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# Event-triggered Risk-sensitive Maximum a Posterior Estimation for Hidden Markov Models

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*Abstract*— **In this work, we consider an event-triggered risk-sensitive state estimation problem for hidden Markov models. The event-triggered condition considered is general, which involves the current measurement and past information. We use the reference probability measure approach in solving this problem. We derive the linear recursion in the unnormalized information state conditioned on the available information of the estimator, based on which risk-sensitive maximum a posterior estimates can be evaluated. The results are extended to the vector measurement scenario as well via a sequential event-triggered approach.**

## I. Problem Formulation

Firstly, we introduce a hidden Markov model (HMM) on the probability space $(\Omega, \mathcal{F}, P)$. The hidden process considered is a discrete-time homogeneous, first-order Markov chain $X$ belonging to a finite set and the state space of $X$ can be identified with $S_X = \{e_1, e_2, \cdots, e_N\}$, where $e_i$ is the unit vector in $\mathbb{R}^N$ with the $i$th element equal to 1. $\{\mathcal{F}_k^X\}$ is the complete filtration generated by $\sigma\{X_0, \cdots, X_k\}$. Due to the Markov property,

$$P(X_{k+1} = e_i | \mathcal{F}_k^X) = P(X_{k+1} = e_i | X_k).$$

Let $A := (a_{ij}) \in \mathbb{R}^{N\times N}$, $a_{ij} =: P(X_{k+1} = e_i | X_k = e_j)$, such that $\sum_{i=1}^N a_{ij} = 1$. Then

$$E[X_{k+1}|\mathcal{F}_k^X] = E[X_{k+1}|X_k] = AX_k. \tag{1}$$

The sensor measurement process is

$$y_k = cX_k + v_k. \tag{2}$$

For simplicity suppose $y$ is scalar. The case of vector $y$ is discussed in Section II-C. $v_k \in \mathbb{R}$ is white noise with a strictly positive density function $\phi$. $c^T = [c_1, \cdots, c_N]^T \in \mathbb{R}^N$ is the observation vector. Similar to $\{\mathcal{F}_k^X\}$, we have $\{\mathcal{F}_k^y\}$.

Now we introduce the state estimation problem for the risk-sensitive maximum a posterior (MAP) for HMMs [1], [2]. Given $\hat{X}_0, \cdots, \hat{X}_{k-1}$, define $\hat{X}_k \in S_X$ recursively as the risk-sensitive MAP estimate of $X_k$ such that

$$\hat{X}_k = \arg\min_{\zeta\in S_X} E[\exp(\theta\Psi_{0,k}(\zeta))|\mathcal{F}_k^y], \ k \geq 0 \tag{3}$$

where $\theta > 0$ is the risk-sensitive parameter and

$$\Psi_{0,k}(\zeta) = \hat{\Psi}_{0,k-1} + \mu(X_k, \zeta), \ k \geq 0$$

where $\hat{\Psi}_{0,k-1} = \sum_{i=0}^{k-1} \mu(X_i, \hat{X}_i)$ for $k \geq 1$ and $\hat{\Psi}_{0,k-1} = 0$ for $k = 0$. Here

$$\mu(u, v) = \begin{cases} 0, & \text{if } u = v \\ 1, & \text{otherwise.} \end{cases}$$

In this work, we consider the risk-sensitive remote state estimation of finite-state HMMs based on event-triggered measurements, which are sent to the remote estimator decided by an event-triggered process $\gamma_k$ taking values in $\{0, 1\}$. If $\gamma_k = 1$, an event is triggered and $y_k$ is sent by the sensor; otherwise $y_k$ will not be sent. Define

$$\mathcal{I}_k := \{\gamma_0, \cdots, \gamma_k, \gamma_0 y_0, \gamma_k y_k\}.$$

Here we consider a general event-triggered condition such that we only need to assign a probability of idle for the sensor when given $y_k$ and past information, i.e.

$$P(\gamma_k = 0|y_k, \mathcal{I}_k) = \rho(y_k, \mathcal{I}_{k-1}). \tag{4}$$

The objective in this paper is to evaluate the risk-sensitive MAP estimate of the hidden state $\hat{X}_k$ conditioned on the available information set $\mathcal{I}_k$, i.e.,

$$\hat{X}_k = \arg\min_{\zeta\in S_X} E[\exp(\theta\Psi_{0,k}(\zeta))|\mathcal{I}_k], \ k \geq 0 \tag{5}$$

## II. Main Results

In this section, we use the reference probability approach [3] to solve the considered risk-sensitive estimation problem. To do this, we first introduce a new measure and link it with the original measure, based on which the reformulated cost index is obtained and the recursive estimation problem is further solved.

