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On LQG control of stochastic port-Hamiltonian systems on infinite-dimensional spaces

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Abstract— **This paper presents an ongoing research on the infinite horizon Linear Quadratic Gaussian (LQG for short) control problem for stochastic port-Hamiltonian systems on infinite-dimensional spaces with bounded input, output and noise operators. An adapted version of the separation principle is stated for this specific class of systems. Under suitable conditions, the LQG controller is shown to preserve the stochastic port-Hamiltonian structure. Finally, we propose some perspectives and open tracks to follow. The theory is illustrated on an example of vibrating string subject to a Hilbert space valued random forcing.**

Index Terms— **Infinite-dimensional system - Port-Hamiltonian system - Stochastic partial differential equation - LQG method**

***AMS Subject Classifications*— 93E03, 60H15**

I. Introduction

In the deterministic setting, an extensive literature can be found on the Linear Quadratic optimal control problem on infinite-dimensional spaces in [CZ95] and the references therein.

The stochastic Linear Quadratic control problem was first addressed by Wonham [Won68a] and [Won68b] and Kushner [Kus62] by means of a dynamic programming approach. For an overview of the existing literature in finite dimension, interested readers are referred e.g. to [FR75].

A generalization to infinite-dimensional spaces was undertaken by Ichikawa in [Ich79], where the author studies the stochastic LQ control with bounded control, observation and noise operators and its related Riccati equation by using a semigroup framework and a dynamic programming approach. In other works, a study of the LQG control problem and a generalization of the separation principle can be found in [Cur78] and [CI77], which is illustrated by several examples and applications.

It is worth noticing that the literature cited so far deals either with general infinite-dimensional systems or very specific classes of systems. To the best of our knowledge, no attempt has already been made to develop an adapted approach in order to study and solve the LQG control problem for stochastic port-Hamiltonian systems on infinite-dimensional spaces (SPHSs).

The class of nonlinear time-varying SPHSs was introduced in [SF13] on finite-dimensional spaces as the stochastic extension of [MvdS92]. Recently, the authors generalized SPHSs on infinite-dimensional spaces with boundary control and observation operators in [LW17] as the stochastic extension of deterministic linear infinite-dimensional first order port-Hamiltonian systems, see [LZM05].

Here, the semigroup approach is preferred to the variational one. Since the early work of Ito in the mid-1940s, the theory of stochastic differential systems (SDEs) has been the object of a considerable attention. Background material for the study of stochastic partial differential equations (SPDEs) using a semigroup approach and the theory of stochastic integration with respect to Hilbert space valued stochastic processes can be found in [Cho14], [DPZ08] and [Liu05]. For space reasons, a recall will not be given here.

The present paper reports on current research to address and solve the LQG control problem for stochastic port-Hamiltonian systems (SPHSs) with bounded control, observation and noise operators. Moreover, conditions are derived to preserve the stochastic port-Hamiltonian structure of the LQG controller and thus the closed-loop dynamic can be interpreted as the interconnection of infinite-dimensional stochastic port-Hamiltonian systems. Here the system noise is assumed to be an infinite-dimensional Gaussian white noise process whereas the measurement noise is of finite dimension. This paper is meant to be a first attempt to address the LQG control problem for infinite-dimensional SPHSs in a stochastic context.

The content of the paper is as follows. In Section II the class of stochastic port-Hamiltonian systems is presented as in [LW17] except for the fact that here the control and observation operators are bounded. The LQG control problem is introduced for this specific class of systems. Section III is devoted to the solution of the LQG control problem by using the separation principle. In Section IV a LQG controller preserving the stochastic port-Hamiltonian framework is proposed. Eventually, the results outlined in this paper are illustrated by an example of an inhomegenous vibrating string subject to a space and time dependent Gaussian white noise process.

