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Feedback Stabilization of Switched Differential Algebraic Systems

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***Abstract*— A stabilization problem for a switched differential-algebraic system is investigated. We propose an approach for solving effectively the stabilization problem for an autonomous linear switched differential-algebraic system based on projector and flow matrices. In switched DAEs, the switches can induce jumps in certain state-variables, and it has been shown that the formulation as a switched DAE already implicitly defines these jumps, no additional jump map must be given. These jumps can be calculated in terms of the consistency projectors. The essence of this method is to design a stabilizing controller for switched differential-algebraic systems, i.e., the continuous dynamics of each subsystem are described by sets of differential-algebraic equations, using an averaging switched DAE model based on consistency projector and flow matrices to guarantee convergence towards an equilibrium point via fast switching**

***Index Terms*— Differential-algebraic equations, Moments, Optimal Control, Switched Systems**

I. Introduction

In most all areas of electrical, mechanical, or chemical engineering the modeling and control of the dynamics of complex systems is nowadays highly modularized, thus allowing the efficient generation of mathematical models for substructures and link them together via constraints. Differential-algebraic equations (DAEs) arise in the modeling of physical systems where the state variables satisfy certain algebraic constraints alongside some differential equations that govern the evolution of these state variables. In complex systems naturally arise the switched control systems, which are characterized by a set of several continuous nonlinear state dynamics with a logic-based controller, which determines simultaneously a sequence of switching times and a sequence of modes [4], [11], [9].

In the last years several researchers have considered the control of switched systems (see, for example [3], and [2], [16], [1], [8]). In switched systems, a switching signal and an external control input may be designed together to guarantee closed-loop asymptotically stabilization [11], [9]. In the particular case of switched linear systems, several effective techniques have been presented and summarized in books such as in [4], [11]. The research field of switched differential-algebraic systems has been growing in the last decade, some work has been done in stability analysis and numerical solutions, see e.g. [5], [6], [12], [13], and recently in [10] it is presented an averaging model for a switched DAEs and it is studied convergence and stability of the proposed model. In switched DAEs, the switches can induce jumps in certain state-variables, and it has been shown that the formulation as a switched DAE already implicitly defines these jumps, no additional jump map must be given. These jumps can be calculated in terms of the consistency projectors. Control algorithm considering switched DAE systems has no been widely studied. In general, the work has been focus on constrained cases, such as switched DAEs with lower indexes (index-1 to index-3 DAE) [6] or stabilization via single switching signal based on a approximation parameter [7]. However, this condition is not satisfied because the averaging approach will result in switching signals depending on this parameter. The main limitation of the stabilizing algorithm proposed in [7] is the destabilizing effect of the consistency projectors, the nonexistence of an averaged model for the switched DAE is not considered [15].

The main contribution of this paper is to design a stabilizing controller for switched differential-algebraic systems, i.e., the continuous dynamics of each subsystem are described by sets of differential-algebraic equations in general, using an averaging switched DAE model based on consistency projector and flow matrices to guarantee convergence towards an equilibrium point via fast switching. The proposed approach overcomes the limitation of the destabilizing effect of the consistency projector using the properly defined averaged model.

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The rest of the paper is organized as follows. Section II presents a brief description of the main concepts of switched DAEs. In Section III, some theoretical results are shown for feedback stabilizing control design. Section IV presents simulation results and finally in Section V some conclusions and future work are drawn.

II. Switched Differential Algebraic Systems

A. Basic Definitions

The switched differential algebraic system adopted in this work is described by sets of differential algebraic equations for each mode of operation and a switching signal that determines which mode is operating at each point of time. Consider a general mathematical model described by switched differential algebraic equations (DAE) of the form

$$E_\sigma \dot{x}(t) = A_\sigma x_1(t) + B_\sigma u(t), \qquad (1)$$

where x is the state, the switching signal $\sigma : [t_0, t_F] \to \mathcal{Q} \in \{1, 2, ..., q\}$ is a *piecewise constant* function of time, with t_0 and t_F as the initial and final times respectively, and (q) is the number of subsystems. Every mode of operation corresponds to a specific subsystem (E_i, A_i), for some $i \in \mathcal{Q}$, and the *switching signal* σ determines which subsystem is followed at each point of time into the interval $[t_0, t_F]$, E_i, $A_i \in \mathbb{R}^n \times \mathbb{R}^n$, $B_i \in \mathbb{R}^n \times \mathbb{R}^m$ are constant matrices for each subsystem, and $x(0) = x_0$ are fixed initial values. In addition, we consider a non-Zeno behavior, i.e., we exclude an infinite switching accumulation points in time. In order to obtain a solution in the new mode, the initial value obtained by the algorithm have to be consistent with the DAE in the next mode. For each subinterval of $[t_0, t_F]$ we assume that consistent initial values for the currently mode exist, such that a solution exists. Furthermore, we assume that the integration interval have nonzero measure, i.e., an instantaneous multiple mode change is not allowed [6]. Finally, we assume that the state does not have jump discontinuities. Moreover, for the interval $[t_0, t_F]$, the control functions must be chosen such that the initial and final conditions are satisfied.

