{10}$$

After adopting the notation

$$\begin{aligned} & T_\delta(x_0, u(\cdot)) \\ & \quad \stackrel{\text{def}}{=}\; \inf\, \{T \in [0, +\infty) \,:\, x(T;\, 0, x_0, u(\cdot)) \;\in\; \bar{B}_\delta(0_n)\} \\ & \forall x_0 \in G \quad \forall u(\cdot) \in \mathcal{U}, \end{aligned} \tag{11}$$

the value function (10) can be equivalently rewritten as

$$\begin{aligned} & V_\delta(x_0) \\ & \quad = \inf_{\substack{u(\cdot) \in \mathcal{U}\,: \\ T_\delta(x_0,\, u(\cdot)) \,<\, +\infty}} \left\{ \int\limits_0^{T_\delta(x_0,\, u(\cdot))} g(x(t;\, 0, x_0, u(\cdot)),\, u(t))\, dt \right\} \\ & \forall x_0 \in G \quad \forall \delta \in (0, r]. \end{aligned} \tag{12}$$

Here the convention $\inf \emptyset = +\infty$ is used. Similarly to (7), introduce the function

$$G \,\ni\, x_0 \;\longmapsto\; v_\delta(x_0) \;\stackrel{\text{def}}{=}\; 1 \,-\, e^{-V_\delta(x_0)} \;\in\; [0, 1]. \tag{13}$$

The next theorem indicates the validity of the proposed approximation and can be proved by using Theorem 2.14 and the dynamic programming principle for the transformed value function $v(\cdot)$ in the original problem (6) (a general formulation of dynamic programming principles can be found in [23]).

Theorem 3.1: Under Assumptions 2.1, 2.3, 2.9 and 2.12, the following properties hold:

1) for any $\delta \in (0, r]$, the domain of asymptotic null-controllability can be represented as

$$\begin{aligned} \mathcal{D}_0 \;&=\; \{x_0 \in G \,:\; V_\delta(x_0) \,<\, +\infty\} \\ &=\; \{x_0 \in G \,:\; v_\delta(x_0) \,<\, 1\} \end{aligned}$$

(it does not depend on δ due to Definition 2.6);

2) $v_\delta(x_0) \to v(x_0)$ uniformly on $\mathcal{D}_0$ as $\delta \to +0$.

One more assumption is needed for establishing the existence of an optimal control strategy in the problem (10).

Assumption 3.2: The set

$$\big\{ (f(x,u),\, g(x,u)) \;\in\; \mathbb{R}^{n+1} \,:\; u \in U \big\}$$

is convex for every $x \in G$.

The following result can be verified with the help of the general existence theorem in [24, §9.3].

Theorem 3.3: Let Assumptions 2.1, 2.3, 2.9, 2.12 and 3.2 hold. For any fixed initial state $x_0 \in \mathcal{D}_0$ and parameter $\delta \in (0, r]$, the minimization problem (10) admits an optimal control strategy.

Some smoothness properties are also needed in order to use necessary optimality conditions (Pontryagin's principle) for the problem (10).

Assumption 3.4: The functions $f(\cdot, u)$ and $g(\cdot, u)$ are continuously differentiable on $\bar{G}$ for any $u \in U$.

Theorem 3.5: (Pontryagin's principle; see, e. g., [24, §5.1, §4.2 (emphasize Remark 10), §4.4.B]) Let Assumptions 2.1, 2.3, 2.9, 2.12, 3.2 and 3.4 hold. Suppose that $u^*(\cdot) \in \mathcal{U}$ is an optimal control strategy in the problem (10) for a fixed initial state $x_0 \in \mathcal{D}_0$ and a fixed parameter $\delta \in (0, r]$. Let $x^*\colon\ [0, T_\delta(x_0, u^*(\cdot))] \to G$ be the corresponding optimal state trajectory. Denote $T^*_\delta(x_0) \overset{\text{def}}{=} T_\delta(x_0, u^*(\cdot)) < +\infty$ for the sake of brevity. Introduce also the Hamiltonian:

$$\begin{gathered} H(x, u, p, \tilde{p}) \overset{\text{def}}{=} \langle p, f(x, u)\rangle + \tilde{p}\, g(x, u) \\ \forall\ (x, u, p, \tilde{p}) \in G \times U \times \mathbb{R}^n \times \mathbb{R}. \end{gathered} \tag{14}$$

