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On the Notion of Moment at a Pole of a Nonlinear System

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***Abstract*— The notion of moment at a pole of a single-input, single-output, continuous-time, nonlinear, time-invariant system is studied. It is argued that the existing notion of moment at pole can be extended to wider classes of systems. The moment at a pole of a class of systems in feedback form is shown to be uniquely determined by the steady-state impulse response of the system under weaker assumptions than those of previous works. With a class of asymptotically autonomous systems as a guiding example, it is shown that general invariant manifolds (which do not necessarily pass through the origin) can be also used to characterise moments at a pole. Finally, a dual notion of moment at a pole is introduced and shown to be "natural" for systems in feedforward form.**

I. Introduction

Model reduction methods have received considerable attention in recent years, primarily as a result of ever-demanding requirements for simulation tools in analysis and design. The aim of model reduction methods is to construct simplified models of a given system while retaining prescribed features of the original system [1]. This is instrumental, for example, to simulate systems with a large number of state variables or to cope with time and storage constraints. The need of accurate, inexpensive mathematical models frequently leads to a fundamental modelling trade-off: the choice between the simplicity and versatility of linear, time-invariant models [2 – 4] and the complexity and richness of nonlinear models [5 – 8]. While model reduction methods for linear, time-invariant systems are nowadays a classical topic in control theory [1], their nonlinear counterpart is far less understood and deserves further study.

A model reduction method which has been extended to nonlinear systems over the past decades is known in the literature as model reduction by moment matching [1]. Starting from the seminal contribution [9], the concept of moment matching has been re-visited and extended in a series of works, see, *e.g.*, [10 – 12]. The key observation is that moment matching can be interpreted as an equivalence condition on the steady-state response of the original system and of the reduced order model. Recently, the domain of definition of moments of a linear, time-invariant system has been extended to poles of the transfer function [13 , 14], where moments are classically not defined [1]. This, in turn, has led to nonlinear enhancements of the notions of eigenvalue, of pole and of moment at a pole [15], which have been shown to be intimately connected to the steady-state *impulse response* of the underlying system. The significance of these results has been subsequently demonstrated in the solution of the model reduction problem at poles both for linear and nonlinear systems [16], a problem which in the linear case is closely related to finding stable projections of a given transfer matrix [17]. An important consequence of these works is that the classical model reduction method by modal approximation [1] can be regarded as moment matching method in which the moment matching condition is imposed at poles [16].

The main contributions of the paper are threefold. First, it is shown that the steady-state impulse response of a nonlinear system can be characterised in terms of the moment at a pole of the system under more general circumstances than those outlined in [16]. Second, it is argued that general invariant manifolds, which do not necessarily pass through the origin, can be used to characterise the moment at a pole of a system, thus extending the notion of moment first proposed in [15]. To support this claim, moments of a class of asymptotically autonomous systems are studied. Third, an alternative notion of moment at a pole, "dual" with respect to that of [15], is introduced and shown to be "natural" for systems in feedforward form. With these results the authors intend to suggest that the existing notion of moment at pole can be extended to wider classes of systems.

The remainder of this work is organised as follows. Section II provides basic definitions, preliminary results and the problem formulation. Section III contains the main results of the paper, where the moments at a pole are studied for systems in feedback form, for a class of asymptotically autonomous systems and for systems in feedforward form. Section IV concludes the paper with a summary of the results and an outlook for future research directions.

Notation: $\mathbb{N}$ denotes the set of non-negative integer numbers. $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and the set of n-dimensional vectors with real entries, respectively. $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote the set of positive real numbers and the set of non-negative real numbers, respectively. $\operatorname{Re}\lambda$ denotes the real part of the complex number λ. $\operatorname{col}(x_1, x_2)$ denotes the (n_1+n_2)-dimensional vector obtained by stacking the vectors $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ one above the other. $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$. δ_0 denotes the Dirac δ-distribution. $L_f h$ denotes the Lie derivative of the mapping h along the vector field f [6, p. 8].

