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Robust H_∞ time-varying formation tracking for high-order multi-agent systems with external disturbances

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***Abstract*— Robust H_∞ time-varying formation tracking problems for high-order multi-agent systems with matched and mismatched disturbances under directed topologies are investigated. In this paper, the followers are required to form a formation specified by a time-varying piecewise continuously differentiable vector while tracking the trajectory of the leader containing unknown control input with a H_∞ disturbance attenuation performance. Firstly, a distributed formation tracking protocol is put forward which is constituted by using local neighboring information. An approach to designing the protocol for high-order multi-agent systems to achieve the robust H_∞ time-varying formation tracking is proposed, where the formation tracking feasibility constraint is also derived. Stability of the closed-loop multi-agent system under the protocol is proved. Finally, the effectiveness of the theoretical results are verified by a third-order simulation example.**

I. Introduction

In the last decade, the formation control of multi-agent systems have been extensively studied in both scientific and engineering fields owing to their broad applications, such as spacecraft formation [1], unmanned aerial vehicle (UAV) formation [2], robot formation [3], autonomous underwater vehicle (UAV) formation [4], etc. With the development of the cooperative control, researchers are showing more interest in the ways to study the formation control problems of multi-agent systems by introducing the consensus theory [5], [6], which has good scalability and less computation. Therefore, studying the formation control problems for multi-agent systems by combining the consensus theories is of vital importance.

A basic structure of formation control method based on consensus theories was put forward by Ren [7] for second-order multi-gent systems. Based on the work in [7], further studies have been made about the formation control problems [8], [9]. The leader-follower formation tracking is another form. In [10], a consensus tracking protocol was proposed and the theoretical results were verified by an experiment on the multiple mobile robot systems. In [11], the robust consensus tracking problems were discussed for second-order multi-agents with uncertainty nonlinearities. Distributed adaptive backstepping methods were introduced to deal with the output consensus tracking [12], [13] and formation tracking [14] problems for high-order nonlinear multi-agent systems.

Note that disturbances exist in the most practical systems. For high-order multi-agent systems, it is significant and challenging to investigate the formation control strategy with good anti-disturbance performance. One knows that the matched disturbances to the control input can be dealt with completely by using strong robust methods, such as the sliding mode control method. As for mismatched disturbances, it is hard to compensate it. H_∞ control theory is suitable to solve this problem to get a desired disturbance attenuation performance [23]. Recently, researchers are focusing on the formation control and consensus control problems by the combining H_∞ control theory to achieve the disturbance attenuation. Consensus control problems for multi-agent systems with a prescribed H_∞ disturbance attenuation index were studied in [15], [16] and the distributed H_∞ consensus tracking control problems were studied in [17], [18]. Besides, Huang et al. [19] investigated the H_∞ formation tracking problems for multiple spacecraft systems.

Note that the above mentioned works [11], [12], [13], [14], [17], [18], [19] only focused on the consensus tracking and the time-invariant formation tracking problems, and time-varying formation tracking is more precise to describe the practical tasks such as the target enclosing where followers track the trajectory of the leader and the formation between the followers was time-varying. Time-varying formation tracking problems have been investigated in [20], [21], [22]. However, the control inputs of the leaders in [20], [21], [22] were assumed to be zero and disturbances were ignored. To the best of our knowledge, time-varying formation tracking problems for high-order multi-agent systems with leader's unknown control input and disturbance attenuation are still open.

Motivated by the challenges mentioned above, this paper studies the time-varying formation tracking problems for high-order multi-agent systems with matched and mismatched disturbances, and leader's unknown control input based on H_∞ control theories. Compared with the previous works on formation tracking of multi-agent systems, the contributions of the current paper lie in the following three-fold. Firstly, the dynamics of the leader and the followers is high-order, and the formation between the followers is time-varying. In [20], [21], [22], the dynamics was second-order cases. Although the agents have high-order dynamics [12], [13], [14], the formations between the followers were time-invariant. Secondly, the leader can have unknown control input, and the designed protocol can deal with the unknown control input and matched disturbances simultaneously. Time-varying formation tracking problems were studied in [20], [21], [22], however, they do not considered the effects of the disturbances and the leaders control input was

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ignored. Thirdly, the multi-agent system under the protocol can achieve the H_∞ disturbance attenuation performance. Although H_∞ disturbance attenuation can be achieved in [17], [18], [19], they only studied the consensus tracking or time-invariant formation tracking problems.