Then there exist an absolutely continuous function $p^*\colon\ [0, T^*_\delta(x_0)] \to \mathbb{R}^n$ and a constant $\tilde{p}^* \geqslant 0$ such that the following properties hold:

- $(p^*(t), \tilde{p}^*) \neq 0_{n+1}$ for every $t \in [0, T^*_\delta(x_0)]$;
- $(x^*(\cdot), p^*(\cdot))$ is a solution of the characteristic boundary value problem

$$\begin{cases} \dot{x}^*(t) = f(x^*(t), u^*(t)), \\ \dot{p}^*(t) = -\mathrm{D}_x H(x^*(t), u^*(t), p^*(t), \tilde{p}^*), \\ t \in [0, T^*_\delta(x_0)], \\ x^*(0) = x_0, \quad \|x^*(T^*_\delta(x_0))\| \leqslant \delta, \\ p^*(T^*_\delta(x_0)) \in \mathcal{N}\left(x^*(T^*_\delta(x_0));\, \bar{B}_\delta(0_n)\right) \\ \qquad = \{\lambda\, x^*(T^*_\delta(x_0)) : \lambda \geqslant 0\} \end{cases} \tag{15}$$

  (here $\mathcal{N}(\xi; \Xi)$ denotes the normal cone to a convex set $\Xi \subseteq \mathbb{R}^n$ at a point $\xi \in \Xi$);
- $u^*(\cdot)$ satisfies the Hamiltonian minimum condition

$$\begin{aligned} &H(x^*(t), u^*(t), p^*(t), \tilde{p}^*) \\ &\quad = \min_{u \in U} H(x^*(t), u, p^*(t), \tilde{p}^*) \end{aligned} \tag{16}$$

  for almost all $t \in [0, T^*_\delta(x_0)]$;
- one also has $H(x^*(t), u^*(t), p^*(t), \tilde{p}^*) \equiv 0$ for all $t \in [0, T^*_\delta(x_0)]$.

Remark 3.6: Note that the Hamiltonian (14) is positive homogeneous of degree 1 with respect to $(p, \tilde{p})$. Then, in Theorem 3.5, the set of possible values of $\tilde{p}^*$ can be reduced to $\{0, 1\}$. The case $\tilde{p}^* = 0$ is called abnormal.

Furthermore, denote

$$\begin{gathered} U^*(x, p, \tilde{p}) \overset{\text{def}}{=} \operatorname{Arg}\min_{u \in U} H(x, u, p, \tilde{p}) \\ \forall\ (x, p, \tilde{p}) \in G \times \mathbb{R}^n \times \mathbb{R}. \end{gathered} \tag{17}$$

Remark 3.7: For numerical purposes, it is reasonable to parametrize the characteristic field with respect to $(p_0, \tilde{p}^*)$, so that Cauchy problems can be solved instead of the boundary value problems (15), (16). Indeed, the latter may have multiple solutions, leading to the practical dilemma of obtaining a solution that provides the optimal cost. In order to have the uniqueness of solutions of characteristic Cauchy problems, one needs to ensure that the extremal control map (17) is singleton, or to specify an appropriate singleton selector of $U^*(x^*(t), p^*(t), \tilde{p}^*)$ for almost all $t \in [0, T^*_\delta(x_0)]$, which may also require an additional analysis.

The next result describes the aforementioned parametrization of the characteristic field and reduces the problem of computing the transformed value function (13) at any selected state in $\mathcal{D}_0$ to finite-dimensional optimization with respect to $(p_0, \tilde{p}^*)$ (see also [13, Theorem 3.8]).