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II. Problem formulation

Consider a continuous-time, single-input, single-output, nonlinear, time-invariant system described by the equations[1]

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1}$$

in which $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and the mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ are such that $f(0) = 0$ and $h(0) = 0$, respectively. Throughout the paper all mappings are assumed to be smooth, *i.e.* infinitely many times differentiable. We also assume that system (1) is minimal, *i.e.* strongly observable and strongly accessible [5, Chapter 2].

Definition 1. [15] Consider system (1). The vector field $f_2 : \mathbb{R}^m \to \mathbb{R}^m$ is said to be a pole of system (1) if there exist (local) coordinates $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ in which system (1) reads as

$$\dot{x}_1 = f_1(x_1, x_2), \ \dot{x}_2 = f_2(x_2) + g_2(x_2)u, \ y = h_1(x_1), \tag{2}$$

and a unique mapping $\pi : \mathbb{R}^m \to \mathbb{R}^{n-m}$, locally defined in a neighbourhood of the origin and such that $\pi(0) = 0$, which solves the partial differential equation

$$f_1(\pi(x_2), x_2) = \frac{\partial \pi}{\partial x_2} f_2(x_2) \tag{3}$$

System (1) is said to have a pole at zero if the vector field f_2 is identically zero.

Remark 1. Definition 1 establishes that a pole completely specifies the dynamics of a system along a given subset of the state space, in analogy with the case of linear systems [14]. Note that different, non-equivalent notions of pole of a nonlinear system exist in the literature [19 – 21]. The advantage of the notion of pole given in Definition 1 is that it extends to systems which do not admit a representation in terms of transfer function. Moreover, it allows to pose and solve the model reduction problem at poles both for linear and nonlinear systems within a unified framework [16]. △

Remark 2. Definition 1 requires system (1) to have an inherent cascade decomposition and the existence of a (local) invariant manifold with specific properties. Necessary and sufficient conditions for system (1) to possess such a structure can be given using tools from geometric control theory [16]. △

Definition 2. [15] Consider system (1). Suppose (2) holds with respect to the (local) coordinates $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ and let $f_2 : \mathbb{R}^m \to \mathbb{R}^m$ be a pole of system (1). The moment of system (1) at f_2 is defined as the function $h_1(\pi(\cdot))$, with π the unique solution of (3). The moment of system (1) at f_2 is said to be the moment of system (1) at zero if the vector field f_2 is identically zero.

Remark 3. Definition 2 is motivated by the analysis provided in [16], where the notion of moment at a pole has been defined for linear systems and subsequently extended to nonlinear systems. The reader is referred to [16] for further detail. △

[1] The choice of focusing on input-affine systems is made to ensure that the impulse response of the system, and its derivatives, are well-defined (see [9, 18] for more detail).

The following result states that the moment of a system at a pole uniquely determines the steady-state *impulse response*[2] of the output of the system, provided that certain assumptions hold.

Theorem 1. [16] Consider system (1). Suppose (2) holds with respect to the (local) coordinates $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ and let $f_2 : \mathbb{R}^m \to \mathbb{R}^m$ be a pole of system (1). Assume that the equilibrium at the origin of the system $\dot{x}_1 = f_1(x_1, 0)$ is locally exponentially stable, that the system $\dot{x}_2 = f_2(x_2)$ is Poisson stable and that the pair $(f_2, g_2(0))$ is exciting[3]. Then the (well-defined) moment of system (1) at f_2 uniquely determines the steady-state impulse response of the output of system (1).

The goal of the paper is to show that the existing notion of moment at a pole can be extended to wider classes of systems. With this goal in mind, we provide evidence for the following assertions.

(A1) Theorem 1 provides only sufficient conditions for the moment of a system at a pole to uniquely specify the steady-state impulse response of the system.
(A2) General invariant manifolds can be used to extend the notion of moment at a pole.
(A3) A “dual” notion of moment at pole is more suited to certain classes of nonlinear systems.