The rest of this paper is arranged as follows. In Section II, some introductions to the graph theories are given and the problem formulation is provided. The main results are summarized in Section III and the simulation results is shown with a example in Section IV. Finally, the conclusion is drawn in Section V.

Throughout this paper, for simplicity of notation, let $\mathbb{R}$ be the set of real numbers. Let I_N be an identity matrix with dimension N. $\mathbf{1}_N$ represents a column vector of size N with 1 as its elements. The matrix inequality $A > B$ means that $A - B$ is positive definite. Let $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ be the maximum and the minimum eigenvalues of a real symmetric matrix P, respectively. The norm $\|x\|$ of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$. Let $\otimes$ be the Kronecker product.

II. Preliminaries and problem description

In this section, some basic concepts and results on graph theory are introduced and the problem description is presented.

A. Preliminaries

Let $G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a directed graph of order N, where $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ is the node set, $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}, i \neq j\}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix with elements a_{ij}. An edge of G is denoted by $e_{ij} = (v_i, v_j)$, where node v_i is called a neighbor of node v_j. If not otherwise specified, for all $i, j \in \{1, 2, \cdots, N\}$, $a_{ij} = 1$ if and only if $e_{ji} \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Let $\mathcal{N}_i = \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$ represent the neighbor set of node v_i. Let $\deg_{in}(v_i) = \sum_{j=1}^{N} a_{ij}$ denote the in-degree of node v_i. Denote by $\mathcal{D} = \text{diag}\{\deg_{in}(v_i), i = 1, 2, \cdots, N\}$ the degree matrix of G. The Laplacian matrix L associated with G is defined by $L = \mathcal{D} - \mathcal{A}$. A directed path from node v_{i_1} to node v_{i_r} is a sequence of distinct edges of the form (v_{i_1}, v_{i_2}),(v_{i_2}, v_{i_3}),$\cdots$,$(v_{i_k}, v_{i_{k+1}})$ where $v_{i_k} \in \mathcal{V}$ $(k = 1, 2, \cdots, r-1)$. More details on graph theory can refer to [24].

Lemma 1: ([23]) For any vectors $x(t)$, $y(t)$ of appropriate dimensions and any positive definite matrix Z of appropriate dimension, the following inequality holds:

$$\pm 2x^T(t)y(t) \leq x(t)^T Z x(t) + y^T(t) Z^{-1} y(t). \tag{1}$$

Lemma 2: (Schur Complement Formula) ([23]) Given a symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, the following statements are equivalent:

(i) $S < 0$;

(ii) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;

(iii) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

B. Problem description

Consider a high-order multi-agent system with N agents. The directed interaction topology is described by G. For $i, j \in \{1, 2, \cdots, N\}$ the interaction strength is denoted by a_{ij}.

Definition 1: An agent is called a leader if it has no neighbors. An agent is called a follower if it has at least one neighbor.

Let $F = \{1, 2, \cdots, N-1\}$ be the follower subscript set. The dynamics of the follower i is described by

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t) + B_1 w(t) + B_2 \Delta(t), \\ y_i(t) = Cx_i(t), \end{cases} \tag{2}$$

where $x_i(t) = [x_{i,1}(t), x_{i,2}(t), \cdots, x_{i,n}(t)]^T \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^q$ and $y_i(t) \in \mathbb{R}^m$ are states, control input and output vectors of agent i $(i \in F)$, respectively, $w_i(t) \in R^p$ and $\Delta(t) \in R^q$ are the mismatched and matched disturbances, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{m\times n}$, $B_1 \in \mathbb{R}^{p\times p}$, $B_2 = BM$, $M \in \mathbb{R}^{q\times q}$. The dynamics of leader labeled by N is described by

$$\begin{cases} \dot{x}_N(t) = (A - BK_1)\, x_N(t) + Bu_N(t), \\ y_N(t) = Cx_N(t), \end{cases} \tag{3}$$

where $K_1 \in \mathbb{R}^{q\times n}$, $x_N(t)$, $u_N(t)$ and $y_N(t)$ are states, control input and output vectors. Suppose that $u_N(t)$ is bounded and $x_N(t)$ is bounded under $u_N(t)$.