Theorem 3.8: Let Assumptions 2.1, 2.3, 2.9, 2.12, 3.2 and 3.4 hold. For any $x_0 \in \mathcal{D}_0$ and $\delta \in (0, r]$, the value $v_\delta(x_0)$ is the minimum of

$$1 - \exp\left\{ - \int_0^{T_\delta(x_0,\, u^*(\cdot))} g(x^*(t), u^*(t))\, dt \right\} \tag{18}$$

over the solutions of the characteristic Cauchy problems

$$\begin{cases} \dot{x}^*(t) = f(x^*(t), u^*(t)), \\ \dot{p}^*(t) = -\mathrm{D}_x H(x^*(t), u^*(t), p^*(t), \tilde{p}^*), \\ u^*(t) \in U^*(x^*(t), p^*(t), \tilde{p}^*), \\ t \in I_\delta(x_0, u^*(\cdot)) \\ \quad \overset{\text{def}}{=} \begin{cases} [0, T_\delta(x_0, u^*(\cdot))], & T_\delta(x_0, u^*(\cdot)) < +\infty, \\ [0, +\infty), & T_\delta(x_0, u^*(\cdot)) = +\infty, \end{cases} \\ x^*(0) = x_0, \quad p^*(0) = p_0, \end{cases} \tag{19}$$

for all vectors

$$(p_0, \tilde{p}^*) \in \{(p, \tilde{p}) : p \in \mathbb{R}^n,\ \tilde{p} \in \{0, 1\}\}. \tag{20}$$

Moreover, the optimization over the bounded set

$$(p_0, \tilde{p}^*) \in \{(p, \tilde{p}) \in \mathbb{R}^n \times \mathbb{R} : \|(p, \tilde{p})\| = 1,\ \tilde{p} \geqslant 0\} \tag{21}$$

(instead of the unbounded set (20)) leads to the same minimum cost.

Remark 3.9: Theorems 3.3 and 3.5 yield that, for any $x_0 \in \mathcal{D}_0$ and $\delta \in (0, r]$, the problem (10) admits an optimal control strategy and the related characteristic curve satisfying Pontryagin's principle. Hence, the set (21) in Theorem 3.8 can be replaced with the whole unit sphere in $\mathbb{R}^{n+1}$, i. e., with

$$(p_0, \tilde{p}^*) \in \{(p, \tilde{p}) \in \mathbb{R}^n \times \mathbb{R} : \|(p, \tilde{p})\| = 1\}. \tag{22}$$

This eliminates the need for handling the additional constraint $\tilde{p}^* \geqslant 0$ in numerical optimization. The sphere (22) can be parametrized in a standard way:

$$\begin{cases} \tilde{p}^* = \cos\theta_n, \quad p_{01} = \prod_{i=1}^{n} \sin\theta_i, \\ p_{0j} = \cos\theta_{j-1} \prod_{i=j}^{n} \sin\theta_i, \quad j = \overline{2, n}, \\ 0 \leqslant \theta_1 < 2\pi, \quad 0 \leqslant \theta_j \leqslant \pi, \quad j = \overline{2, n}. \end{cases} \tag{23}$$

Due to the periodicity in the angles θ_i, $i = \overline{1, n}$, one can perform unconstrained optimization over them by using an appropriate numerical method (see, e. g., [25, Chapter 10]).