III. Main results

This section contains the main results of the paper. To support the assertions in Section II, we characterise the moments of three classes of systems: a class of systems in feedback form, a class of asymptotically autonomous systems and systems in feedforward form.

A. Moments of a class of systems in feedback form

Consider a system described by the equations

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = \mu_0 u, \quad y = h_1(x_1), \tag{4}$$

in which $x(t) = \mathrm{col}(x_1(t), x_2(t)) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $f_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $h_1 : \mathbb{R}^{n-1} \to \mathbb{R}$ are such that $f_1(0, 0) = 0$, $h_1(0) = 0$ and $\mu_0 \in \mathbb{R}_+$ is a given constant such that $f_1(0, \mu_0) = 0$.

The purpose of this section is to show that Theorem 1 provides only sufficient conditions for the moment of a system at a pole to uniquely determine the steady-state impulse response of the system. To this end, we show that it is possible to characterise the moment of system (4) at zero in terms of the steady-state impulse response of its output using the theory of (local) bifurcations [24], even when the assumptions of Theorem 1 are violated. Note, preliminarily, that the solution of the system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = 0,$$

[2] The notion of steady-state response is borrowed from [22]. By steady-state impulse response we mean the steady-state response corresponding to the input $u = \delta_0$.

[3] The pair (f, x_0) is said to be *exciting* if $\dim E(x_0) = n$, in which $E(x) = \mathrm{span}\,\{\theta_k(x),\ k \in \mathbb{N}\}$, with $\theta_{k+1}(x) = \frac{\partial \theta_k}{\partial x} f(x)$ for $k \in \mathbb{Z}_+$ and θ_0 the identity map on $\mathbb{R}^n$. The reader is referred to [23] for further detail.

with $x(0) = \mathrm{col}(x_1(0), \mu_0)$ coincides with the solution of system (4), with $u = \delta_0$ and $x_2(0) = 0$, since

$$x_1(t) = x_1(0) + \int_0^t f_1(x_1(\zeta), \mu_0) d\zeta, \quad x_2(t) = \mu_0.$$

The following result is also useful for the development of our analysis.

Lemma 1. System (4) has a pole at zero if and only if there exists a unique mapping $\pi : \mathbb{R} \to \mathbb{R}^{n-1}$, locally defined in a neighbourhood of the origin and such that $\pi(0) = 0$, which solves the equation $f_1(\pi(x_2), x_2) = 0$.

Proof. The claim is a direct consequence of the definition of pole at zero, since system (4) has an obvious cascade decomposition. □

The following statement characterises the case in which the equilibrium point at the origin of the system

$$\dot{x}_1 = f_1(x_1, \mu_0) \tag{5}$$

is hyperbolic, *i.e.* the case in which the matrix

$$F_1 = \left. \frac{\partial f_1}{\partial x_1} \right|_{x=0} \tag{6}$$

has no eigenvalues on the imaginary axis.

Theorem 2. Consider system (4) and system (5). Assume the equilibrium at the origin of system (5) is hyperbolic. Then system (4) has a pole at zero.

Proof. By hypothesis, the origin is an equilibrium point of system (5) and the matrix F_1 is invertible. By the implicit function theorem this implies that there exists a unique mapping $\pi : \mathbb{R} \to \mathbb{R}^{n-1}$, locally defined in a neighbourhood of the origin and such that $\pi(0) = 0$, which solves the equation $f_1(\pi(x_2), x_2) = 0$. By Lemma 1, system (4) has therefore a pole at zero. □

Remark 4. The hyperbolic equilibrium at the origin of system (5) persists for every μ sufficiently close to μ_0. This implies that the moment of system (4) at zero is well-defined for every μ sufficiently close to μ_0. However, in this case it is not necessarily possible to relate the moment of system (4) at zero with the steady-state impulse response of the output of the system. △

