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Modeling collective behaviors: A moment-based approach*

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***Abstract*— In this note we introduce a new approach for modeling and analyzing collective behavior of a group of agents using moments. We represent the group of agents by their moments and show how the dynamics of the moments can be modeled. Then approximate trajectories of the moments can be computed and an inverse problem is solved to recover macro-scale properties of the group of agents. To illustrate the theory, a numerical example with interactions between the agents is given.**

I. Introduction

In a large number of applications it is essential to understand the collective macro-scale behavior of a group of particles or agents, while the dynamics of each agent is modeled on a micro-scale. This occurs in applications in biology [6], material science [19], [23] and modeling of human behavior [5], [13], [18]. In the latter, an important open question is how to intervene in moving crowds, especially in situations of panic [3], [26], [20]. However, such systems typically contain a large number of agents, often too large for considering each agent individually. Moreover, in many cases the agents are exchangeable and tracking of each agent may not even be desirable.

In this work we study model reduction techniques of systems containing a large number of identical nonlinear subsystems. We will utilize a lifting technique, combined with approximations, by which one instead of directly addressing the nonlinear systems, considers the action of a transfer operator similar to the Koopman operator [22]. This technique allows for formulating a nonlinear problem into infinite dimensional problems (cf. [22], [2]), which have structures that allow for natural approximations in terms of finite dimensional linear or quadratic systems. We propose to develop this framework for applications such as crowd dynamics and other macroscopic control problems. The approach builds on approximating the infinite dimensional system (i.e., the distribution of subsystems) in terms of moments, thereby allowing for analysis, observation and control of the overall system.

II. Representing multi-agent systems by moments

Consider a multi-agent system consisting of N identical agents. Let $K \subset \mathbb{R}^d$ be a compact set and let the state of agent i be $x_i \in K$, for $i = 1, \dots, N$. The distribution of the agents can be described in a concise way by

$$d\mu(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i)dx, \tag{1}$$

where δ denotes the Dirac delta function. Note that such $d\mu \in \mathcal{M}_+(K)$ is a nonnegative measure on K. This is a time-dependent measure which conveys all information about the current states of the agents in the system. We will use an approximation of this measure in order to avoid having to compute the dynamics of each individual agent, which would be to expensive when the number of agents is large.

Given a set of continuous kernel functions $\phi_k \in C(K)$, for $k = 1, \dots, M$, the corresponding moments of the measure (1) are defined by

$$m_k = \int_K \phi_k(x) d\mu(x) = \frac{1}{N}\sum_{i=1}^{N} \phi_k(x_i) \tag{2}$$

for $k = 1, \dots, M$. In order to capture the collective behavior composed by the individuals, we will in this work approximate the dynamics of $d\mu(x)$ by considering the dynamics of a finite set of moments $\{m_k\}_{k=1}^M$. The approximated dynamics can then be used to estimate the moments at a given time and finally the measure representing the particle distribution at the given time is reconstructed via equation (2). The approximation of the moment dynamics is discussed in section III and IV-A, and the reconstruction is discussed in the following subsection and in section IV-B.

A. *From moments to measure - the inverse problem*

The problem of recovering a nonnegative measure $d\mu$ from a sequence of numbers $m := [m_1, \dots, m_M]^T$ is an ill-posed inverse problem, since not all such sequences are bone fide moment sequences. The problem of characterizing all sequences m that are moments of some nonnegative measures is a classical problem in mathematics [1], [12], [14]. However, even if m is a bona fide moment sequence there are typically an infinite set of nonnegative measures that matches it. Although this is the case, the moments still carry valuable information. For a given moment sequence m it is for example possible to bound the mass of a matching measure, both from above and below, on any compact set [16], [10]. From a point of view of multi-agent systems this means that, e.g., in an evacuation scenario we could still answer questions regarding how many agents that at most are in a certain area.

Polynomial moments have also been used previously in the literature on crowd control [27], as they can be used in order

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to achieve a certain polygon shape [17]. Moment problems also occur in many other application areas such as spectral estimation [25], optimal control [9], [7], and modeling the distribution of stochastic processes, e.g., in a chemical plant [24] or in an electrical or mechanical system [8].

III. Modeling multi-agent systems using moments

In this section a methodology to approximate the moments dynamics is proposed. This gives a moment-based model for collective behaviors since $d\mu$ can be reconstructed from these moments.