### A. Change of Measure

Now we introduce a new measure $\bar{P}$, under which we still have

$$\begin{aligned} \bar{E}[X_{k+1}|\mathcal{F}_k^X] = \bar{E}[X_{k+1}|X_k] = AX_k \\ \bar{P}(\gamma_k = 0|y_k, \mathcal{I}_k) = \rho(y_k, \mathcal{I}_{k-1}), \end{aligned} \tag{6}$$

but $\{y_k\}$,$k \geq 0$ is a sequence of independent and identically distributed (i.i.d.) random variable with density function $\phi$ satisfying

$$\begin{aligned} \bar{P}(y_k \leq t|\mathcal{F}_k^X \cup \mathcal{F}_{k-1}^y \cup \mathcal{F}_{k-1}^\gamma) &= \bar{P}(y_k \leq t) \\ &= \int_{-\infty}^t \phi(y_k)dy_k. \end{aligned} \tag{7}$$

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Write $\mathcal{G}_k := \mathcal{F}_k^X \cup \mathcal{F}_k^y \cup \mathcal{F}_k^\gamma$. Then the new measure $\bar{P}$ is defined by the restriction of the Radon-Nikodym derivative over $\mathcal{G}_k$:

$$\left.\frac{dP}{d\bar{P}}\right|_{\mathcal{G}_k} = \prod_{l=0}^{l=k} \frac{\phi(y_k - cX_k)}{\phi(y_k)}. \quad (8)$$

Using a version of Bayes' Theorem, we have

$$E[\exp(\theta\Psi_{0,k}(\zeta))|\mathcal{I}_k] = \frac{\bar{E}[\bar{\Lambda}_k \exp(\theta\Psi_{0,k}(\zeta))|\mathcal{I}_k]}{\bar{E}[\bar{\Lambda}_k|\mathcal{I}_k]}. \quad (9)$$

Thus we can modify the problem in (5) as follows:

$$\hat{X}_k = \arg\min_{\zeta\in S_X} \bar{E}[\bar{\Lambda}_k \exp(\theta\Psi_{0,k}(\zeta))|\mathcal{I}_k],\ k \geq 0. \quad (10)$$

*B. Recursive Estimates*

*Definition 1:* Define $\alpha_k = [\alpha_k(e_1), \cdots, \alpha_k(e_N)]^T$ as the unnormalized information state such that for $r \in \mathbb{N}_{1:N}$

$$\alpha_k(e_r) = \bar{E}[\bar{\Lambda}_k \exp(\theta\hat{\Psi}_{0,k-1})\langle X_k, e_r\rangle|\mathcal{I}_k],\ k \geq 0. \quad (11)$$

Notice that $\alpha_k$ is not an unnormalized conditional distribution [3], [4] since it includes not only the actual state of the system but also part of the risk-sensitive cost.

*Theorem 1:* For the HMM (1)-(2) and the event-triggered condition (4), the information state $\alpha_k$ has the following linear recursion:

$$\alpha_k = \mathrm{diag}\{b_k\}A\mathrm{diag}\{d_{k-1}\}\alpha_{k-1} \quad (12)$$

where $d_{k-1} = \left[\exp(\theta\mu(e_i, \hat{X}_{k-1}))\right]_{i\in\mathbb{N}_{1:N}}$ and

$$b_k = \frac{1}{\bar{P}(\gamma_k = 0|\mathcal{I}_{k-1})}\left[\int_{\mathbb{R}} \phi(y_k - c_i)\rho(y_k, \mathcal{I}_{k-1})dy_k\right]_{i\in\mathbb{N}_{1:N}}$$

if $\gamma_k = 0$ and $b_k = \left[\frac{\phi(y_k-c_i)}{\phi(y_k)}\right]_{i\in\mathbb{N}_{1:N}}$ if $\gamma_k = 1$.

*Theorem 2:* The event-triggered risk-sensitive MAP estimation problem (5) is solved by

$$\hat{X}_k = e_{i^*},\ i^* = \arg\max_{i\in\mathbb{N}_{1:N}} \alpha_k(e_i),\ k \in \mathbb{N}. \quad (13)$$

*Example 1:* Consider a general deterministic event-trigged condition [5]

$$\gamma_k = \begin{cases} 0, & \text{if } |y_k - \beta_k| \leq \delta_k \\ 1, & \text{otherwise.} \end{cases} \quad (14)$$

where $\delta_k \geq 0$ is a fixed threshold and $\beta_k$ is a known parameter based on the information set $\mathcal{I}_{k-1}$ to the remote estimator. In this case, if $\gamma_k = 0$,

$$\int_{\mathbb{R}} \phi(y_k - c_i)\rho(y_k, \mathcal{I}_{k-1})dy_k = \int_{\delta_k+\beta_k}^{-\delta_k+\beta_k} \phi(y_k - c_i)dy_k$$

for $i \in \mathbb{N}_{1:N}$. Combining Theorem 1 and 2, one can evaluate the risk-sensitive MAP estimate for event-triggered condition (14).