We now focus on the more interesting situation in which the equilibrium point at the origin of system (5) is *not* hyperbolic. We show that, under certain assumptions, the steady-state impulse response of the output of system (4) can be characterised in terms of the moment of the system at zero, despite the fact that the system $\dot{x}_1 = f_1(x_1, 0)$ is not locally exponentially stable (as required by Theorem 1). To this end, note that if the matrix (6) has two purely imaginary eigenvalues and no other eigenvalues with zero real part, then by the center manifold theorem [25] there exists a mapping $\pi : \mathbb{R} \to \mathbb{R}^{n-1}$, locally defined in a neighbourhood of the origin, such that $\pi(0) = 0$ and $f_1(\pi(x_2), x_2) = 0$. Moreover, the third order Taylor series approximation of the restriction to the center manifold of system (4) is governed by equations of the form [24, p. 151]

$$\dot{\xi}_1 = \psi_1(\xi)\xi_1 - \psi_2(\xi)\xi_2, \tag{7a}$$

$$\dot{\xi}_2 = \psi_2(\xi)\xi_1 + \psi_1(\xi)\xi_2, \tag{7b}$$

in which

$$\psi_1(\xi) = d\mu + a\left(\xi_1^2 + \xi_2^2\right)$$

and

$$\psi_2(\xi) = \omega + c\mu + b\left(\xi_1^2 + \xi_2^2\right),$$

with $\xi = \mathrm{col}(\xi_1, \xi_2) \in \mathbb{R}^2$ and $a, b, c, d \in \mathbb{R}$. Note also that in this case the eigenvalues $\lambda(\mu)$ and $\bar{\lambda}(\mu)$ of the matrix

$$F_1(\mu) = \left. \frac{\partial f_1}{\partial x_1} \right|_{x_1 = \pi(\mu)} \tag{8}$$

which are imaginary at $\mu = \mu_0$ vary smoothly with μ [24, p. 151]. We are now ready to establish the following result.

Theorem 3. Consider system (4), system (5) and system (7). Assume $a \in \mathbb{R}_-$ and $d \in \mathbb{R}_-$. Moreover, assume that the condition

$$\left. \frac{d \,\mathrm{Re}\, \lambda}{d\mu} \right|_{\mu = \mu_0} \neq 0, \tag{9}$$

holds, in which λ is an eigenvalue of the matrix (8) which is imaginary at $\mu = \mu_0$. Then system (4) has a pole at zero. Moreover, the (well-defined) moment of system (4) at zero uniquely determines the steady-state impulse response of the output of system (4).

Proof. Under the stated assumptions the Hopf bifurcation theorem [24, p.151] implies the existence of a unique center manifold for system (5) passing through the origin. This, in turn, implies that there exists a unique mapping $\pi : \mathbb{R} \to \mathbb{R}^{n-1}$, locally defined in a neighbourhood of the origin and such that $\pi(0) = 0$ and $f_1(\pi(x_2), x_2) = 0$. By Lemma 1, this implies that system (4) has a pole at zero. Moreover, the moment of system (4) at zero is well-defined since π is unique. The Hopf bifurcation theorem also implies that the origin is unstable and surrounded by an attractive periodic orbit γ [24, p.151], since by hypothesis $a \in \mathbb{R}_-$, $d \in \mathbb{R}_-$ and condition (9) holds. Hence, every non-zero initial condition in a neighbourhood of the origin generates a solution of system (5) which converges to the periodic orbit γ as $t \to \infty$. Thus the steady-state impulse response of system (4) is well-defined and the output of system (4) can be written as

$$y(t) = h_1(\pi(x_2(t))) + \varepsilon(t),$$

in which ε is an exponentially decaying function. Thus, the steady-state impulse response of the output of system (4) is

$$y_{ss}(t) = h_1(\pi(x_2(t))).$$

By Definition 2, this implies that the moment of system (4) at zero uniquely specifies the steady-state impulse response of the output of system (4) and, hence, the claim is proved. □

Theorem 3 states that the moment of system (4) at zero uniquely determines the steady-state impulse response of the output of system (4) even if the assumptions of Theorem 2 are violated. This confirms that the assumptions of Theorem 2 are only sufficient for this to happen.