Assumption 1: The item $w_i(t) \in R^p$ is the unknown external disturbance that belongs to $L_2[0, \infty)$, that is, $\int_0^\infty (w^T(t)w(t))dt < +\infty$.

A time-varying formation is specified by a vector $h(t) = [h_1^T(t), h_2^T(t), \cdots, h_{N-1}^T(t)]^T \in \mathbb{R}^{n\times(N-1)}$ with $h_i(t) \in \mathbb{R}^n$ piecewise continuously differentiable.

From Definition 1, the Laplacian matrix L of a directed topology G can be formed as follows

$$L = \begin{bmatrix} L_1 & L_2 \\ \mathbf{0} & 0 \end{bmatrix}, \tag{4}$$

where $L_1 = [L_{1,i,j}] \in \mathbb{R}^{(N-1)\times(N-1)}$ and $L_2 = [L_{2,i,N}] \in \mathbb{R}^{(N-1)\times 1}$ $(i, j \in F)$.

Consider the following robust H_∞ time-varying formation tracking protocol:

$$\begin{cases} u_i(t) = -K_1 x_i(t) - \kappa(t)\text{sgn}(s_i(t)) \\ \qquad\qquad - \sum\limits_{j=1}^{N-1} a_{ij} K_2 \left((x_i(t) - h_i(t)) - (x_j(t) - h_j(t))\right) \\ \qquad\qquad - a_{iN} K_2 \left(x_i(t) - h_i(t) - x_N(t)\right), \\ s_i(t) = \sum\limits_{j=1}^{N-1} a_{ij} K_2 \left((x_i(t) - h_i(t)) - (x_j(t) - h_j(t))\right) \\ \qquad\qquad + a_{iN} K_2 \left(x_i(t) - h_i(t) - x_N(t)\right), \end{cases} \tag{5}$$

where $i \in F$, $K_2 \in \mathbb{R}^{q\times n}$ is a constant gain matrix, and $\kappa(t)$ is a positive function which is to be designed later, $\text{sgn}(\cdot)$ represents the sign function.

Let $\varsigma_i(t) = x_i(t) - h_i(t)$, $\xi_i(t) = \varsigma_i(t) - x_N(t)$ and $\xi(t) = [\xi_1(t), \xi_2(t), \cdots, \xi_{N-1}(t)]^T$. Under protocol (5), the closed loop dynamics of multi-agent system (2) and (3) can be described by

$$\begin{cases} \dot{\xi}_i(t) = (A - BK_1)\xi_i(t) - \sum\limits_{j=1}^{N-1} a_{ij} BK_2 \left(\xi_i(t) - \xi_j(t)\right) \\ \qquad -a_{iN} BK_2 \xi_i(t) + B_1 w(t) + (A - BK_1) h_i(t) - \dot{h}_i(t) \\ \qquad -\kappa(t) B \mathrm{sgn}(s_i(t)) - Bu_N(t) - BM\Delta(t), \\ \dot{x}_N(t) = (A - BK_1)x_N(t) + Bu_N(t), \end{cases} \tag{6}$$

Rewrite (6) to a compact form as:

$$\begin{cases} \dot{\xi}(t) = \left(I_{N-1} \otimes (A - BK_1) - L_1 \otimes BK_2\right) \xi(t) \\ \qquad + \left(I_{N-1} \otimes B_1\right) w(t) - \left(I_{N-1} \otimes B\right) \left(\mathbf{1}_{N-1} u_N(t)\right) \\ \qquad -\kappa(t) \left(I_{N-1} \otimes B\right) \mathrm{sgn}(s(t)) - \left(I_{N-1} \otimes BM\right) \bar{\Delta}(t) \\ \qquad + \Big(\left(I_{N-1} \otimes (A - BK_1)\right) h(t) - \dot{h}(t)\Big), \\ \dot{x}_N(t) = (A - BK_1)x_N(t) + Bu_N(t), \end{cases} \tag{7}$$

where $w(t) = [w_1(t), w_2(t), \cdots, w_{N-1}(t)]^T$ and $\bar{\Delta}(t) = [\Delta(t), \Delta(t), \cdots, \Delta(t)]^T = 1_{N-1}\Delta(t)$.

Definition 2: Multi-agent system described by (2) and (3) is said to achieve the robust H_∞ time-varying formation tracking if the following two conditions hold.