*Example 2:* Consider a general stochastic event-trigged condition [6]

$$\gamma_k = \begin{cases} 0, & \text{if } \tau_k \leq \exp(-\frac{1}{2}w_k(y_k - \xi_k)^2) \\ 1, & \text{otherwise.} \end{cases} \quad (15)$$

Assume the measurement noise $v(k)$ is Gaussian with zero-mean and covariance $\sigma > 0$ such that $\phi(\cdot)$ is the $N(0, \sqrt{\sigma})$ density. In this case, if $\gamma_k = 0$,

$$\int_{\mathbb{R}} \phi(y_k - c_i)\rho(y_k, \mathcal{I}_{k-1})dy_k = \eta_k \exp\left(-\frac{(c_i - \xi_k)^2}{2(\sigma + w_k{}^{-1})}\right)$$

for $i \in \mathbb{N}_{1:N}$, where $\eta_k$ is unrelated to $c_i$ and $\xi_k$. Again, combining Theorems 1 and 2, one can evaluate the risk-sensitive MAP estimate for event-triggered condition (15).

*C. Extension to the Vector Measurement Scenario*

Suppose the sensor measurement process where $y_k$ is $m$-dimensional with elements

$$\begin{aligned} y_k^1 &= c^1X_k + v_k^1 \\ y_k^2 &= c^2X_k + v_k^2 \\ &\vdots \\ y_k^m &= c^mX_k + v_k^m,\ k \geq 0. \end{aligned} \quad (16)$$

*Remark 1:* If elements of the measurement noise $v(k)$ are correlated, in the case where $v(k)$ is Gaussian we can use a nonsingular linear transformation approach to get the uncorrelated measurement noise.

We use a sequential event-triggered approach to sequentially decide whether $y_k^i$ is to be transmitted to the remote estimator. Likewise, let $\gamma_k^i \in \{0, 1\}$ be the transmission decision variable of $y_k^i$ and define

$$\mathcal{I}_k^i := \{\gamma_0^1, \gamma_0^2 \cdots, \gamma_k^i, \gamma_0^1 y_0^1, \gamma_0^2 y_0^2, \cdots, \gamma_k^i y_k^i\},\ 1 \leq i \leq m.$$

Then transmission probability functions for each element of $y_k$ can be assigned as

$$P^i(\gamma_k^i = 0|y_k^i, \mathcal{I}_k^{i-1}) = \rho^i(y_k^i, \mathcal{I}_k^{i-1}). \quad (17)$$

*Theorem 3:* For the HMM (1) and (16) and the event-triggered condition (17), the information state $\alpha_k$ has the following linear recursion:

$$\alpha_k = \mathrm{diag}\{b_k\}A\mathrm{diag}\{d_{k-1}\}\alpha_{k-1} \quad (18)$$

where $d_{k-1} = \left[exp(\theta\mu(e_i, \hat{X}_{k-1}))\right]_{i\in\mathbb{N}_{1:N}}$ and

$$\begin{aligned} b_k = \Bigg[\prod_{l=1}^{m}\Bigg(&\frac{1-\gamma_k^l}{\bar{P}(\gamma_k^l = 0|\mathcal{I}_k^{l-1})}\int_{\mathbb{R}} \phi^l(y_k^l - c_i^l)\rho^l(y_k^l, \mathcal{I}_k^{l-1})dy_k^l \\ &+ \gamma_k^l \frac{\phi^l(y_k^l - c_i^l)}{\phi^l(y_k^l)}\Bigg)\Bigg]_{i\in\mathbb{N}_{1:N}} \end{aligned} \quad (19)$$

and the convention $\mathcal{I}_k^0 = \mathcal{I}_{k-1}^m$ is assumed.

Likewise, the risk-sensitive MAP estimate $\hat{X}_k$ can be evaluated using Theorem 2.

## III. Conclusions

We have investigated a risk-sensitive state estimation problem for finite-state HMMs based on event-triggered measurements in this work. By utilizing the change of measure approach, the linear recursive unnormalized information state under a general event-triggering condition and the result for risk-sensitive MAP estimates are obtained. The corresponding results for the vector measurement scenario are also achieved. The state variable considered here is discrete (finite values), and an interesting extension is to consider systems with continuous-range states. We speculate that the change of measure approach and the provided frame in this work will be also helpful to study this case.

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