A. Modeling basic systems of agents

We start with deriving the moment dynamics for systems where the dynamics of each individual is governed by a spacial vector field. This basic scenario is involved in many applications, such as the crowd evacuation in a domain with obstacles and movement analysis for a particle accelerator. Let the dynamics of each individual be governed by

$$\dot{x}_i(t) = f(x_i(t)), \qquad i = 1, \dots, N. \tag{3}$$

Correspondingly, the dynamics of the moments satisfies

$$\begin{aligned}\dot{m}_k(t) &= \frac{1}{N}\sum_{i=1}^{N} \frac{d\phi_k(x_i(t))}{dt} = \frac{1}{N}\sum_{i=1}^{N} \frac{\partial \phi_k(x_i(t))}{\partial x_i(t)} f(x_i(t)) \\ &= \int_{x\in K} \frac{\partial \phi_k(x)}{\partial x} f(x) d\mu(x). \end{aligned}\tag{4}$$

If the function $\frac{\partial \phi_k}{\partial x} f(x)$ is well approximated by $\sum_{\ell=1}^{M} a_\ell^k \phi_\ell(x)$, where $a_\ell^k \in \mathbb{R}$ are some coefficients, then by the linearity of the integral and definition (2), the dynamics of the moments system (4) can be written as

$$\dot{m}(t) = \frac{d}{dt}\begin{bmatrix} m_1 \\ \vdots \\ m_M \end{bmatrix} \approx \begin{bmatrix} a_1^1 & \cdots & a_M^1 \\ \vdots & & \vdots \\ a_1^M & \cdots & a_M^M \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_M \end{bmatrix} = Am, \tag{5}$$

where $m = [m_1, \dots, m_M]^T$. We denote the error of the approximation for function $\frac{\partial \phi_k}{\partial x} f(x)$ as

$$\varepsilon_k(x) := \frac{\partial \phi_k(x)}{\partial x} f(x) - \sum_\ell a_\ell^k \phi_\ell(x). \tag{6}$$

Introducing the notation $\hat{\lambda}(A) := \lambda_{\max}(\frac{A+A^T}{2})$, we have the following result on the error bound of moment system (5).

Theorem 1: Let $\overline{m}(t)$ be the solution of approximated system (5) starting from an initial condition $\overline{m}(0)$, and $m(t)$ be the solution of (4) starting from $m(0)$. Then the difference of two solutions $\Delta m(t) = m(t) - \overline{m}(t)$ is bounded by

$$\|\Delta m(t)\| \leq \|\Delta m(0)\| \cdot e^{\hat{\lambda}(A)t} + \frac{e^{t\hat{\lambda}(A)} - 1}{\hat{\lambda}(A)} \sqrt{\sum_k \max_{x\in K} \varepsilon_k^2(x)},$$

if $\hat{\lambda}(A) \neq 0$, and by

$$\|\Delta m(t)\| \leq \|\Delta m(0)\| + t \cdot \sqrt{\sum_k \max_{x\in K} \varepsilon_k^2(x)}$$

if $\hat{\lambda}(A) = 0$, where $\varepsilon_k(x)$ is defined in (6).

B. Modeling multi-agent systems with interactions

In multi-agent systems, besides a spacial vector field, the interactions between each pair of individuals sometimes play a even more essential role for its collective behaviors. The individuals possessing nonlinear interactions is governed by the dynamics

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j=1}^{N} g(x_i(t), x_j(t)).$$

Then the exact moments dynamics is given by

$$\begin{aligned}\dot{m}_k(t) &= \frac{1}{N}\sum_{i=1}^{N} \frac{d\phi_k(x_i(t))}{dt} \\ &= \frac{1}{N}\sum_{i=1}^{N} \frac{\partial \phi_k(x_i(t))}{\partial x_i(t)} \frac{1}{N}\sum_{j=1}^{N} g(x_i(t), x_j(t)) \\ &= \int_{x\in K}\int_{y\in K} \frac{\partial \phi_k(x)}{\partial x} g(x,y) d\mu(x) d\mu(y). \end{aligned}\tag{7}$$

Similarly to the previous case, provided that for $(x, y) \in K^2$ we have a good approximation for the function $\frac{\partial \phi_k(x)}{\partial x} g(x, y)$ in the basis $\{\phi_j(x)\phi_\ell(y)\}_{j,\ell}$, i.e.,

$$\frac{\partial \phi_k(x)}{\partial x} g(x, y) \approx \sum_{\ell,j=1}^{M} \beta_{\ell,j}^k \phi_\ell(x)\phi_j(y), \tag{8}$$

by the linearity of the integral and (2), the moment dynamics (7) can be approximated as

$$\dot{m}_k(t) \approx \sum_{\ell,j=1}^{N} \beta_{\ell,j}^k m_\ell m_j =: m(t)^T B_k m(t). \tag{9}$$

The approximation error in the approximation (8) is denoted as

$$\varepsilon_k(x, y) := \frac{\partial \phi_k(x)}{\partial x} g(x, y) - \sum_{\ell,j=1}^{M} \beta_{\ell,j}^k \phi_\ell(x)\phi_j(y). \tag{10}$$

Bounds similar to the ones in Theorem 1 can also be derived for these systems, but due to lack of space we defer the results to a forthcoming paper.