(i) Define $\bar{\zeta}_i(t) = C\xi_i(t)$ and $\bar{\zeta}(t) = [\bar{\zeta}_1(t), \bar{\zeta}_2(t), \cdots, \bar{\zeta}_n(t)]^T$. The following inequality holds:

$$\begin{aligned} \left\|T_{\bar{\zeta}w}(s)\right\|_\infty &= \sup_{v\in} \bar{\sigma}(T_{\bar{\zeta}w}(jv)) \\ &= \sup_{\mathbf{0}\neq w(t)\in L_2[0,\infty)} \frac{\left\|\bar{\zeta}(t)\right\|_2}{\left\|w(t)\right\|_2} < \gamma, \end{aligned} \tag{8}$$

where $i \in F$, $T_{\bar{\zeta}w}(s)$ represents the closed-loop transfer function matrix from $w(t)$ to $\bar{\zeta}(t)$, $\gamma > 0$ is a given H_∞ performance index, and $\bar{\sigma}(\cdot)$ denotes the largest singular value.

(ii) For any given bounded initial states $x_i(0) \in \mathbb{R}^n$, there exists a nonnegative constant ε such that

$$\lim_{t\to\infty} \|x_i(t) - x_N(t) - h_i(t)\| \leq \varepsilon, \forall i \in F, \tag{9}$$

where ε is called the robust H_∞ time-varying formation tracking error bound.

The current paper are mainly focusing on the following two problems for the multi-agent system described by (2) and (3) under protocol (5): (i) what conditions should be satisfied to achieve the robust H_∞ time-varying formation tracking, and (ii) how to design the protocol (5) to achieve the robust H_∞ time-varying formation tracking.

III. Main results

In this section, an algorithm is firstly put forward to give the procedures to design the protocol (5) to achieve the robust H_∞ time-varying formation tracking. Then a theorem is presented to prove the stability of the algorithm.

Assumption 2: For each follower, there exists at least one leader that has a directed path to it.

With Assumption 2, one can obtain the following lemmas, which are useful in analyzing the robust H_∞ time-varying formation tacking issues.

Lemma 3: ([9]) For the directed interaction topology G, all eigenvalues λ_i of L_1 have positive real parts, which can be written as $\lambda(L_1) = \{\lambda_1, \lambda_2, \cdots, \lambda_{N-1}\}$, $\mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) \leq \cdots \leq \mathrm{Re}(\lambda_{N-1})$; each entry of $-L_1^{-1}L_2$ is nonnegative and each row of $-L_1^{-1}L_2$ has a sum equal to one, that is $-L_1^{-1}L_2 = \mathbf{1}_{N-1}$.

Lemma 4: ([9]) Let L_1 be the Laplacian matrix of a directed interaction topology between the followers with a spanning tree, the root of which is the leader N. There exists a positive definite matrix Θ such that

$$L_1^T\Theta + \Theta L_1 \geq 2\delta_0\Theta, \tag{10}$$

where δ_0 is a positive scalar and $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_{N-1}\} = L_1^{-1}\mathbf{1}_{N-1}$.

Based on Assumption 2, and Lemmas 3 and 4, in the following, a design procedure with three steps is given to determine the control parameters of protocol (5).

Algorithm 1: To achieve the robust H_∞ time-varying formation tacking for multi-agent system (2) and (3), protocol (5) can be designed in the following procedure.

Step 1: Choose proper gain matrix K_1 design the motion modes of the leader and the followers. For the predefined $h_i(t)$, check the following time-varying formation tracking feasibility condition

$$\Big((A - BK_1)h_i(t) - \dot{h}_i(t)\Big) \in L_2[0, \infty). \tag{11}$$

If condition (11) holds, then continue; else, if one cannot find a proper K_1 to make (11) hold, the robust H_∞ time-varying formation tracking cannot be achieved for multi-agent system (2) and (3) under protocol (5).

Step 2: Estimate the maximum bound of the control input of the leader and the mismatched disturbances, and choose the positive function and choose the positive function $\kappa(t)$ to satisfy $\kappa(t) \geq \|u_N(t)\|_\infty + \|M\Delta(t)\|_\infty$.