Remark 3.10: Before implementing a numerical algorithm based on Theorem 3.8, one does not know whether a particular initial state $x_0 \in G$ belongs to the domain $\mathcal{D}_0$ of asymptotic null-controllability or not. Furthermore, there may be a nonempty set of characteristic processes $(u^*(\cdot), x^*(\cdot), p^*(\cdot), \tilde{p}^*)$ for which $T_\delta(x_0, u^*(\cdot)) =$

$+\infty$ and the cost (18) equals 1. Thus, in order to implement the algorithm for any particular $\delta \in (0, r]$ and $x_0 \in G \setminus \bar{B}_\delta(0_n)$ (the case $x_0 \in \bar{B}_\delta(0_n)$ is trivial and leads to $v_\delta(x_0) = 0$), it is reasonable to specify a sufficiently small parameter $\delta_1 \in (0, 1)$ and to stop integrating the characteristic system (19) at such instants $t = T$ for which either

$$\begin{cases} \|x^*(t)\| \,>\, \delta \;\; \forall t \in [0, T), \quad \|x^*(T)\| \,=\, \delta, \\ \exp\left\{ -\int\limits_0^T g(x^*(t),\, u^*(t))\, dt \right\} \;\geqslant\; \delta_1, \end{cases} \tag{24}$$

or

$$\begin{cases} \|x^*(t)\| \,>\, \delta \quad \forall t \in [0, T], \\ \exp\left\{ -\int\limits_0^T g(x^*(t),\, u^*(t))\, dt \right\} \;=\; \delta_1. \end{cases} \tag{25}$$

Indeed, Item 3 of Assumption 2.12 guarantees that the function

$$I_\delta(x_0, u^*(\cdot)) \ni t \longmapsto 1 - \exp\left\{ -\int\limits_0^t g(x^*(s),\, u^*(s))\, ds \right\}$$

strictly increases if $x_0 \in G \setminus \bar{B}_\delta(0_n)$ (in the trivial case $x_0 \in \bar{B}_\delta(0_n)$, one has $T_\delta(x_0, u(\cdot)) = 0$ for all $u(\cdot) \in \mathcal{U}$). If the additional stopping conditions (25) hold, one can approximate the cost (18) by the value $1 - \delta_1$ (assigning the maximum value 1 may cause unwanted discontinuities). Then, in line with Theorems 2.14 and 3.1, the domain $\mathcal{D}_0$ can be approximated by

$$\mathcal{D}_{0,\delta,\delta_1} \;\stackrel{\text{def}}{=}\; \{x_0 \in G :\; v_{\delta,\delta_1}(x_0) \,<\, 1 - \delta_1\}, \tag{26}$$

where $v_{\delta,\delta_1} \colon G \to [0, 1)$ is the approximation of $v_\delta(\cdot)$ obtained in the aforementioned way.

Remark 3.11: The optimization with respect to $(p_0, \tilde{p}^*)$ in Theorem 3.8 may be essentially multi-extremal. In particular, the sphere (22) may contain a relatively large region such that the target ball $\bar{B}_\delta(0_n)$ is not reached by the corresponding state trajectories (according to Remark 3.10, the approximate cost is constant and equals $1 - \delta_1$ in this region). In order to find an appropriate initial guess for an iterative method regarding the main optimization problem, one can consider an auxiliary shooting problem. Fix some $T_{\max} > 0$. The aim is to find a reverse-time characteristic

$$[0, T_{\max}] \ni \tau \;\longmapsto\; (\hat{x}^*(\tau),\, \hat{p}^*(\tau),\, \tilde{p}^*) \in G \times \mathbb{R}^n \times \mathbb{R} \tag{27}$$

that emanates from

$$\begin{gathered} \|(\hat{p}^*(0),\, \tilde{p}^*)\| \,=\, 1, \quad \hat{p}^*(0) \,\neq\, 0_n, \quad \tilde{p}^* \geqslant 0, \\ \hat{x}^*(0) \,=\, \delta\, \frac{\hat{p}^*(0)}{\|\hat{p}^*(0)\|} \,\in\, \bar{B}_\delta(0_n) \end{gathered} \tag{28}$$