B. Moments of a class of asymptotically autonomous systems

We now argue that moments at a pole can be defined exploiting general invariant manifolds. We illustrate this idea by studying a class of asymptotically autonomous systems and by removing the requirement in Definition 2 that the invariant manifold which defines the moment of the system has to pass through the origin.

Theorem 4. Consider system (2). Let $x_1(t) \in \mathbb{R}^{n-1}$, $x_2(t) \in \mathbb{R}_+$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$. Assume there exists a vector field $\varphi_1 : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that the following hold.

(i) The integral curves of the vector field f_1 are bounded and $f_1(x_1, x_2) \to \varphi_1(x_1)$, as $\|x_2\| \to \infty$, uniformly on compact subsets of $\mathbb{R}^{n-1}$.

(ii) The vector field f_2 is such that $f_2'(0)$ is not an eigenvalue of the matrix
$$F_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_{x=0}.$$

(iii) The system
$$\dot{x}_1 = \varphi_1(x_1) \tag{10}$$
has an unstable equilibrium at the origin and a unique globally attractive periodic orbit.

Then the vector field f_2 is a pole of system (2) and the (well-defined) moment of system (2) at f_2 uniquely determines the steady-state impulse response of the output of system (2).

Proof. We first show that f_2 is a pole of system (2). Note that the system admits an obvious cascade decomposition and, thus, we only need to show that there exists a unique mapping $\pi : \mathbb{R} \to \mathbb{R}^{n-1}$ which solves the partial differential equation (3). However, this is implied by assumption (ii) and hence f_2 is a pole of system (2). Note also that the moment of system (2) is well-defined, since the mapping π is unique.

We now show that the moment of system (2) at f_2 uniquely determines the steady-state impulse response of the output of system (2). By assumption (i), system (2) is asymptotically autonomous with limit equation (10) [26]. Moreover, the integral curves of the vector field f_1 are bounded [27], and every solution of system (2) approaches its ω-limit set, which is non-empty, compact and connected. Thus, by assumption (iii), the ω-limit set of system (2) coincides with the unique globally attractive periodic orbit of system (10), since every point of the ω-limit set of system (2) lies on a solution that is contained in the ω-limit set of system (10) [27]. This implies that the steady-state impulse response of system (2) is well-defined and the output of system (2) can be written as
$$y(t) = h_1(\pi(x_2(t))) + \varepsilon(t),$$
in which ε is an exponentially decaying function. Thus, the steady-state impulse response of the output of system (2) is
$$y_{ss}(t) = h_1(\pi(x_2(t))).$$
By Definition 2 and assumption (ii), this implies that the moment of system (4) at zero uniquely specifies the steady-state impulse response of the output of system (4) and, hence, the claim is proved. □

We now illustrate Theorem 4 with an academic example.

Example 1. Consider the system
$$\dot{x}_{11} = \left(1 - \sqrt{x_{11}^2 + x_{12}^2}\right) x_{11} - \Big(\Omega + \psi(x_2)\Big) x_{12}, \tag{11a}$$
$$\dot{x}_{12} = \left(1 - \sqrt{x_{11}^2 + x_{12}^2}\right) x_{12} + \Big(\Omega + \psi(x_2)\Big) x_{11}, \tag{11b}$$
$$\dot{x}_2 = \alpha x_2 + u, \tag{11c}$$
$$y = \sqrt{x_{11}^2 + x_{12}^2} + \arctan\left(\frac{x_{12}}{x_{11}}\right), \tag{11d}$$
with $x(t) = \mathrm{col}(x_{11}(t), x_{12}(t), x_2(t)) \in \mathbb{R}^2 \times \mathbb{R}_+$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\Omega \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+ \setminus \{1, \Omega\}$, $x_{11}^2(0) + x_{12}^2(0) \neq 0$, and $\psi : \mathbb{R}_+ \to \mathbb{R}$ a strictly decreasing function such that
$$\lim_{x_2 \to \infty} \psi(x_2) = 0.$$
The purpose of this example is to show that the vector field $f_2 : \mathbb{R} \to \mathbb{R}$, defined as $f_2(x_2) = \alpha x_2$ for every $x_2 \in \mathbb{R}$, is a pole of system (11) and that the (well-defined) moment of system (11) at f_2 uniquely determines the steady-state impulse response of the output of system (11). To this end, we exploit Theorem 4 to obtain the desired conclusion.