Definition 1: A control for the switched differential-algebraic system in (1) is a duplet consisting of

(a) a finite sequence of modes, and
(b) a finite sequence of switching times such that $t_0 < t_1 < \cdots < t_F$.

Locally in each mode $i \in \mathcal{Q}$ and for every small subinterval of $[t_0, t_F]$, we consider the classical solution concept. The solution of a switched differential-algebraic system depends on the initial mode, initial conditions, and mode switching control rule. We assume also that all solutions of each mode evolve within a consistency space [14]. An important issue when dealing with switched DAEs is the possible jumps after a switch, which makes necessary to allow solutions with jumps leading to problems in obtaining the solution of (1). To handle this behavior we use the distributional solution space introduced in [13] for switched DAEs.

Assumption 2: The matrix pairs (E_i, A_i) are regular in the sense that each matrix pair are of the same dimension n and the characteristic polynomial $\det(sE_i - A_i) \neq 0$.

In the next subsection, it is introduced the concept of regularity and consistency projector that are fundamental to guarantee existence and uniqueness of solution of the switched DAEs.

B. Regularity and Consistency Projectors

In Assumption 2 is presented the regularity conditions needed for each matrix pair (E_i, A_i). Based on this assumption, the following result presents the characterization of regularity in terms of Quasi-Weierstrass form (it is not assumed that the matrices N_i, and J_i are in Jordan canonical form), which provides a decoupling of each DAE into purely differential and algebraic parts [13]

Proposition 3: If each matrix pairs $(E_i, A_i) \in \mathbb{R}^n \times \mathbb{R}^n$ is regular, then there exist invertible matrices S_i, T_i $\in \mathbb{R}^n \times \mathbb{R}^n$ such that each (E_i, A_i) is transformed into a Quasi-Weierstrass form

$$(S_i E_i T_i, S_i A_i T_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N_i \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix} \right) \qquad (2)$$

where I is the identity matrix of the appropriate dimension, N_i are Nilpotent matrices, and J_i are some matrices.

In [13], it is presented a useful method to obtain matrices S_i and T_i using Wong sequences. Consider a regular matrix pair (E_i, A_i) with index η, the associated Wong sequences of subspaces are as follows

$$\upsilon_0 := \mathbf{R}^n, \quad \upsilon_{k+1} = A_k^{-1}(E_k \upsilon_k), \quad \text{for} \quad k = 0, 1, ..., \upsilon^*,$$

with

$$\upsilon^* := \bigcap_k \upsilon_k$$

$$\omega_0 := \{0\}, \quad \omega_{l+1} = E_l^{-1}(A_l \omega_l), \text{ for } \quad l = 0, 1, ..., \omega^*,$$

with

$$\omega^* := \bigcup_l \omega_l,$$

then we choose full rank matrices V and W, such that im$V = \upsilon^*$ and im$W = \omega^*$ and obtain invertible matrices

$$T := [V, W], \quad S := [EV, AW]^{-1}.$$

Wong sequences are nested and get stationary after exactly η steps. Once we have obtained the Quasi-Weierstrass form of regular matrices E_i, A_i, we can define the consistency projectors and flow matrices as follows

Definition 4: For each pair (E_i, A_i), the consistency projectors are given by

$$M_i = T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1},$$

the flow matrices are given by

$$A_i^d = T_i \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1},$$

and the differential projectors are given by

$$M_i^d = T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S_i,$$

.

It should be guaranteed that the design is impulse and jump free. Assume that the matrix pair (E_i, A_i) are regular and that there is a switch at time t from mode $\sigma(t^-) = p$ to mode $\sigma(t^+) = r$, with $p,\ r \in \mathcal{Q}$. If for all $p, r \in \mathcal{Q}$

$$E_r(I - M_r)M_p = 0$$

holds, then all distributional solutions $x(t)$ of (1) are impulse-free for any switching signal. And if also

$$(I - M_r)M_p = 0$$

holds, then all solutions are jump and impulse free for any switching time, i.e, $x(t^+) = x(t^-)$.