and minimizes the lowest deviation

$$\min_{\tau \,\in\, [0, T_{\max}]} \|\hat{x}^*(\tau) \,-\, x_0\|^2 \tag{29}$$

from the initial state x_0. Here τ is the reverse time variable. The conditions (28) are called forth by the terminal condition in Pontryagin's principle (see (15)). Thus, one arrives at optimization with respect to $(\hat{p}^*(0),\, \tilde{p}^*)$. It is reasonable to implement it with a certain number of randomly generated starting points. Suppose that the numerical optimization over the whole unit sphere of the vectors $(\hat{p}^*(0),\, \tilde{p}^*)$ leads to a sought-after reverse-time characteristic (27) satisfying (28) (in particular, suppose that neither of the executed iterations returns a negligibly small $\|\hat{p}^*(0)\|$). Then $(\hat{p}^*(\tau'),\, \tilde{p}^*)$ with τ' being a minimizer in (29) can serve as a starting point for the main optimization problem.

Remark 3.12: Let $\delta \in (0, r]$. If one can explicitly find a local stabilizing feedback control strategy for all initial states $x_0 \in \bar{B}_\delta(0_n) \subseteq \bar{B}_r(0_n)$ (recall Assumption 2.3 and Remark 2.4), then it is reasonable to use it directly after the optimal feedback control law for the problem (10) brings a state trajectory to the ball $\bar{B}_\delta(0_n)$. Note that $\bar{B}_\delta(0_n)$ may not be invariant with respect to the state trajectories of the differential inclusion generated by this local stabilizing feedback policy. If $\tilde{V}_{\mathrm{loc}}(\cdot)$ is the corresponding local control Lyapunov function defined on some open domain $\Omega \supset \bar{B}_\delta(0_n)$ and the target set $\bar{B}_\delta(0_n)$ in the problem (10) is replaced with an appropriate level set

$$\Omega_c \;\stackrel{\text{def}}{=}\; \left\{ x \in \Omega :\; \tilde{V}_{\mathrm{loc}}(x) \leqslant c \right\} \;\subseteq\; \bar{B}_r(0_n)$$

for some $c > 0$, then the mentioned invariance property will hold for Ω_c. One may also think of using $\tilde{V}_{\mathrm{loc}}(\cdot)$ in order to specify a nonzero terminal part of the cost functional for the new target set Ω_c. However, this may lead to a difference between the final time of an optimal process and the first time of reaching the target set by the related state trajectory, i. e., to a violation of the value function representation through the first entry times (see (11), (12)). Since the latter plays a significant role for Theorems 3.1, 3.3 as well as in the numerical algorithm based on Theorem 3.8 and Remarks 3.9–3.11, the terminal part of the cost functional in (10) is set to zero.

Remark 3.13: In some situations, Pontryagin's principle (Theorem 3.5) allows verification of the absence of singular regimes, so that the nonuniqueness in the choice of extremal control values may occur only at isolated time instants, and the set of the vectors $(p_0, \tilde{p}^*)$ in Theorem 3.8 can be accordingly reduced (for example, by excluding the abnormal case $\tilde{p}^* = 0$) in order to have unique solutions of the characteristic Cauchy problems (19). If one cannot exclude singular regimes from consideration, the multi-valued extremal control map (17) may yield multiple solutions for some of the Cauchy problems (19), which is rather difficult to handle in a numerical algorithm (see also [13, Remark 3.12]). For particular classes of optimal control problems, characteristics based algorithms of computing the value functions can be modified so as to handle singular regimes and to avoid the nonuniqueness in the choice of extremal control values (see [13, Examples 3.14 and 3.15]).

Remark 3.14: Characteristics based approaches (such as the ones proposed in this work and also in [13], [16], [17]) have a number of important advantages over well-known grid-based methods (noted in Remark 2.18). First, the method of characteristics allows the value functions to be approximated separately at different initial states. Therefore, the curse of dimensionality can be attenuated, and parallel computations can significantly increase the numerical efficiency. For a high state space dimension $n \geqslant 4$, constructing global (or semi-global) value function approximations suffers from the curse of complexity. Sparse grid techniques [18] may help to overcome the latter if n is not too high ($n \leqslant 6$) and if the possibly multi-extremal optimization with respect to the initial adjoint vector ($(p_0, \tilde{p}^*)$ in Theorem 3.8) is successfully implemented for all considered initial states (which is not a priori guaranteed). Furthermore, treating different initial states via the method of characteristics makes it possible to select arbitrary bounded regions and grids in the state space for computations. Next, the optimal feedback control strategies at any isolated states can be obtained directly from integrated optimal characteristics, and one does not need to approximate partial derivatives of the value functions, which may be an unstable procedure. However, the developed characteristics based approaches still have a limited range of practical applicability, as follows from the aforementioned curse of complexity, Remark 3.13 and [13, Remarks 2.14, 2.15, 3.11–3.13]. That is why further extensions are worth investigating.