II. Stochastic port-Hamiltonian systems

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, wherein the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual assumptions of completeness and right continuity. Let us consider a Hilbert space Z and let us denote by $L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)$ the Hilbert space of $\mathbb{R}^m$-valued and $\mathbb{F}$-adapted stochastic processes with norm $\|\cdot\|^2_{L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)} := \int_0^\infty \mathbb{E}\|\cdot\|^2_{\mathbb{R}^m} ds$. In this paper the class of first order stochastic port-Hamiltonian systems driven by a Z-valued white noise Gaussian process $(\eta(t))_{t\geq 0}$

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with bounded control and noise operators is considered. It is governed by the following stochastic partial differential (SPDE)

$$\frac{\partial X}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)X(\zeta,t)) + P_0\mathcal{H}(\zeta)X(\zeta,t) \\ +Bu(t) + (H\eta(t))(\zeta), \tag{1}$$

where $P_1 = P_1^T$ is invertible, $P_0 = -P_0^T$ and $\mathcal{H}(\zeta) \in L^\infty([a,b];\mathbb{R}^{n\times n})$ is symmetric and $m_1 I \leq \mathcal{H}(\zeta) \leq m_2 I$ for a.e. $\zeta \in [a,b]$ for some constants $m_1, m_2 > 0$. The state space $\mathcal{X} := L^2([a,b];\mathbb{R}^n)$ is endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}} = \langle\cdot,\mathcal{H}\cdot\rangle_{L^2}$. Here, the input $u \in L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)$ and the state process $X(\zeta,t)$ can be regarded as a random variable $X(\zeta,t):\Omega\to\mathbb{R}^n$, $\zeta\in[a,b]$ and $t\geq 0$.

The SPDE (1) is completed by a set of boundary port variables given by the flow $f_\partial(t)$ and the effort $e_\partial(t)$. These port variables are usually expressed as the combination of the co-energy variables $(\mathcal{H}X(t))$ restricted to the extremities of $[a,b]$ in the following way:

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = R_0 \begin{pmatrix} (\mathcal{H}X(t))(b) \\ (\mathcal{H}X(t))(a) \end{pmatrix}, \tag{2}$$

where $R_0 \in \mathbb{R}^{2n\times 2n}$ is defined as $R_0 := \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ 0 & 0 \end{pmatrix}$.

The SPDE (1) can be rewritten under the Ito form :

$$dX(t) = (AX(t) + Bu(t))dt + Hdw(t), \\ X(0) = X_0 \tag{3}$$

in which $(w(t))_{t\geq 0}$ stands for a Z-valued Wiener process with incremental covariance Q and intensity $H \in \mathcal{L}(Z,\mathcal{X})$. Here the operator $Q \in \mathcal{L}(Z)$ is assumed to be symmetric and nonnegative and to satisfy $\mathrm{Tr}[Q] < \infty$, where Tr denotes the trace operator of Q. The operator A is defined by

$$Ax := P_1\frac{d}{d\zeta}(\mathcal{H}x) + P_0\mathcal{H}x \tag{4}$$

on the domain

$$D(A) = \Big\{x \in \mathcal{X} : \mathcal{H}x \in H^1([a,b];\mathbb{R}^n), \\ W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0\Big\}, \tag{5}$$

where W_B is a $n\times 2n$ full rank real matrix. The observation model is taken to be

$$dZ(t) = B^*\mathcal{H}X(t)dt + Fdv(t), \tag{6}$$

where $v(t)$ is a $\mathbb{R}^m$-valued Wiener process with invertible incremental covariance matrix V and intensity $F \in \mathbb{R}^{m\times m}$. Note that the observation process has to be finite dimensional since the covariance matrix of the measurement noise is assumed to be invertible and of trace class. Nonetheless, this is not restrictive from a physical point of view since one can only hope to have a (possibly large) finite number of observations. The initial condition X_0 and the noise processes $w(t)$ and $v(t)$ are assumed to be mutually independent.

The next result ensures the generation of a C_0-semigroup by the operator A.

Theorem 1: [Vil07, Theorem 2.13] Consider the operator A with domain $D(A)$ given by (4) and (5). Assume that W_B is a full rank matrix of size $n\times 2n$. Then A is the infinitesimal generator of a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$ if and only if W_B satisfies $W_B\Sigma W_B^T \geq 0$, where $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. Furthermore, A is the infinitesimal generator of a unitary group if and only if $W_B\Sigma W_B^T = 0$.