We observe that the dynamics of system (11) is best understood in the cylindrical coordinates $x_{11} = \rho \cos \vartheta$ and $x_{12} = \rho \sin \vartheta$. Differentiating with respect to time one obtains
$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\rho\sin\vartheta \\ \sin\vartheta & \rho\cos\vartheta \end{bmatrix} \begin{bmatrix} \dot{\rho} \\ \dot{\vartheta} \end{bmatrix}.$$
Inverting the above relation, system (11) can be rewritten as
$$\dot{\rho} = \rho(1 - \rho), \tag{12a}$$
$$\dot{\vartheta} = \Omega + \psi(x_2), \tag{12b}$$
$$\dot{x}_2 = \alpha x_2 + u, \tag{12c}$$
$$y = \rho + \vartheta. \tag{12d}$$

We are now in the position to establish the following result.

Proposition 1. The vector field f_2 is a pole of system (11) and the (well-defined) moment of system (12) at f_2 uniquely determines the steady-state impulse response of the output of system (12).

Proof. To prove the claim we show that assumptions (i)-(iii) of Theorem 4 hold. To this end, define the vector fields
$$\varphi_1(\rho, \vartheta) = \begin{bmatrix} \rho(1-\rho) \\ \Omega \end{bmatrix}, \; f_1(\rho, \vartheta, x_3) = \begin{bmatrix} \rho(1-\rho) \\ \Omega + \psi(x_2) \end{bmatrix},$$
for every $(\rho, \vartheta, x_2) \in \overline{\mathbb{R}}_+ \times \mathbb{R} \times \mathbb{R}_+$.

(i) The integral curves of the vector field f_1 are clearly bounded. Thus, to prove that (i) holds we only need to show that $f_1(\rho, \vartheta, x_2) \to \varphi_1(\rho, \vartheta)$, as $x_2 \to \infty$, uniformly on compact subsets of $\overline{\mathbb{R}}_+ \times \mathbb{R}$. In other words, we need to show that there exists a constant $\delta \in \mathbb{R}_+$ such that for every

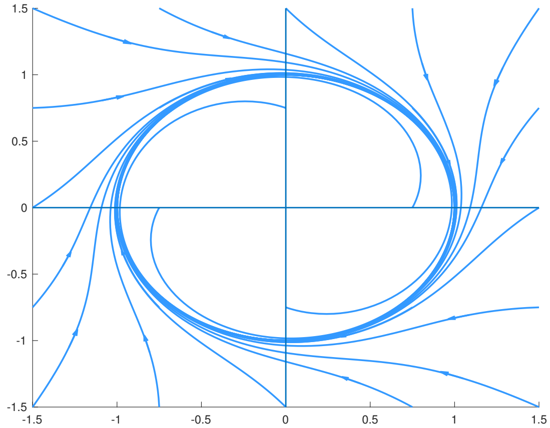

Fig. 1. The projection on the (x_{11}, x_{12}) plane of the phase portrait of system (11) and the circumference of unitary radius (solid), with $\Omega = 1$, $\alpha = 2$ and $\psi : \mathbb{R}_+ \to \mathbb{R}$ defined as $\psi(x_2) = x_2^{-2}$ for every $x_2 \in \mathbb{R}$.

