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Optimal control problems with oscillations, concentrations, and discontinuities

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***Abstract*— Optimal control problems with oscillation (chattering controls) and concentration (impulsive controls) can have integral performance criteria such that concentration of the control signal occurs at a discontinuity of the state signal. Techniques from functional analysis (anisotropic parametrized measures) are applied to give a precise meaning of the integral cost and to allow for the sound application of numerical methods. In the oral presentation of this work, we show how this can be combined with the Lasserre hierarchy of semidefinite programming relaxations. This includes in particular the use of compactification techniques allowing for unbounded time, state and control.**

I. Introduction

As a consequence of optimality, various limit behaviours can be observed in optimal control: minimizing control law sequences may feature increasingly fast variations, called oscillations (chattering controls [9]), or increasingly large values, called concentrations (impulsive controls [8]). The simultaneous presence of oscillations and concentrations in optimal control need careful analysis and specific mathematical tools, so that the numerical methods behave correctly. Previous work of two of the authors [2] combined tools from partial differential equation analysis (DiPerna-Majda measures [3]) and semidefinite programming relaxations (the moment-sums-of-squares or Lasserre hierarchy [7]) to describe a sound numerical approach to optimal control in the simultaneous presence of oscillations and concentrations. To overcome difficulties in the analysis, a certain number of technical assumptions were made, see [2, Assumption 1, Section 2.2], so as to avoid the simultaneous presence of concentrations (in the control signals) and discontinuities (in the system trajectories).

In the present contribution we would like to remove these technical assumptions and accommodate the simultaneous presence of concentrations and discontinuities, while allowing oscillations as well. For this, we exploit a recent extension of the notion of DiPerna-Majda measures [5], so that it makes sense mathematically while allowing for an efficient numerical implementation with semidefinite programming relaxations.

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II. Relaxing Optimal Control

Let $L : [0,1]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ and $F : [0,1]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ be continuous functions. For initial and final conditions $y_0, y_T \in \mathbb{R}^n$ and some integer $1 \le p \le \infty$, the formulation of the classical optimal control problem is

$$\begin{array}{ll} \inf_u & \int_0^1 L(t,y(t),u(t))dt \\ \text{s.t.} & \dot{y}(t) = F(t,y(t),u(t)),\ y(0)=y_0,\ y(1)=y_1, \\ & y\in W^{1,1}([0,1];\mathbb{R}^n), \quad u \in L^p([0,1];\mathbb{R}^m). \end{array} \tag{1}$$

Consider a minimizing sequence of controls whose p-th power is Lebesgue integrable $(u_k)_{k\in\mathbb{N}} \subset L^p([0,1];\mathbb{R}^m)$ for problem (1) and the corresponding sequence of absolutely continuous trajectories $(y_k)_{k\in\mathbb{N}} \subseteq W^{1,1}([0,1];\mathbb{R}^n)$. Then the infimum in (1) might not be attained because $(u_k)_{k\in\mathbb{N}}$ might not converge in $L^p([0,1];\mathbb{R}^m)$ and $(y_k)_{k\in\mathbb{N}}$ might not converge in $W^{1,1}([0,1];\mathbb{R}^m)$. To overcome this issue it has been proposed to relax the regularity assumptions on the control and state signals. We discuss some of the approaches now in detail.

III. Oscillation

The limit of a minimizing sequence for (1) might fall out of the feasible space because of oscillation effects of $(u_k)_{k\in\mathbb{N}}$. Consider for example the optimal control problem

$$\begin{array}{ll} \inf_u \int_0^1 & (u(t)^2-1)^2 + y(t)^2 dt \\ \text{s.t.} & \dot{y}(t) = u(t),\ y(0)=0,\ y(1)=0, \\ & y\in W^{1,4}([0,1]), \quad u\in L^4([0,1]). \end{array} \tag{2}$$

As the integrand in the cost is a sum of squares, the infimum of (2) is at least zero. To see that the infimum actually is zero, consider the sequence of controls $(u_k)_{k\in\mathbb{N}} \subset L^4([0,1])$ defined by

$$u_k(t) := \left\{ \begin{array}{ll} 1, & \text{if } t\in\left[\frac{2l+1}{2^k}, \frac{l+1}{2^{k-1}}\right],\ 0\le l\le k-1 \\ -1, & \text{otherwise} \end{array} \right. \tag{3}$$

for $k > 1$ and $u_1 := 0$. For the corresponding sequence of trajectories $(y_k)_{k\in\mathbb{N}}$ defined by $y_k(t) := \int_0^t u_k(t)dt$ it holds that $y_k \in W^{1,4}([0,1])$ and $y_k(1) = 0$ as desired. Hence, $(u_k)_{k\in\mathbb{N}}$ is a sequence of feasible controls. A short calculation shows that using this sequence the cost in (2) converges to zero. While the limit $y_\infty := 0$ of $(y_k)_{k\in\mathbb{N}}$ stays in $W^{1,4}([0,1])$, the sequence of controls $(u_k)_{k\in\mathbb{N}}$ however does not converge in $L^4([0,1])$.