In this paper we will therefore develop the theory under the following conditions.

Assumption 1:
1) The matrix W_B is assumed to be full rank and to satisfy $W_B\Sigma W_B^T \geq 0$.
2) The stochastic convolution product $\int_0^t T(t-s)Hdw(s)$ is assumed to be well-defined, i.e. $\int_0^t \mathrm{Tr}[T(s)HQH^*T^*(s)]ds < \infty$ for all $t \geq 0$.

For further details on the theory of stochastic integration in the Ito sense, the reader is referred to [DPZ08], [MPBL14] or [Cho14].

Following [DPZ08], for a given control $u \in L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)$ and an initial condition $X_0 \in L^2_{\mathcal{F}_0}(\Omega;\mathcal{X})$, the SDE (3) has a unique mild solution given by

$$X(t) = T(t)X_0 + \int_0^t T(t-s)Bu(s)ds \\ + \int_0^t T(t-s)Hdw(s). \tag{7}$$

For further details on stochastic port-Hamiltonian systems, we refer to [LW17].

The LQG control problem is to minimize the functional

$$J(u) = \lim_{T\to\infty}\mathbb{E}\int_0^T \|B^*\mathcal{H}X(t)\|^2_{\mathbb{R}^m} + \|\tilde{R}^{1/2}u(t)\|^2_{\mathbb{R}^m}dt, \tag{8}$$

over the admissible controls $u \in L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)$ and subject to (3) and (6). The weight matrix $\tilde{R}$ is assumed to be symmetric and positive definite and $\mathcal{H}BB^*\mathcal{H}$ is assumed to be positive semi-definite.

III. Separation principle

The aim of this section is to state the separation principle for SPHSs on the basis of [CI77] and [CP78]. For an admissible class of controls, it is known that the LQG control problem (8) under (3) and (6) can be divided into two separate problems, namely the estimation of the state process $(X(t))_{t\geq 0}$ based on the observation process $(Z(t))_{t\geq 0}$ and the LQ control problem with complete observation on the estimated state process $(\hat{X}(t))_{t\geq 0}$. Notice that in [BV75], a different approach is considered and necessary and sufficient conditions are derived for optimality with a convex differentiable cost functional.

Definition 1:
1) (A,B) is said to be exponentially (exp.) stabilizable if there exists an operator $K \in \mathcal{L}(\mathcal{X},\mathbb{R}^m)$ such that $A-BK$ generates an exp. stable C_0-semigroup $(T_{A-BK}(t))_{t\geq 0}$.
2) $(B^*\mathcal{H},A)$ is said to be exponentially (exp.) detectable if there exists an operator $L \in \mathcal{L}(\mathbb{R}^m,\mathcal{X})$ such that $A-LB^*\mathcal{H}$ generates an exp. stable C_0-semigroup $(T_{A-LB^*\mathcal{H}}(t))_{t\geq 0}$.

The solution of the optimal feedback control problem is known to be closely related to the Riccati equation, which involves the adjoint operator of A. The computation of this adjoint can be found in [Vil07] for N order port-Hamiltonian systems. For the sake of self-containedness, we present a proof for the specific first order case ($N=1$).

Proposition 1: Let W_B be a $n\times 2n$ full rank matrix written as $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. Let us consider the operator A with its associated domain $D(A)$ given by (4) and (5). Then, its adjoint A^* is given by

$$A^*x = -P_1\frac{d}{d\zeta}(\mathcal{H}x) - P_0(\mathcal{H}x) = -\mathcal{J}\mathcal{H}x \tag{9}$$

for all x in

$$\begin{aligned} D(A^*) = \Big\{ & x\in L^2([a,b];\mathbb{R}^n) : \mathcal{H}x \in H^1([a,b];\mathbb{R}^n), \\ & \begin{bmatrix} -(I+M^T) & (I-M^T) \end{bmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = 0 \Big\}, \end{aligned} \tag{10}$$

where I denotes the identity matrix and $M = (W_1+W_2)^{-1}(W_1-W_2)$.