The solution of the switched DAE can now be characterized in terms of projectors and flow matrices as follows [10].

Theorem 5: Consider the system (1) with Assumption 2 and consistency projectors, differential projectors, and flow matrices introduced in Definition 4. Then $x(t)$ is the impulse free part of any distributional solution of (1) if and only if $x(t)$ is a solution of the switched system with jumps given by

$$\dot{x}(t) = A_i^d x(t) + M_i^d B_i u(t), \tag{3}$$

$$x(t_{k,i}^+) = M_i x(t_{k,i}^-),$$

where $t_{k,i}^-$ and $t_{k,i}^+$ are the instant time before and after the switch, respectively. The initial conditions before the switched DAE is activated $x(0^-)$ is assumed consistent and

$$x(0^-) = x_0.$$

Finally, the following projector assumptions are introduced. Consider the set of projectors $\{M_i\}_{i=1}^q$, then we have the following Projector Assumptions

- im$M_\cap \subseteq$ imM_i and
- $\ker M_\cap \supseteq \ker M_i$,

$\forall i \in \mathcal{Q}$, where im and $\ker$ are the image and kernel of matrices, respectively. The product projector $M_\cap$ is given by

$$M_\cap = \prod_{i=1}^q M_i. \tag{4}$$

Theorem 5 provides tools to control design since using projectors and flow matrices we are able to use previous result for switched linear systems. In the next section, we extend available stabilization result for switched systems to switched DAEs systems.

III. State Feedback Switching Control

In this section, we present a state feedback switching control algorithm for switched DAEs system (1). The main objective is to design a feedback stabilizing control law, which involves finding appropriate switching signals as well as continuous state feedback controllers to make the closed-loop system asymptotically stable.

Consider the switched DAEs system (1), we propose a definition of stabilizability as follows

Definition 6: System (1) is said to be linear feedback stabilizable, if there exist a switching signal σ, and state feedback control inputs

$$u_i(t) = -K_i x(t) \tag{5}$$

such that the closed-loop switched DAEs system

$$E_\sigma \dot{x}(t) = (A_\sigma - B_\sigma K_\sigma) x(t)$$

is well-posed and uniformly asymptotically stable.

In order to develop a control algorithm to satisfy Definition 6, we apply the average technique to approximate the switching DAEs system based on Theorem 5. First, let us define the simplex set as

$$\Delta = \left\{ \alpha_i \in \mathbf{R} | \sum_{i=1}^{q} \alpha_i = 1, \alpha_i \geq 0 \right\}.$$

where q is the number of subsystems and the set of subsystems indexes is defined as $i \in \mathcal{Q} = \{1, 2, \ldots, q\}$. The average matrices are defined as convex combinations of the subsystems matrices. Recall that the solutions of the switched DAE (1) can be expressed by solutions of a switched system with jumps (3). Recently in [10], it is proposed an averaged model that approximates the behavior of the switched DAE. Considering that a time-varying system can be viewed as a time-invariant averaged system with a small perturbation, an averaging model has been proposed. Consider a periodic switching signal σ for the switched DAE system (1), the averaging model converges to the switched system for a fast switching with a small period. The proposed average model is as follows

$$\dot{x}_\alpha(t) = M_\cap \left(\sum_{i=1}^{q} \alpha_i A_i^d \right) M_\cap + M_\cap \left(\sum_{i=1}^{q} \alpha_i B_i^d \right)$$

$$x_\alpha(0) = M_\cap x_0 \tag{6}$$

where $i \in \mathcal{Q}$, α_i are defined in the simplex, and $M_\cap$ is defined in (4).

Consider that assumptions in Theorem 5 are satisfied and there exists an average model (6), then we can establish the quadratically stabilization conditions for switched DAE.

Lemma 7: System (1) is quadratically stabilizable if there exist gain matrices K_i, $i \in \mathcal{Q}$ such that the matrix

$$\left\{ M_\cap \left(\sum_{i \in \mathcal{Q}} \alpha_i M_i^d (A_i - B_i K_i) \right) M_\cap : \alpha_i \in \Delta \right\}$$

contains a Hurwitz matrix.

Let us define

$$A_{K\alpha} = M_\cap \left(\sum_{i \in \mathcal{Q}} \alpha_i M_i^d (A_i - B_i K_i) \right) M_\cap$$

as the average closed-loop matrix. The following proposition presents the stabilizability condition.