$x_2 > \delta$ the inequality $\|f_1(\rho, \vartheta, x_2) - \varphi_1(\rho, \vartheta)\| < \varepsilon$ holds for every $(\rho, \vartheta) \in \mathcal{K}$, where $\varepsilon \in \mathbb{R}_+$ is a given constant and $\mathcal{K}$ is a given compact subset of $\overline{\mathbb{R}}_+ \times \mathbb{R}$. To this end, recall that ψ is, by assumption, a strictly decreasing function such that

$$\lim_{x_2 \to \infty} \psi(x_2) = 0.$$

This implies that there exists $\bar{x}_2 \in \mathbb{R}_+$ such that for every $x_2 > \bar{x}_2$ the inequality $|\psi(x_2)| < \varepsilon$ holds. Thus, selecting $\delta = \bar{x}_2$ we see that $\|f_1(\rho, \vartheta, x_2) - \varphi_1(\rho, \vartheta)\| = |\psi(x_2)| < \varepsilon$, as desired.

(ii) This assumption is verified by direct computation. The hypothesis $\alpha \notin \{1, \Omega\}$ implies $\alpha^2 - (\Omega + 1)\alpha + \Omega \neq 0$. This, in turn, can be equivalently expressed as

$$\det(\alpha I - F_1) = \det \begin{bmatrix} \alpha - 1 & -\Omega \\ -1 & \alpha - \Omega \end{bmatrix} \neq 0.$$

This implies that α is not an eigenvalue of F_1.

(iii) Consider the planar system

$$\dot{\rho} = \rho(1 - \rho), \quad \dot{\vartheta} = \Omega, \tag{13}$$

and note that there exist only two invariant sets: the equilibrium point at the origin and the circumference of unitary radius. Note also that $\dot{\rho} = \rho(1 - \rho)$ implies that if $\rho(0) > 1$ then $\dot{\rho}(t) < 0$, while if $\rho(0) < 1$ then $\dot{\rho}(t) > 0$. As a result, the origin is unstable and the solution lying on the circumference of unitary radius is the unique globally attractive periodic solution of the system. ▲

Fig. 1 displays the projection on the (x_{11}, x_{12}) plane of the phase portrait of system (11) and the circumference of unitary radius (solid), with $\Omega = 1$, $\alpha = 2$ and $\psi : \mathbb{R}_+ \to \mathbb{R}$, defined as $\psi(x_2) = x_2^{-2}$ for every $x_2 \in \mathbb{R}$. Note that the origin is unstable and that the solution lying on the circumference of unitary radius is a globally attractive periodic orbit of system (13). ▲

C. *Moments of systems in feedforward form*

We now consider an alternative notion of moment at a pole. Motivated by the discussions in [28] and [29], we introduce a notion of moment which can be regarded as "dual" to that of Definition 2 and which is more suited to systems admitting a cascade decomposition which consists of a locally exponentially stable system driving a stable system. For simplicity, our analysis is restricted to systems in feedforward form, though similar considerations can be applied to a wider class of systems.

Consider a system described by the equations

$$\dot{x}_1 = f_1(x_2), \quad \dot{x}_2 = f_2(x_2) + g_2(x_2)u, \quad y = h_1(x_1), \tag{14}$$

in which $x(t) = \text{col}(x_1(t), x_2(t)) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and the mappings $f_1 : \mathbb{R}^m \to \mathbb{R}^{n-m}$, $f_2 : \mathbb{R}^m \to \mathbb{R}^m$, $g_2 : \mathbb{R}^m \to \mathbb{R}^m$, $h_1 : \mathbb{R}^{n-m} \to \mathbb{R}$ are such that $f_1(0) = 0$, $f_2(0) = 0$ and $h_1(0) = 0$. Note that the vector field f_2 is a pole of system (14) as long as the linear approximation of f_2 has no eigenvalues on the imaginary axis, since this implies the existence of a unique solution of the partial differential equation (3). In this case, however, the local asymptotic stability requirement of Theorem 1 is violated. Thus it is *not* possible to characterise the steady-state impulse response of system (14) in terms of the moment of system (14) at the pole f_2. For this reason, we introduce the following "dual" notion of moment at a pole for system (14).