Proof: The adjoint of an unbounded operator is given by

$$A^*y = z \Leftrightarrow \forall x\in D(A), \langle Ax, y\rangle_{\mathcal{X}} = \langle x, z\rangle_{\mathcal{X}} \tag{11}$$

with domain defined by:

$$y\in D(A^*) \Leftrightarrow \exists z\in\mathcal{X} \text{ s.t. } \forall x\in D(A),\ \langle Ax,y\rangle_{\mathcal{X}} = \langle x,z\rangle_{\mathcal{X}}$$

On one hand, by integrating by parts, we have that

$$\begin{aligned} \langle Ax, y\rangle_{\mathcal{X}} &= [y^T(\zeta)\mathcal{H}(\zeta)P_1\mathcal{H}(\zeta)x(\zeta)]_a^b \\ &\quad - \int_a^b \frac{d}{d\zeta}(y^T(\zeta)\mathcal{H}(\zeta))P_1(\mathcal{H}x)(\zeta)d\zeta \\ &\quad + \int_a^b y^T(\zeta)\mathcal{H}(\zeta)P_0(\mathcal{H}x)(\zeta)d\zeta, \end{aligned} \tag{12}$$

and, on the other hand,

$$\langle x, A^*y\rangle_{\mathcal{X}} = \int_a^b (A^*y(\zeta))^T\mathcal{H}(\zeta)x(\zeta)d\zeta. \tag{13}$$

Since the equality between (12) and (13) must hold, we deduce that

$$A^*y = -P_0\mathcal{H}y - P_1\frac{d}{d\zeta}(\mathcal{H}y) \tag{14}$$

and

$$[y^T(\zeta)\mathcal{H}(\zeta)P_1\mathcal{H}(\zeta)x(\zeta)]_a^b = 0. \tag{15}$$

The relation (15) can be rewritten as

$$\left(\begin{bmatrix} 0 & P_1 \\ -P_1 & 0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}y)(a) \\ (\mathcal{H}y)(b)\end{bmatrix}\right)^T \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix} = 0. \tag{16}$$

The boundary term (16) can be rewritten as

$$\left(\begin{bmatrix} 0 & P_1 \\ -P_1 & 0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}y)(a) \\ (\mathcal{H}y)(b)\end{bmatrix}\right)^T \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{bmatrix} \tag{17}$$

$$= \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a)\end{pmatrix}^T R_0^T\Sigma R_0 \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a)\end{pmatrix} \tag{18}$$

From [JZ12, Lemma 7.3.2], these holds

$$\ker \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \text{Ran} \begin{bmatrix} I-M \\ I+M \end{bmatrix}, \tag{19}$$

where $M = (W_1+W_2)^{-1}(W_1-W_2)$ with $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. Using (19) in (18) and defining $\begin{pmatrix} f_{\partial,y} \\ e_{\partial,y}\end{pmatrix} = R_0\begin{bmatrix} (\mathcal{H}y)(a) \\ (\mathcal{H}y)(b)\end{bmatrix}$ we deduce that

$$\begin{pmatrix} f_{\partial,y} \\ e_{\partial,y}\end{pmatrix}^T \Sigma \begin{pmatrix} I-M \\ -(I+M)\end{pmatrix} l = 0, \tag{20}$$

for all $l\in\mathbb{R}^n$, which is equivalent to

$$\begin{pmatrix} f_{\partial,y} \\ e_{\partial,y}\end{pmatrix} \in \text{ Ker } \begin{pmatrix} -(I+M^T) & I-M^T \end{pmatrix}. \tag{21}$$

■

In the following, under suitable stabilizability and detectability assumptions (see Theorem 2 below), we derive the filter and control Riccati equations for SPHSs by using the expression of the adjoint operator A^* given by (9) and (10). The best estimate $\hat{X}(t)$ of the state $X(t)$ is governed by the Kalman filter equation, i.e.