IV. NUMERICAL SIMULATIONS

In the following two examples, the running cost $g(\cdot,\cdot)$ is chosen in the form

$$g(x,u) \;=\; \frac{\lambda_1}{2}\,\|x\|^2 \,+\, \frac{\lambda_2}{2}\,\|u\|^2 \quad \forall\,(x,u)\,\in\,\mathbb{R}^n \times U \tag{30}$$

with positive constants λ_1, λ_2, and the conditions of Theorem 3.8 hold. The Kruzhkov transformed CLFs and corresponding feedback control strategies have been approximated via the algorithm based on Theorem 3.8 and Remarks 3.9–3.11. The numerical simulations have been conducted on a relatively weak machine with 1.4 GHz Intel 2957U CPU via a C++ code (without algorithm parallelization), and the related average runtimes per node on a state grid are mentioned below. More powerful machines can provide essentially lower runtimes, especially when implementing parallelization. Note also that the highest runtimes have been observed for states $x_0 \in \mathcal{D}_0$ located near the boundary $\partial\mathcal{D}_0$.

Example 4.1: Consider the linearized single-link inverted pendulum with stationary pivot described by the system

$$\begin{cases} \dot{x}_1(t) \,=\, x_2(t), \\ \dot{x}_2(t) \,=\, x_1(t) \,+\, u(t), \\ x(0) \,\in\, \mathbb{R}^2, \quad u(t) \,\in\, U \,=\, [-a, a], \quad t \geqslant 0. \end{cases} \tag{31}$$

Then the controllability domain $\mathcal{D}_0$ is the strip $|x_1 + x_2| < a$ [26, §1.2]. Take $a = 0.5$, $\lambda_1 = 1$, $\lambda_2 = 0.1$, $\delta = 10^{-2}$, $\delta_1 = 10^{-3}$, and set $T_{\max} = 5$ for the auxiliary finite-dimensional optimization problem of Remark 3.11. The reverse- and forward-time characteristic systems have been integrated numerically via the second-order Heun method with the stepsize $5 \cdot 10^{-3}$. The preliminary shooting optimization has been performed via the Powell routine of [25] with the tolerance parameter 10^{-7}, and 5 starting points on the unit sphere of the extended terminal adjoint vectors have been randomly generated for each selected initial state. For the main optimization with respect to the extended initial adjoint vector (recall Theorem 3.8), the same Powell routine has been used with the tolerance parameter 10^{-5} (for each selected initial state, there has been only one starting point obtained from the shooting optimization). The numerical simulation results in Fig. 1 show that the approximate controllability domain is in good agreement with the exact one. The average runtime per state has been around 8 s.

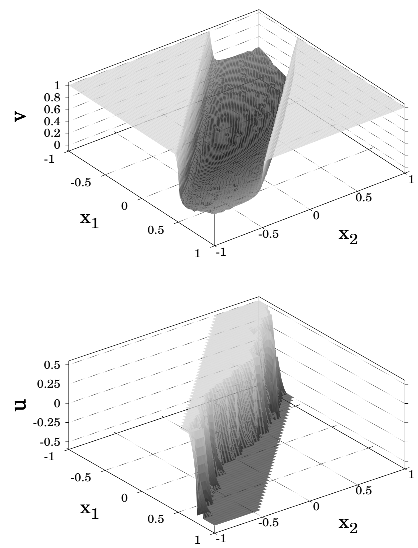

Fig. 1. *Example* 4.1: *the Kruzhkov transformed CLF and related feedback control law on the square* $[-1, 1]^2$ *of the state space.*