Definition 3. Consider system (14) and let $f_2 : \mathbb{R}^m \to \mathbb{R}^m$ be a pole of system (14). The moment of system (14) at f_2 is defined as $h_1(L_{g_2}\pi(0))$, with π the unique solution of the partial differential equation

$$f_1(x_2) = \frac{\partial \pi}{\partial x_2} f_2(x_2). \tag{15}$$

The moment of system (14) at f_2 is said to be the moment of system (14) at zero if the vector field f_2 is identically zero.

The following result provides conditions under which the moment of system (14) at f_2, in the sense of Definition 3, uniquely determines the steady-state impulse response of the output of system (14).

Theorem 5. Consider system (14). Assume that the equilibrium at the origin of system $\dot{x}_2 = f_2(x_2)$ is locally exponentially stable and globally asymptotically stable. Then the vector field f_2 is a pole of system (14) and the (well-defined) moment of system (14) at f_2, in the sense of Definition 3, uniquely determines the steady-state impulse response of the output of system (14).

Proof. Under the stated assumptions, there exists a unique mapping $\pi : \mathbb{R}^m \to \mathbb{R}^{n-m}$, locally defined in a neighbourhood of the origin and such that $\pi(0) = 0$, which solves the partial differential equation (15). Since system (14) has an obvious cascade decomposition, this implies that the vector field f_2 is a pole of system (14) and that the moment of system (14) at f_2, in the sense of Definition 3, is well-defined. Set

$$z = x_1 - \pi(x_2)$$

and note that z and x_2 qualify as (local) coordinates for system (14). In such coordinates, system (14) is described by the equations

$$\begin{aligned}\dot{z} &= L_{g_2}\pi(x_2)u,\\ \dot{x}_2 &= f_2(x_2) + g_2(x_2)u,\\ y &= h_1(z + \pi(x_2)).\end{aligned}$$

Setting $u = \delta_0$ and recalling that, by assumption, the equilibrium at the origin of the system $\dot{x}_2 = f_2(x_2)$ is globally asymptotically stable, shows that the steady-state impulse response of system (14) is well-defined and the output of system (14) can be written as

$$y(t) = h_1(L_{g_2}\pi(0) + \varepsilon(t)),$$

in which ε is an exponentially decaying function. Thus, the steady-state impulse response of the output of system (14) is

$$y_{ss}(t) = h_1(L_{g_2}\pi(0)).$$

By Definition 3, this implies that the moment of system (14) at zero uniquely specifies the steady-state impulse response of the output of system (14) and, hence, the claim is proved. □

Remark 5. Theorem 5 establishes that steady-state impulse response of a system in feedforward form can be characterised in terms of the moment of the system at a pole in the sense of Definition 3. Note that under the assumptions of Theorem 5 this is *not* the case if the notion of moment at a pole of Definition 2 is used. This demonstrates that different, possibly non-equivalent, notions of moment at a pole should be considered depending on the class of systems under consideration. △

Remark 6. Definition 3 and Theorem 5 can be extended to systems of the form (2), provided all the eigenvalues of the linear approximation of the system $\dot{x}_1 = f_1(x_1, x_2)$ lie on the imaginary axis. △

IV. Conclusion

The notion of moment at a pole of a nonlinear system introduced in [16] has been studied. Alternative notions of moment at a pole have been explored and moments at a pole of systems in feedback form, of a special class of asymptotically autonomous systems and of systems in feedforward form have been characterised in terms of steady-state impulse responses. The results of this paper indicate that different notions of moment at a pole should be used when considering different classes of systems. An interesting direction for future research is the extension of these notions to multi-input, multi-output systems.

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