$$\begin{aligned} d\hat{X}(t) &= (A\hat{X}(t) + Bu(t))dt + L(dZ(t) - B^*\mathcal{H}\hat{X}(t)dt) \\ \hat{X}(0) &= \mathbb{E}[X_0], \end{aligned} \tag{22}$$

where $L := P_f\mathcal{H}B(FVF^*)^{-1}$ in which P_f is the stabilizing self-adjoint nonnegative solution of the filter operator Riccati equation (FORE) given by

$$\begin{aligned} [\mathcal{J}\mathcal{H}P_f - P_f\mathcal{J}\mathcal{H} - P_f\mathcal{H}B(FVF^*)^{-1}B^*\mathcal{H}P_f \\ +HQH^*]x = 0, \end{aligned} \tag{23}$$

for all $x\in D(A^*)$ with $P_f(D(A^*))\subset D(A)$. In the case of complete observation of the state process, it is known that the optimal control can be written in feedback form by solving the control operator Riccati equation (CORE)

$$[-\mathcal{J}\mathcal{H}P_c + P_c\mathcal{J}\mathcal{H} + \mathcal{H}BB^*\mathcal{H} - P_cB\tilde{R}^{-1}B^*P_c]x = 0, \tag{24}$$

for all $x\in D(A)$ with $P_c(D(A))\subset D(A^*)$. Then, the optimal control $u^*(t)$ is given in terms of the optimal state $X^*(t)$ by

$$u^*(t) = -\tilde{R}^{-1}B^*P_cX^*(t). \tag{25}$$

Observe that the Riccati equation (24) is deterministic and thus the relation between the optimal control and the state process, which are random, is completely deterministic.

In the case of partially observed state process given by (6), by following [CI77], the cost is minimized over a certain class of admissible controls $\mathcal{U}_{ad}\subset L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)$, which roughly consists in controls depending only on the observation in order to avoid the dependence on the control law. The admissible set is chosen as

$$\begin{aligned} \mathcal{U}_{\text{ad}} = \{u : u(t) \text{ is adapted to } \sigma(Z(s) : 0\le s\le t) \\ \text{and to } \sigma(\gamma(s) : 0\le s\le t)\}, \end{aligned} \tag{26}$$

where $(\gamma(t))_{t\ge 0}$ is the innovation process given by

$$d\gamma(t) = dZ(t) - B^*\mathcal{H}\hat{X}(t)dt \tag{27}$$

with incremental covariance matrix FVF^*.

By adapting [CI77, Theorem 2.3] to our framework, we can now state the separation principle for SPHSs.

Theorem 2: (Separation principle)
Suppose that (A, B) and $(A, HQ^{1/2})$ are exp. stabilizable and that $(B^*\mathcal{H}, A)$ is exp. detectable. Consider the problem of minimizing $J(u)$ given by (8) subject to (3) and (6) over the class of admissible controls $\mathcal{U}_{\text{ad}}$. Then there exists a unique optimal control $u_* \in \mathcal{U}_{\text{ad}}$ given by

$$u_*(t) = -\tilde{R}^{-1}B^*P_c\hat{X}(t), \tag{28}$$

$$\hat{X}_*(t) = S(t)\hat{X}_0 + \int_0^t S(t-s)P_f\mathcal{H}B(FVF^*)^{-1}dZ(s), \tag{29}$$

where $(S(t))_{t\geq 0}$ is the C_0-semigroup generated by $(A - B\tilde{R}^{-1}B^*P_c - P_f\mathcal{H}B(FVF^*)^{-1}B^*\mathcal{H})$, P_c and P_f are the solutions of the CORE (24) and the FORE (23), respectively. The minimizing problem of $J(u)$ given by (8) subject to (3) and (6) is equivalent to the problem of minimizing

$$\hat{J}(u) = \lim_{T\to\infty} \mathbb{E}\int_0^T \|B^*\mathcal{H}\hat{X}(s)\|^2_{\mathbb{R}^m} + \|\tilde{R}^{1/2}u(s)\|^2_{\mathbb{R}^m}ds. \tag{30}$$

subject to (22).