Example 4.2: Consider the five-dimensional nonholonomic integrator

$$\begin{cases} \dot{x}_1(t) \,=\, u_1(t), \\ \dot{x}_2(t) \,=\, u_2(t), \\ \dot{x}_3(t) \,=\, x_1(t)\,u_2(t) \,-\, x_2(t)\,u_1(t), \\ \dot{x}_4(t) \,=\, x_1^2(t)\,u_2(t), \\ \dot{x}_5(t) \,=\, -x_2^2(t)\,u_1(t), \\ x(0) \,\in\, \mathbb{R}^5, \quad u(t) \,\in\, U \,=\, [-a, a]^2, \quad t \geqslant 0, \end{cases} \tag{32}$$

which is locally null-controllable due to [27, Exercise 1.12, p. 212]. Take the parameters $a, \lambda_1, \lambda_2, \delta, \delta_1$ as in Example 4.1, and set $T_{\max} = 2$. The reverse- and forward-time characteristic systems have been integrated numerically via the standard fourth-order Runge–Kutta scheme with the stepsize $5 \cdot 10^{-3}$. The preliminary and main finite-dimensional optimization processes have been implemented similarly to those for Example 4.1. Fig. 2 illustrates the computed reductions of the Kruzhkov transformed CLF and stabilizing feedback control law to the square $[-0.3, 0.3]^2$ on the plane $x_3 = x_4 = x_5 = 0$ of the state space (for points on this plane outside the selected square and therefore closer to the boundary ∂D_0, the convergence rate of the optimization methods has appeared to dramatically decrease). The average runtime per state has been around 19 s.

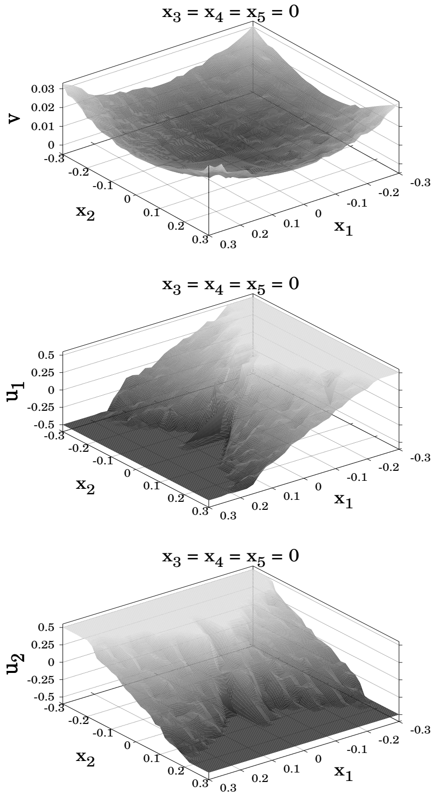

Fig. 2. *Example* 4.2: *the reductions of the Kruzhkov transformed CLF and related feedback control law to the square* $[-0.3, 0.3]^2$ *on the plane* $x_3 = x_4 = x_5 = 0$ *of the state space.*

V. CONCLUSION

As was already mentioned in the introduction and in Remark 3.14, it seems promising to combine the characteristics based approach of this work with the sparse grid framework of [18] in order to obtain global approximations of CLFs and stabilizing feedback strategies in regions of relatively high dimensions. Furthermore, a conceptually similar method allowing parallel implementation was developed in [28], [29] for a different problem of computing finite-time reachable sets.

As a future perspective, it is also relevant to design stabilizing feedbacks for more complex nonlinear mechanical systems [26] than the ones considered in Section IV. Then it is reasonable to solve the preliminary optimization problem of Remark 3.11 via an efficient numerical library that implements a robust direct collocation method [18]. This may significantly reduce computational errors.

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