Step 3: Choose proper κ_0, δ_1 and δ_2 to make the following LMI has positive solution Q,

$$\begin{bmatrix} \Xi_0 & Q \\ * & -\left(\delta_2 I + c_1 C^T C\right)^{-1} \end{bmatrix} < 0, \tag{12}$$

where $\Xi_0 = Q\bar{A}^T + \bar{A}Q - \delta_0\kappa_0 BB^T + c_0 B_1 B_1^T + \delta_1 I$, $\bar{A} = A - BK_1$, $Q = Q^T > 0$, $c_0 = \gamma^{-2}\lambda_{\max}(\Theta)\lambda_{\max}(L_1 L_1^T)$, $c_1 = \lambda_{\max}(\Theta)$ and γ is a designed H_∞ disturbance attenuation performance index. Then K_2 can be given by $K_2 = \kappa_0 B^T Q^{-1}$.

Theorem 1: Suppose that Assumptions 1 and 2 and the formation tracking feasibility condition (11) are satisfied. Multi-agent system described by (2) and (3) with one leader of unknown control input under the protocol (5) designed in Algorithm 1 achieves the robust H_∞ time-varying formation tracking.

Proof: Let $\zeta(t) = L_1\xi(t)$. It follows from (7) that

$$\begin{aligned}\dot{\zeta}(t) =& \left(I_{N-1} \otimes (A - BK_1) - L_1 \otimes BK_2\right)\zeta(t) \\ &+ (L_1 \otimes B_1)\, w(t) + \mu(t) - \kappa(t)\,(L_1 \otimes B)\,\mathrm{sgn}(s(t)) \\ &-(L_1 \otimes BM)\left(\mathbf{1}_{N-1}\Delta(t)\right) - (L_1 \otimes B)\left(\mathbf{1}_{N-1}u_N(t)\right),\end{aligned} \tag{13}$$

where $\mu_i(t) = (A - BK_1)h_i(t) - \dot{h}_i(t)$ and $\mu(t) = [\mu_1(t), \mu_2(t), \cdots, \mu_{N-1}(t)]^T$.

Choose a Lyapunov function candidate as $V(t) = \zeta^T(t)(\Theta \otimes P)\zeta(t)$, where $P = P^T > 0$ and Θ is defined in Lemma 4. Choose $K_2 = \kappa_0 B^T P$. Taking the derivatives of $V(t)$ gives

$$\begin{aligned}\dot{V}(t) =& \zeta^T(t)\left(\Theta \otimes (\bar{A}^T P + P\bar{A})\right. \\ &\left.-(L^T\Theta + \Theta L_1) \otimes \kappa_0 PBB^T P\right)\zeta(t) \\ &+2\zeta^T(t)\,(\Theta L_1 \otimes PB_1)\, w(t) + 2\zeta^T(t)\,(\Theta \otimes P)\,\mu(t) \\ &-2\kappa(t)\zeta^T(t)\,(\Theta L_1 \otimes PB)\,\mathrm{sgn}(s(t)) \\ &-2\zeta^T(t)\,(\Theta L_1 \otimes PBM)\left(\mathbf{1}_{N-1}\Delta(t)\right) \\ &-2\zeta^T(t)\,(\Theta L_1 \otimes PB)\left(\mathbf{1}_{N-1}u_N(t)\right).\end{aligned} \tag{14}$$

By Lemma 4, one gets

$$\begin{aligned}&-\zeta^T(t)\left((L^T\Theta + \Theta L_1) \otimes \kappa_0 PBB^T P\right)\zeta(t) \\ &\le -\zeta^T(t)\left(\Theta \otimes \kappa_0\delta_0 PBB^T P\right)\zeta(t).\end{aligned} \tag{15}$$

It follows from Lemma 1 that

$$\begin{aligned}&2\zeta^T(t)\,(\Theta L_1 \otimes PB_1)\, w(t) \\ &\le \gamma^{-2}\zeta^T(t)\left(\Theta L_1 L_1^T \Theta \otimes PB_1 B_1^T P\right)\zeta(t) + \gamma^2 w^T(t) w(t) \\ &\le \gamma^{-2}\lambda_{\max}(\Theta)\lambda_{\max}(L_1 L_1^T)\zeta^T(t)\left(\Theta \otimes PB_1 B_1^T P\right)\zeta(t) \\ &+\gamma^2 w^T(t) w(t),\end{aligned} \tag{16}$$

and

$$\begin{aligned}&2\xi^T(t)\,(\Theta \otimes P)\,\mu(t) \\ &\le \xi^T(t)\,(\Theta \otimes \delta_1 PP)\,\xi(t) + \frac{\lambda_{\max}(\Theta)}{\delta_1}\mu^T(t)\mu(t),\end{aligned} \tag{17}$$

where γ is the H_∞ disturbance attenuation performance index, δ_1 is a positive scalar.