In contrast to that, the sequence of measures defined by $d\nu_k(t,u) := \delta_{u_k(t)}(du|t)dt$ converges to $d\nu(t,u) :=$

$\frac{1}{2}(\delta_{-1}+\delta_1)(du)dt$ in the sense that for all $f \in C([0,1])$ and $g \in C_p(\mathbb{R})$:

$$\lim_{k\to\infty}\int_0^1\int_{\mathbb{R}} f(t)g(u)d\nu_k(t,u) = \int_0^1\int_{\mathbb{R}} f(t)g(u)d\nu(t,u) \tag{4}$$

where $C_p(\mathbb{R}) := \{g \in C(\mathbb{R}) : g(u) = o(|u|^p) \text{ for } |u| \to \infty\}$ is the set of continuous functions of less than p-th growth. Integration with respect to ν shows

$$\int_0^1\int_{\mathbb{R}} u d\nu(t,u) = \int_0^1\int_{\mathbb{R}} u\tfrac{1}{2}(\delta_{-1}+\delta_1)(du)dt = 0 = y_\infty(1).$$

A similar reasoning shows that the cost with respect to ν is zero.

More generally, this observation motivates to relax the regularity assumptions on the control u in (1) and also allow for limits $d\nu(t,u) = d\omega(u|t)dt$ of control sequences $(u_k)_{k\in\mathbb{N}} \subseteq L^p([0,1];\mathbb{R}^m)$. In general the measure ω depends on time, i.e., we have a family of probability measures $\omega(.|t)_{t\in[0,1]} \subset P(\mathbb{R}^m)$. Such parametrized measures obtained as limits of a sequence of functions $(u_k)_{k\in\mathbb{N}} \subseteq L^p([0,1];\mathbb{R}^m)$ have been called *L^p-Young measures*. For an explicit characterization of these measures see e.g. [6]. The relaxed version of (1) that now takes into account oscillating control sequences can be written as

$$\begin{array}{ll} \inf_\omega & \int_0^1\int_{\mathbb{R}^m} L(t,y(t),u)\,\omega(du|t)dt \\ \text{s.t.} & \dot{y} = \int_{\mathbb{R}^m} F(t,y(t),u)\,\omega(du|t)dt, y(0)=y_0, y(1)=y_1, \\ & y \in W^{1,1}([0,1];\mathbb{R}^n),\ \omega(.|t) \in P(\mathbb{R}^m). \end{array} \tag{5}$$

For a comprehensive reference on Young measures and their use in the control of ordinary and partial differential equations, see [4, Part III].

IV. Concentration

Oscillation of the control sequence is not the only reason that prevents the infimum in (1) from being attained. As a second example consider the following problem of optimal control:

$$\begin{array}{ll} \inf_u & \int_0^1 (t-\frac{1}{2})^2 u(t)dt \\ \text{s.t.} & \dot{y}(t) = u(t) \geq 0,\ y(0)=0,\ y(1)=1, \\ & y \in W^{1,1}([0,1]), \quad u \in L^1([0,1]). \end{array} \tag{6}$$

Note that the control enters into the problem linearly. Again the infimum of the optimal control problem is zero as the integrand is positive and using the sequence of controls

$$u_k(t) := \begin{cases} k, & \text{if } t \in \left[\frac{k-1}{2k}, \frac{k+1}{2k}\right] \\ 0, & \text{else} \end{cases} \tag{7}$$

the cost converges to zero. As in the previous section neither $(u_k)_{k\in\mathbb{N}}$ nor any subsequence converges in $L^1([0,1])$. In contrast to the previous example this time $(y_k)_{k\in\mathbb{N}}$ does not converge in $W^{1,1}([0,1])$ neither. We hence use the extension $BV([0,1])$, i.e. the space of functions with bounded variation, as a relaxed space for the trajectory. Following the same approach as before, we replace the control function by a measure. But now, instead of interpreting the control as a measure on the control space, we exploit the fact that u appears linearly in (6) and hence that we can directly integrate with respect to u and define a sequence of probability measures $(\tau_k)_{k\in\mathbb{N}} \subset P([0,1])$ by $\tau_k(dt) := u_k(t)dt$. A short calculation shows that this sequence has the weak star limit $\tau := \delta_{1/2}$ i.e. for all $f \in C([0,1])$:

$$\lim_{k\to\infty}\int_0^1 f(t)\tau_k(dt) = \int_0^1 f(t)\tau(dt).$$

Note that by integrating before passing to the limit we transfer the unboundedness of the control into the measurement of time and only keep the direction (i.e. $+1$ in this example) of the control. Whereas we observed a superposition of two different controls in the previous example, here we see a concentration of the control in time. A relaxation of (1), that can take into account concentration effects of the control, is hence given by

$$\begin{array}{ll} \inf_\tau & \int_0^1 L(t,y(t))\tau(dt) \\ \text{s.t.} & dy(t) = F(t,y(t))\tau(dt),\ y(0)=y_0,\ y(1)=y_1, \\ & y \in BV([0,1];\mathbb{R}^n),\ \tau \in P([0,1]). \end{array} \tag{8}$$

See [1] for an application of the moment-sums-of-squares hierarchy for solving numerically non-linear control problems in the presence of concentration.