Sketch of the proof: Since (A, B) and $(A, HQ^{1/2})$ are exp. stabilizable and $(B^*\mathcal{H}, A)$ is exp. detectable, (24) and (23) have unique exp. stabilizing self-adjoint nonnegative solutions P_c and P_f, respectively, see [CP78]. The optimal control is characterized among the class of $\sigma(\gamma(s) : 0 \leq s \leq t)$-adapted control. By using a similar argument as in [CI77], we deduce (28) and (29). The relation (29) entails that $\hat{X}_*(t)$ is $\sigma(Z(s) : 0 \leq s \leq t)$-adapted and then, $u_*(t) \in \mathcal{U}_{\text{ad}}$. ∎

IV. SPHS STRUCTURE PRESERVING FOR THE LQG CONTROLLER

An important problem is the preservation of the SPHS structure in the control process. From the filter equation (22), one can observe that the dynamics of the LQG controller consists of a prediction $(AX(t) + Bu(t))dt$ and a correction $(dZ(t) - B^*\mathcal{H}\hat{X}(t)dt)$. Since the prediction term conserves the stochastic port-Hamiltonian framework, this leads to the natural question whether the LQG controller could conserve the port-Hamiltonian framework.

The LQG controller is given by

$$u_c(t) = -K\hat{X}(t) := -\tilde{R}^{-1}B^*P_c\hat{X}(t),$$
$$d\hat{X}(t) = A\hat{X}(t)dt + P_f\mathcal{H}B(FVF^*)^{-1}d\gamma(t) + Bu_c(t)dt$$

where $L := P_f\mathcal{H}B(FVF^*)^{-1}$. Since the output of the controlled system is the input of the controller, the LQG controller is described by

$$u_c(t) = -K\hat{X}(t) := -\tilde{R}^{-1}B^*P_c\hat{X}(t), \tag{31}$$

$$d\hat{X}(t) = \left[A - LB^*\mathcal{H} - BK\right]\hat{X}(t)dt + LdZ(t), \tag{32}$$

$$\hat{X}(0) = \mathbb{E}\left[X(0)\right], \tag{33}$$

$$Z_c(t) = K\hat{X}(t). \tag{34}$$

Now we add a bounded dissipation term to the general SDE (3) describing SPHSs and we define the class of stochastic dissipative port-Hamiltonian systems.

Definition 2: Stochastic dissipative port-Hamiltonian systems are governed by the following SPDE

$$\frac{\partial X}{\partial t}(\zeta, t) = (\mathcal{J} - \mathcal{R})\mathcal{H}(\zeta)X(\zeta, t) + Bu(t) + (H\eta(t))(\zeta), \tag{35}$$

where

$$\mathcal{J}x := P_1\frac{d}{d\zeta}x + P_0x \tag{36}$$

and $\mathcal{R} = \mathcal{R}^* \in \mathcal{L}(\mathcal{X})$ is a positive semi-definite self-adjoint operator representing the energy dissipation along the spatial domain.

In Theorem 3, a generalization of [WHGM14, Proposition 5] to infinite-dimensional spaces is proposed. Under suitable conditions, the LQG controller is proved to describe a stochastic dissipative port-Hamiltonian system w.r.t. Definition 2.

Theorem 3: If we assume that the following link

$$\tilde{R} = FVF^* \tag{37}$$

holds and if the Riccati operators satisfy the relation

$$\mathcal{H}P_fx = P_c\mathcal{H}^{-1}x \tag{38}$$

for all $x \in D(A) \cap D(A^*)$, then the LQG controller given by (31)-(34) describes a dissipative stochastic port-Hamiltonian system. Moreover, the covariance operator Q and the weighing operator $\mathcal{H}BB^*\mathcal{H}$ are related by

$$\begin{aligned}\mathcal{H}BB^*\mathcal{H}x = [&\mathcal{H}HQH^*\mathcal{H} + (\mathcal{J}\mathcal{H} + \mathcal{H}\mathcal{J})P_c \\ &-P_c(\mathcal{J}\mathcal{H} + \mathcal{H}^{-1}\mathcal{J}\mathcal{H}^2)]x,\end{aligned} \tag{39}$$

for all $x \in D(A) \cap D(A^*)$.