IV. Algorithm for moment based modelling

Left to describe is how the approximation of the dynamics is done in (6) and (10), and how to reconstruct a nominal distribution from the moments.

A. Moment dynamics approximation and regularization

In a first attempt to approximate the dynamics in (6) and (10) we use L_2 approximation. This makes the approximation easy and cheap to compute, since for any linearly independent kernels the solution can be found by solving a finite linear system of equations (see, e.g., [15, Sec. 3.6]).

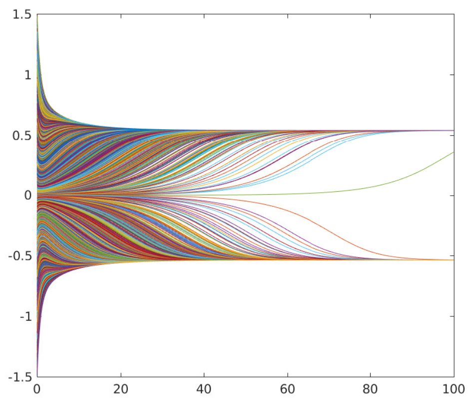


Fig. 1: The behavior of system (14) with 10^4 particles.

The optimization problems we want to solve are thus

$$\min_{a_\ell^k} \quad \sum_{k=1}^{M} \left\| \frac{\partial \phi_k(x)}{\partial x} f(x) - \sum_{\ell} a_\ell^k \phi_\ell(x) \right\|_2, \tag{11}$$

$$\min_{\beta_{\ell,j}^k} \quad \sum_{k=1}^{M} \left\| \frac{\partial \phi_k(x)}{\partial x} g(x,y) - \sum_{\ell,j=1}^{M} \beta_{\ell,j}^k \phi_\ell(x)\phi_j(y) \right\|_2. \tag{12}$$

Remark 1: Another way to approximate the dynamics in (5) and (8) is to use L_∞ approximation. This actually seems more natural, since according to Theorem 1 the L_∞ norm of the approximation error is directly reflected in the moment error Δm. Note also that the moment error Δm depends not only on the instantaneous error in the dynamics, ε_k in (6) and (10), but also on the propagation of the error in time. This propagation is governed by $\hat{\lambda}(A)$ and $\hat{\lambda}(B_{k\ell})$, respectively. Thus it might be of interest to introduce bounds on these quantities when approximating the moment dynamics. These approaches will be further explored in the subsequent research article.

B. Reconstruction of the measure

As mentioned in section II-A, from a finite set of moments we can compute bounds on the measure [16] or obtain a nominal estimate of the measure (1) by solving a convex optimization problem (see, e.g., [4], [11], [21]). An example of such a problem with approximate moment matching is the following total variation minimization problem

$$\min_{d\mu \geq 0, \varepsilon \geq 0} \quad \int_K |\nabla d\mu| + \lambda \cdot \varepsilon \tag{13}$$
$$\text{subject to} \quad |m_k - \int_K \phi_k(x) d\mu(x)| \leq \varepsilon, \quad k = 1, \dots, M,$$

where λ is a regularization parameter.

V. Numerical example

In this section the collective behaviors of a multi-agent system is investigated using the proposed approach. The dynamics of each individual involves nonlinear components, which can lead to intricate collective behaviors on a macroscopic level. In particular, we consider the scenario where the particles are driven by a time-invariant spacial field plus a repulsive force between each pair of individuals. The dynamics of particle i is

$$\dot{x}_i(t) = -x_i + \frac{1}{N} \sum_{j=1}^{N} 2e^{-0.6(x_i - x_j)^2} \cdot (x_i - x_j), \tag{14}$$

for $i = 1, 2, \dots, N$, where $x_i \in \mathbb{R}$ is the state of particle i.