Note that

$$\begin{aligned}&-\kappa(t)\zeta^T(t)\,(\Theta L_1 \otimes PB)\,\mathrm{sgn}(s(t)) \\ &= -\kappa(t)\sum_{i=1}^{N-1}\theta_j\zeta_i^T PB\Big(\sum_{j=1}^{N-1} a_{i,j}\left(\mathrm{sgn}\left(\kappa_0 B^T P\zeta_i(t)\right)\right. . \\ &\quad \left.-\mathrm{sgn}\left(\kappa_0 B^T P\zeta_j(t)\right)\right) + a_{i,N}\left(\mathrm{sgn}\left(\kappa_0 B^T P\zeta_i(t)\right)\right)\Big) \\ &\le -\kappa(t)\sum_{i=1}^{N-1}\theta_j a_{i,N}\left\|\zeta_i^T PB\right\|_1.\end{aligned} \tag{18}$$

Besides, one has

$$\begin{aligned}&-\zeta^T(t)\,(\Theta L_1 \otimes PBM)\left(\mathbf{1}_{N-1}\Delta(t)\right) \\ &= [\zeta_1^T(t), \zeta_2^T(t), \cdots, \zeta_{N-1}^T(t)]\begin{bmatrix}\theta_1 a_{1,N} PBM\Delta(t) \\ \theta_1 a_{2,N} PBM\Delta(t) \\ \vdots \\ \theta_1 a_{N-1,N} PBM\Delta(t)\end{bmatrix} \\ &\le \sum_{i=1}^{N-1}\theta_j a_{i,N}\left\|\zeta_i^T PB\right\|_1 \|M\Delta(t)\|_\infty\end{aligned} \tag{19}$$

and

$$\begin{aligned}&-\zeta^T(t)\,(\Theta L_1 \otimes PB)\left(\mathbf{1}_{N-1}u_N(t)\right) \\ &= [\zeta_1^T(t), \zeta_2^T(t), \cdots, \zeta_{N-1}^T(t)]\begin{bmatrix}\theta_1 a_{1,N} PBu_N(t) \\ \theta_1 a_{2,N} PBu_N(t) \\ \vdots \\ \theta_1 a_{N-1,N} PBu_N(t)\end{bmatrix} \\ &\le \sum_{i=1}^{N-1}\theta_j a_{i,N}\left\|\zeta_i^T PB\right\|_1 \|u_N(t)\|_\infty.\end{aligned} \tag{20}$$

Therefore, it follows from (14) to (20) that

$$\begin{aligned}\dot{V}(t) \le& \zeta^T(t)\left(\Theta \otimes (\ \bar{A}^T P + P\bar{A} - \delta_0\kappa_0 PBB^T P\right. \\ &\left.+c_0 PB_1 B_1^T P + \delta_1 PP)\right)\zeta(t) \\ &+\gamma^2 w^T(t) w(t) + \frac{\lambda_{\max}(\Theta)}{\delta_1}\mu^T(t)\mu(t)\end{aligned} \tag{21}$$

Let $Q = P^{-1}$. By Lemma 2, if LMI (12) is satisfied, one can get the following LMI holds:

$$\begin{aligned}&\bar{A}^T P + P\bar{A} - \delta_0\kappa_0 PBB^T P + c_0 PB_1 B_1^T P + \delta_1 PP \\ &+c_1 C^T C + \delta_2 I < 0.\end{aligned} \tag{22}$$

To studied the H_∞ attenuation performance with respect to the disturbance $w(t)$, $\mu(t)$ is not considered here. Define

$$J(T) = \int_0^T \left(\left\|\bar{\zeta}(\tau)\right\|_2 - \gamma^2\|w(\tau)\|_2\right) d\tau. \tag{23}$$

Then one can get:

$$\begin{aligned}J(T) =& \int_0^T \left(\zeta^T(\tau)\left(I_{N-1} \otimes C^T C\right)\zeta(\tau) - \gamma^2\|w(\tau)\|_2\right) d\tau \\ &+\int_0^T \dot{V}(\tau) d\tau - V(T).\end{aligned} \tag{24}$$

Substituting (21) and (22) into (26) gives

$$J(T) \le -\int_0^T \delta_2\zeta^T(\tau)\zeta(\tau) d\tau - V(T) < 0. \tag{25}$$

Let $T \to \infty$, and one gets

$$\int_0^\infty \left\|\bar{\zeta}(\tau)\right\|_2 dt < \gamma^2\int_0^\infty \|w(\tau)\|_2 dt, \forall w(t) \in L_2[0, \infty), \tag{26}$$

which implies that the closed-loop multi-agent system (2) and (3) under protocol (5) has H_∞ disturbance attenuation performance with an index γ.