V. Oscillation and Concentration

The relaxations proposed so far allow to consider controls that are either oscillating in value or concentrating in time. However it is possible that both effects appear in the same problem. Consider for example

$$\begin{array}{ll} \inf_u & \int_0^1 \frac{u(t)^2}{1+u(t)^4} + (y(t)-t)^2\,dt \\ \text{s.t.} & \dot{y}(t) = u(t) \geq 0,\ y(0)=0,\ y(1)=1, \\ & y \in W^{1,1}([0,1]), \quad u \in L^1([0,1]). \end{array} \tag{9}$$

The infimum zero of (9) can be approached arbitrarily close by a sequence of controls $(u_k)_{k\in\mathbb{N}}$ defined this time by

$$u_k(t) := \begin{cases} k, & \text{if } t \in \left[\frac{l}{k}-\frac{1}{2k^2}, \frac{l}{k}+\frac{1}{2k^2}\right],\ 1 \leq l < k \\ 0, & \text{else} \end{cases} \tag{10}$$

for $k > 1$ and $u_1 := 1$. The idea to capture the limit behaviour of this sequence is to combine a Young measure on the control and replacing the uniform measure on time by a more general measure on time. Note that due to linearity it was possible in Section IV to transfer the limit behaviour of the control into the measurement of time. In the present example the control enters non linearly in the cost, which is why we will need to allow the control to take values at infinity. We consider a metrizable compactification $\beta_U\mathbb{R}$ of the control space corresponding to the ring U of complete and separable continuous functions. Then the sequence of measures $d\nu_k(t,u) := \delta_{u_k(t)}(du|t)dt$ converges to $d\nu(t,u) := \omega(du)\tau(dt)$ with $\omega(du) := \frac{1}{2}(\delta_0+\delta_\infty)(du)$ and $\tau(dt) := 2dt$ understood in the following sense for all $f \in C([0,1])$ and $g_0 \in U$:

$$\begin{array}{l} \lim_{k\to\infty}\int_0^1\int_{\mathbb{R}} f(t)g_0(u)(1+|u|^p)d\nu_k(t,u) \\ = \int_0^1\int_{\beta_U\mathbb{R}} f(t)g_0(u)d\nu(t,u). \end{array} \tag{11}$$

Measures ν obtained as limits of sequences $(u_k)_{k\in\mathbb{N}} \subseteq L^p([0,1];\mathbb{R}^m)$ in the sense of (11) have been called *DiPerna-Majda measures* and they were introduced in [3]. It turns out that every DiPerna-Majda measure ν can be disintegrated into a measure τ on time and an L^p-Young measure ω on $\beta_U\mathbb{R}^m$. A relaxed version of (1) taking into account both oscillation and concentration effects can hence be stated as

$$\begin{array}{ll} \inf_\nu & \int_0^1 \int_{\beta_U\mathbb{R}^m} L_0(t,y(t),u)\,d\nu(t,u) \\ \text{s.t.} & dy(t) = \int_{\beta_U\mathbb{R}^m} F_0(t,y(t),u)d\nu(t,u), \\ & y(0)=y_0,\ y(1)=y_1, \\ & y\in W^{1,1}([0,1];\mathbb{R}^n),\ d\nu(t,u)=\omega(du|t)\tau(dt), \\ & \omega(.|t)\in P(\beta_U\mathbb{R}^m),\ \tau\in P([0,1]) \end{array} \tag{12}$$

where $L_0(t,y,u) := L(t,y,u)/(1+|u|^p)$ and $F_0(t,y,u) := F(t,y,u)/(1+|u|^p)$. In [2], the moment-sums-of-squares hierarchy is adapted to compute numerically DiPerna-Majda measures and solve optimal control problem featuring oscillations and concentrations. However, the approach is valid under a certain number of technical assumptions on the data L and F. These assumptions are enforced to prevent the simultaneous presence of concentration and discontinuity.

VI. Oscillation, Concentration, and Discontinuity

As mentioned in the introduction, the integrals in (1) might not be well defined, as concentration effects of the control are likely to cause discontinuities in the trajectory occurring at the same time. In view of the previous examples we propose to generalize the DiPerna-Majda measures, which themselves are a generalization of Young measures, even further and now also relax the trajectory to a measure valued function depending on time and control. In the oral presentation of this work we explain the notion of *generalized DiPerna-Majda measures* introduced in [5] and then we provide a linear formulation of optimal control problem (1) that can cope with oscillations, concentrations and discontinuities in a unified fashion, while allowing for a numerical implementation using the Lasserre hierarchy.

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