The behaviors of a system consisting of 10^4 homogeneous particles governed by dynamics (14) is simulated, and the trajectories of all particles are shown in Figure 1. As the result of the involved nonlinear interactions, the collective behavior of the system gives rise to a formation consisting of two clusters. The initial position of each agent was draw from a uniform distribution on the interval $[-1.5, 1.5]$, and which of the two clusters a particular agent converges to depends on its initial position.

In this example we choose the kernel functions be polynomials, i.e., $\phi_k(x) = x^{k-1}$, where $k = 1, \dots, 15$, and the compact region is chosen as $K = [-2, 2]$. Then the approximations are carried out for (6) and (10) with $f(x) = -x$ and $g(x,y) = 2e^{-0.6(x-y)^2}(x-y)$, respectively. The L_2 approximation is implemented, which gives an approximated moment systems, and the trajectories of the approximated system are compared with that of the real moment system. For ease of display, the initial moment trajectories for the time interval $t \in [0, 3]$ are shown in Figure 2, and for $t \in [3, 100]$ in Figure 3. The approximated dynamic approximates the true dynamics quite well.

However, judging weather or not these are good approximations is hard by only considering the moment trajectories. Indeed, the importance for the understanding of the collective behavior of the underlying system is the information the approximated moments carry about the distribution of particles. In Figure 4 we present total variation reconstructions (13) performed at times $t = 3$ and $t = 100$. The reconstructions are obtained from the true and approximated moments, and we also present a histogram of the true distribution of the particles. From the reconstructions in Figure 4 we see that the approximated moments seem to capture the essential collective behavior of the underlying dynamical system quite well, and that the difference in the true and approximated moments only gives rise to a small difference between the reconstructed distributions. Note that these are only nominal reconstructions and that a more thorough analysis need to be performed in order to determine, e.g., bounds on the number of agents in a certain region. This will be subject to further research. Moreover, the monomial kernel functions result in ill-conditioning in both the approximation and reconstruction problem. Thus other basis functions such as orthogonal polynomials will also be investigated in the future.

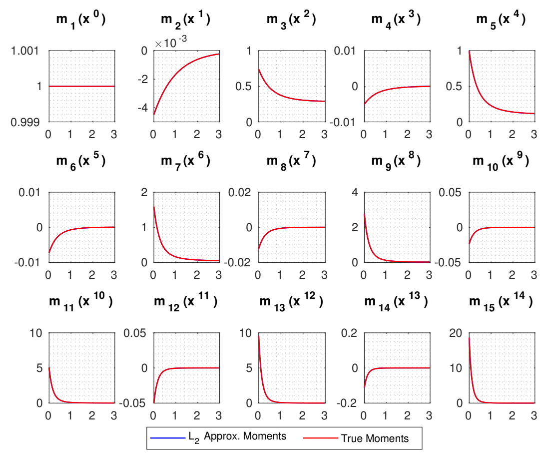


Fig. 2: Trajectories for the moments of the true system and the L_2 approximated system, from time $t = 0$ to 3.

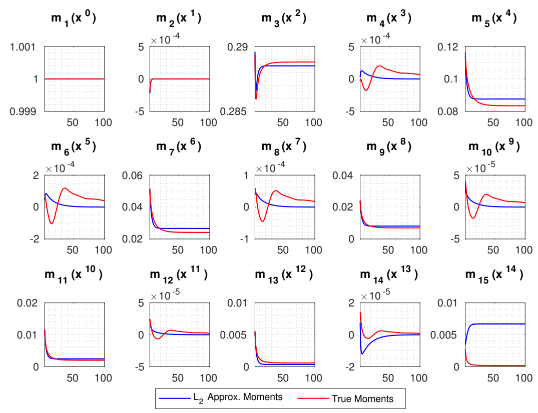
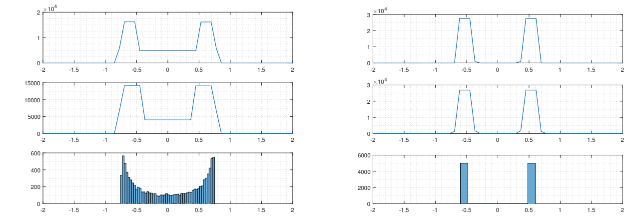


Fig. 3: Moments trajectories of L_2 approximated system and the true moment trajectories, from time $t = 3$ to 100.



(a) Time: $t = 3$. (b) Time: $t = 100$.

Fig. 4: Reconstructions from moments using (13) and histogram. Top row is a reconstruction using the true moments and the middle row is using the moments with dynamics obtained via L_2 approximation. The bottom row is a histogram representation of the true distribution.

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