Moreover, from (23) and (24), it can be obtained that

$$\begin{aligned}\dot{V}(t) \le & -\delta_2 \zeta^T(t)\left(\Theta \otimes I\right)\zeta(t) \\ & +\gamma^2 w^T(t)w(t) + \frac{\lambda_{\max}(\Theta)}{\delta_1}\mu^T(t)\mu(t).\end{aligned} \tag{27}$$

Because $w(t)$, $\mu(t) \in L_2[0,\infty)$, therefore, there exists a positive scalar υ_0, such that

$$\gamma^2 \int_0^t w^T(\tau)w(\tau)d\tau + \frac{\lambda_{\max}(\Theta)}{\delta_1}\int_0^t \mu^T(\tau)\mu(\tau)d\tau \le \upsilon_0. \tag{28}$$

Integrating (27) gives

$$\begin{aligned}V(t) \le & V(0)+\gamma^2 \int_0^t w^T(\tau)w(\tau)d\tau+\frac{\lambda_{\max}(\Theta)}{\delta_1}\int_0^t \mu^T(\tau)\mu(\tau)d\tau \\ \le & V(0)+\upsilon_0.\end{aligned} \tag{29}$$

Denote $\varepsilon = \sqrt{(V(0)+\upsilon_0)/\lambda_{\min}\{\Theta\otimes P\}\bar{\sigma}(L_1)}$ and it can be obtained that

$$\begin{aligned}& \|x_i(t)-x_N(t)-h_i(t)\| \\ & = \|\xi_i(t)\| \le \|\xi(t)\| \le \frac{\|\zeta(t)\|}{\sqrt{\bar{\sigma}(L_1)}} \le \frac{V(t)}{\sqrt{\lambda_{\min}\{\Theta\otimes P\}\bar{\sigma}(L_1)}} \le \varepsilon.\end{aligned} \tag{30}$$

By the Definition 2, one can get that the robust H_∞ time-varying formation tracking is achieved. This completes the proof of Theorem 1. ■

IV. NUMERICAL SIMULATIONS

In this section, a simulation example is provide to illustrate the effectiveness of the proposed robust H_∞ time-varying formation tracking protocol in the previous sections.

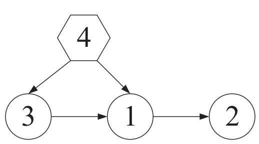


Fig. 1: Directed interaction topology G.

Consider a third-order multi-agent system with four agents with interaction topology G (see Fig. 1) with $n=3$, where three of them are followers and the other one is leader, which means $N=4$. Let $x_i(t)=[x_{i,1}(t),x_{i,2}(t),x_{i,3}(t)]^T$ be the states of agent i ($i \in \{1,2,\cdots,4\}$). The system matrices are described as $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, $B=[0,0,1]^T$, $B_1=[0,0.5,0]^T$ and $C=[1,0,0]$. Choose $K_1=[0,1,0]$, and to satisfy the condition (11), one can design the time-varying formation vector as $h_i(t)$, where $i \in F$ and

$$h_i(t)=\begin{bmatrix}\sin(t+2\pi(i-1)/3)\\ \cos(t+2\pi(i-1)/3)\\ -\sin(t+2\pi(i-1)/3)\end{bmatrix}.$$