Proposition 8: For system (A_i^d, B_i^d), α_i are given convex combination parameters, with $i \in \mathcal{Q}$ such that the pair $(\sum_i \alpha_i A_i^d, \sum_i \alpha_i B_i^d)$ is controllable. Then, the switched affine system is quadratically stabilizable.

We present a state-feedback switching signal based on an appropriate partition of the state space. As a first approach, we assume that the system is single-input

$$B_1 = [1, 0, \ldots, 0]^\top$$

and in the controllable form. Then the single-input system $(\sum_i \alpha_i A_i^d, B_1^d)$ is also quadratically stabilizable. Considering the average matrix A_α and $B_\alpha = \alpha_1 B_1^d$, we can find a feedback K_1 such that

$$A_{K\alpha} = A_\alpha - B_\alpha K_1$$

is Hurwitz with $u_1 = u_2 = -K_1 x$. Supposing assumptions in Lemma (7) holds, we solve the Lyapunov equation

$$A_{K\alpha}^\top P + P A_{K\alpha} = -I_n,$$

and we obtain a symmetric positive definite matrix P, this matrix can be seen as a common Lyapunov function for the switched DAEs system. For each subsystem we denote

$$\forall i \in \mathcal{Q}, Q_i = [M_i^d (A_i - B_1 K_1)]^\top P + P [M_i^d (A_i - B_1 K_1)]. \tag{7}$$

which are positive definite matrices. The main result for the feedback switching algorithm is presented in the following theorem.

Theorem 9: Consider the switched DAEs system (1). If there exist positive matrices Q_i as in (7) such that $V = x^\top P x$. Then the state switching control

$$\sigma(x) = \arg\min_{i \in \mathcal{Q}} \{x^\top Q_i x\}, \tag{8}$$

and the feedback control

$$u(x) = -K_1 x,$$

asymptotically stabilize system (1).

Notice that $\arg\min$ stands for the index which attains the minimum among $\mathcal{Q} = \{1, 2, \cdots, q\}$, and that the eigenvalues of the average matrix $A_{K\alpha} = A_\alpha - B_\alpha K_1$ can be arbitrarily (symmetrically) assigned by appropriately choosing K_1 using techniques such as pole placement.

In this section, we have presented the main theoretical results for stabilization of switched DAE (1). In the next section, some simulation results are presented to illustrate the implementation of Theorem 9 and particularly the switching control (8).

IV. Numerical Example

A simple academic switched differential algebraic model is used as an example to illustrate the approach proposed in this paper. It is established that this system is asymptotically controllable and asymptotically stabilizable. The system has two operation modes described by the following matrices

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and the consistency projectors obtained by some method using Wong sequences are given by

$$M_1 = A_1^d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad M_2 = A_2^d = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

the convex combination parameter are $\alpha_1 = 0.3$, $\alpha_2 = 0.7$. We obtain the product projector

$$M_\cap = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

With these matrices, we choose appropriate eigenvalues for the closed loop system using a simple routine of pole placement following the indications of Theorem 9 and switching control (8).

Fig. 1 shows the trajectories and the switching signal for the switched DAEs applying one single continuous control and a switching signal based on the proposed methodology. It is shown that both controllers acting simultaneously drive the switched DAEs system to the equilibrium point.

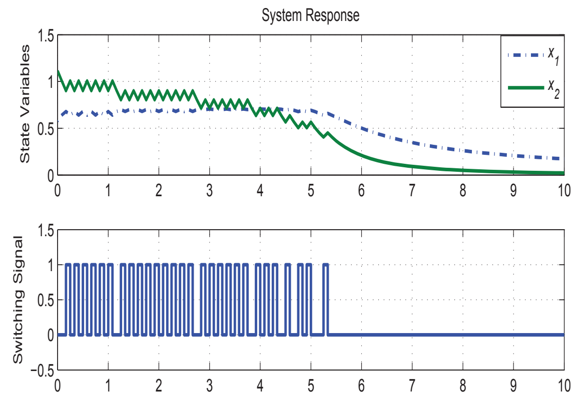

Fig. 1. States and switching signal

V. Conclusions and Future Work

We have presented preliminary results dealing with the stabilization problem of switched differential-algebraic systems. We have introduced a useful tool to solve this problem based on consistency projector and flow matrices to guarantee convergence towards an equilibrium point via fast switching. Some simulation results allow us to observe the effectiveness of the proposed algorithm. There are several possible extension of this work. One direction is towards non-linear systems, in particular DAEs systems of polynomial form. On the other hand, the approach based on consistency projector can be used in linear optimal control problem such as a linear quadratic regulator for switched DAEs.

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