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Model reduction of linear port-Hamiltonian systems: a structure preserving approach

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Extended abstract

Balanced truncation is a well-known *model reduction* method, originally introduced for linear continuous-time invariant (LCTI) systems in the seminal work of Moore [1], where realization theory, *observability* and *controllability* Gramians, and *Hankel singular values* are the main ingredients. We refer the interested reader to [2], [3] for an extensive insight into balanced truncation for LCTI systems, and to [4] for an extension to the nonlinear case.

In the balancing approach, the reduction of the model is based on the elimination of almost non-controllable and almost non-observable states, that is, once the system is *balanced*, the states corresponding to the small Hankel singular values are truncated. Moreover, the reduced system preserves some properties of the original one, such as *asymptotic stability*, observability and controllability. Nonetheless, the structure of the system, *e.g.*, port-Hamiltonian (PH) structure, is not necessarily preserved after the order reduction. Another appealing feature of balanced truncation is that the error bound is well-known and is directly related to the Hankel singular values [5], [6], [7].

An extension of balanced truncation is provided when the controllability and observability Gramians are replaced by the so-called *generalized* Gramians [8], [9], which are solutions to Lyapunov inequalities instead of the traditional equalities. Hence, since the solution of the aforementioned inequalities is not unique, the generalized Gramians provide some degrees of freedom during the balancing process. Furthermore, the model reduction using this approach may lead to a smaller error bound or structure preservation, see [10]. Moreover, a further

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extension to balancing is given by the so-called *extended* Gramians which are characterized by solutions of specific linear matrix inequalities (LMIs). Indeed, these LMIs have solution if and only if the Lyapunov inequalities related to generalized balancing have solution. Accordingly, the extended balanced truncation method provides additional degrees of freedom to impose a structure to the reduced model. For further details on extended balancing and the related LMIs, we refer the reader to [11], [12], [13].
This note is devoted to the model reduction of a particular class of linear systems, namely, LCTI PH systems[1]. This problem has been previously studied for the nonlinear case in [14]. However, for the linear case, the conditions to preserve the PH structure in the reduced model can be relaxed with the use of extended Gramians to balance the system.

Problem formulation

The representation of a linear PH system is given by

$$\Sigma \begin{cases} \dot{x} &= (J-R)\Lambda x + Bu \\ y &= B^\top \Lambda u \\ H &= \frac{1}{2} x^\top \Lambda x \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, for $m \leq n$, $u, y \in \mathbb{R}^m$ are the input and output vectors, respectively. The function H represents the Hamiltonian, with $\Lambda = \Lambda^\top > 0$. The dissipation matrix is denoted by $R = R^\top \geq 0$, $J = -J^\top$ is the interconnection matrix, and B determines the input port. To simplify notation, in the sequel we define $F : J - R$.
To formulate the problem of structure preservation of linear PH systems, we consider that the state can be split as $x = [x_1^\top \ x_2^\top]^\top$. Where, $x_1 \in \mathbb{R}^k$ is the part of the state to be preserved after the reduction of the model and $x_2 \in \mathbb{R}^{k+1}$ is the part to be truncated. Hence, the dynamics and output of system (1) can be rewritten as follows

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y &= \begin{bmatrix} B_1^\top & B_2^\top \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned} \tag{2}$$

Therefore, the objective is to design an algorithm to balance the system and at the same time ensure that $\Lambda_{12} = 0$. If the latter is achieved, the truncation leads to the reduced order system

$$\Sigma_r \begin{cases} \dot{x} &= F_{11}\Lambda_{11} x_1 + B_1 u \\ y_r &= B_1^\top \Lambda_{11} x_1 \\ H_r &= \frac{1}{2} x_1^\top \Lambda_{11} x_1, \end{cases} \tag{3}$$

which is clearly another linear PH system.

[1] For simplicity, henceforth we refer to these systems as linear PH systems.

Balancing of linear PH systems

Assume that system (1) is asymptotically stable. Then, the observability and controllability Lyapunov inequalities

$$QF\Lambda + \Lambda F^\top Q + \Lambda BB^\top \Lambda \leq 0, \quad F\Lambda P + P\Lambda F^\top + BB^\top \leq 0 \tag{4}$$

have solutions $Q = Q^\top > 0$ and $P = P^\top > 0$. The matrices $Q, P \in \mathbb{R}^{n\times n}$ are the generalized observability and controllability Gramians, respectively. Moreover, since for a balanced system both Gramians are diagonal, a solution to the problem of structure preservation takes place when one of the following identities solves one of the inequalities given in (4)

$$Q = \delta\Lambda, \quad Q^{-1} = \delta\Lambda, \quad P = \delta\Lambda, \quad P^{-1} = \delta\Lambda,$$

where δ is a positive scalar. Nevertheless, some straightforward computations show that the solutions above are equivalent to satisfy the condition

$$2\delta R - BB^\top \geq 0, \quad \text{or} \quad \frac{2}{\delta}R - BB^\top \geq 0. \tag{5}$$

Note that the inequality (5) holds only if $R \in \mathrm{Im}\{B\}$, see [14]. Since, R and B are system parameters, this latter condition can be restrictive. On the other hand, in [13] has been proved that the inequalities (4) hold true if and only if the following LMIs have solution

$$\begin{bmatrix} -QA - A^\top Q - \Lambda BB^\top \Lambda & Q - \alpha S - A^\top S \\ Q - \alpha S^\top - S^\top A & S + S^\top \end{bmatrix} \geq 0 \tag{6}$$

$$\begin{bmatrix} -P^{-1}A - A^\top P^{-1} & -P^{-1} + (A^\top + \beta I)T & -2P^{-1}B \\ -P^{-1} + T^\top (A + \beta I) & T + T^\top & 2T^\top B \\ -2B^\top P^{-1} & 2B^\top T & 4I \end{bmatrix} \geq 0 \tag{7}$$

where $S, T \in \mathbb{R}^{n\times n}$, $A := F\Lambda$, and $\alpha, \beta > 0$ are scalars. In this case, the triad (Q, S, α) is the extended observability Gramian and the triad (P^{-1}, T, β) is the inverse of the extended controllability Gramian. Furthermore, the system is balanced when $S = \mathrm{diag}\{\sigma_1, \cdots, \sigma_n\}$, $T = \mathrm{diag}\{\sigma_1^{-1}, \cdots, \sigma_n^{-1}\}$. Therefore, using the extended Gramians, the reduced model preserves the PH structure if (6) or (7) holds for one of the following expressions

$$S = \delta\Lambda, \quad S^{-1} = \delta\Lambda, \quad T = \delta\Lambda, \quad T^{-1} = \delta\Lambda.$$

Note that, the matrices S, T are *free* design parameters and the condition (5) is not required anymore. As a consequence of the latter, the extended Gramians enlarge the family of linear PH systems for which the reduced model preserves the desired structure. Moreover, for balancing with extended Gramians, the error bound is given by

$$\|\Sigma - \Sigma_r\|_\infty \leq 2\sum_{i=k+1}^{n} \sigma_i.$$

In the final presentation of this work, we will include the corresponding proofs, and further details of the development of the theory, along with an illustrative example.

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