For simplicity, the external disturbance are chosen as $\Delta(t)=[\Delta_1(t),\Delta_2(t),\Delta_3(t)]^T = [0.2\sin(t),0.2\sin(t),0.2\sin(t)]^T$ and $w(t)=[w_1(t),w_2(t),w_3(t)]^T=[0.2\omega(t),0.5\omega(t),0.8\omega(t)]^T$, where $\omega(t)$ is the energy-limited white noise which works within the initial two seconds. The unknown control input is designed as $u_4(t)=-x_{4,1}(t)-x_{4,2}(t)-x_{4,3}(t)+0.5t$. Note that the leader is a stable system under $u_4(t)$ and one can get $\|u_4(t)\| \le 0.6$ from simulation results. Therefore, $\kappa(t)$ can be determined as $\kappa(t)=0.8$. Choose $\delta_0=0.4$, $\delta_1=\delta_2=0.01$, $\gamma=5$, $c_0=0.8583$, $c_1=2$, $\kappa_0=100$. Solving (12) gives $P=\begin{bmatrix}4.6801&2.5026&0.7280\\2.5026&1.6516&0.5225\\0.7280&0.5225&0.2222\end{bmatrix}$ and $K_2=[72.7964,52.2504,22.2174]$.

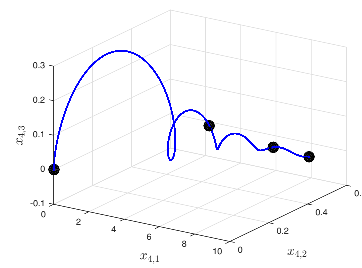
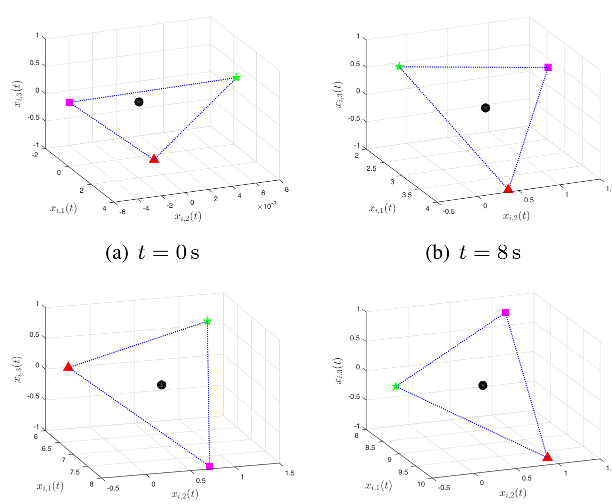


Fig. 2: Curve of the trajectory of the leader.

Fig. 2 shows the trajectory of the leader, where the states are remarked by circles at $t=0s$, $t=8s$, $t=16s$ and $t=20s$, respectively.



(a) $t=0\,$s

(b) $t=8\,$s

(c) $t=16\,$s

(d) $t=20\,$s

Fig. 3: States snapshots of the followers and the leader.

Fig. 3 shows the states snapshots of the three followers and the leader at different time, respectively, where the states of the followers are marked by pink square, green five pointed star and red triangle, respectively. The curve of the robust H_∞ time-varying formation tracking error $\xi_F(t)$ is shown in Fig. 4. Fig 5 displays the H_∞ mismatched disturbance attenuation performance. From Fig. 2 to Fig. 5, one sees that

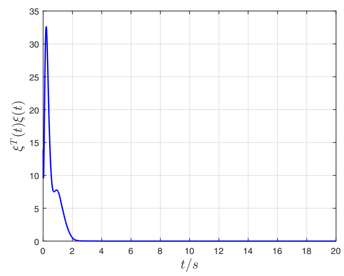


Fig. 4: Curve of the robust H_∞ time-varying formation tracking error $\xi(t)$.

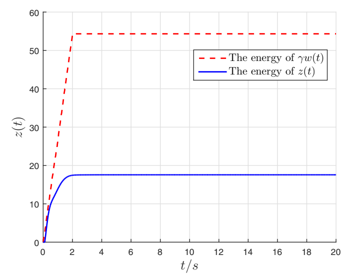


Fig. 5: Curve of H_∞ disturbance attenuation performance.

the state of the leader lies in the center of the triangle, and the moving direction of the leader is time-varying. Besides, from Figs. 4 and 5, one obtains that conditions (8) and (9) hold. Therefore, the desired robust H_∞ time-varying formation tracking is achieved.

V. Conclusions

Robust H_∞ time-varying formation tracking problems for high-order multi-agent systems were studied. Matched and mismatched disturbances were considered simultaneously. A distributed formation tracking protocol was proposed by using local neighboring information. An approach to determine the parameters in the protocol was presented. The formation tracking feasibility constraint was derived. Stability of the closed-loop multi-agent system under the protocol was proved.

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