Proof: In order to describe a dissipative stochastic port-Hamiltonian system, the energy dissipation operator

$$\mathcal{R} := P_f\mathcal{H}B(FVF^*)^{-1}B^* + B\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1} \tag{40}$$

must be self-adjoint and positive semi-definite. On one hand since $\mathcal{H}$ is a self-adjoint operator,

$$\mathcal{R}^* = B(FVF^*)^{-1}B^*\mathcal{H}P_f + \mathcal{H}^{-1}P_cB(\tilde{R}^*)^{-1}B^*.$$

Hence, $\mathcal{R}$ is self-adjoint if the following conditions hold:

$$B(FVF^*)^{-1}B^*\mathcal{H}P_fx = B\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1}x,$$
$$\mathcal{H}^{-1}P_cB(\tilde{R}^*)^{-1}B^*x = P_f\mathcal{H}B(FVF^*)^{-1}B^*x,$$

for all $x \in D(A) \cap D(A^*)$. They will be satisfied if $\tilde{R} = FVF^*$ and $\mathcal{H}P_f = P_c\mathcal{H}^{-1}$. On the other hand, the LQG controller ensures the exp. stability of the closed-loop system. Therefore, all the eigenvalues of the closed-loop system must be in the left of the complex plane and thus the operator $\mathcal{R}$ has to be positive semi-definite. Otherwise, there would exist a vector $d \neq 0$ s.t. $\langle d, \mathcal{R}d\rangle_{\mathcal{X}} = \lambda\|d\|^2 < 0$ for which the dynamic of the closed-loop system would not be exp. stable. By using (37) and (38) in the FORE, we get that

$$[\mathcal{J}P_c\mathcal{H}^{-1} - \mathcal{H}^{-1}P_c\mathcal{H}^{-1}\mathcal{J}\mathcal{H} - \mathcal{H}^{-1}P_cB\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1} + HQH^*]x = 0,$$

for $x \in D(A) \cap D(A^*)$. Factorizing $\mathcal{H}^{-1}$ on both sides and since $\mathcal{H}^{-1}$ is injective, it follows that

$$[\mathcal{H}\mathcal{J}P_c - P_c\mathcal{H}^{-1}\mathcal{J}\mathcal{H}^2 - P_cB\tilde{R}B^*P_c + \mathcal{H}HQH^*\mathcal{H}]x = 0. \tag{41}$$

Subtracting (41) from the CORE given by

$$[-\mathcal{J}\mathcal{H}P_c + P_c\mathcal{J}\mathcal{H} - P_cB\tilde{R}^{-1}B^*P_c + \mathcal{H}BB^*\mathcal{H}]x = 0, \tag{42}$$

we deduce (39), which completes the proof. ∎

Under the conditions (37)-(39) of Theorem 3, the LQG controller conserves the stochastic port-Hamiltonian structure and the LQG control problem should be interpreted as the feedback interconnection of infinite-dimensional stochastic port-Hamiltonian systems.

Remark 1:

1) The relation (38) breaks the separation principle so that the optimal control and the mean-square estimation problem cannot be considered separately anymore. In other words, the focus is either on the control or on the filter, but not on both. If the optimal control problem is considered first, then the weighing operators are chosen first and the covariance operators are set w.r.t. (37) and (39), and vice versa. In this case, the covariance operators do not have a statistical meaning anymore. There are considered as control parameters as the weighing operators in the cost.
2) Since the port-Hamiltonian systems are interconnected in a power conserving way, i.e. $u(t) = -Z_c(t)$ and $u_c(t) = Z(t)$, the feedback interconnection of a stochastic port-Hamiltonian system with a LQG controller is still a stochastic port-Hamiltonian system. Therefore, the structure is conserved in the closed-loop dynamics.
3) The domains of A and A^* are equal when A generates a unitary group, i.e. when $W_B\Sigma W_B^T = 0$. Indeed, this condition is equivalent to M being unitary by [LZM05, Lemma A.1] and since $S = \frac{1}{2}(W_1 + W_2)$ is invertible, we deduce that

$$\begin{aligned}&\text{Ker}\begin{bmatrix} -(I+M^T) & (I-M^T) \end{bmatrix}\\ &= \text{Ker}\begin{bmatrix} -M^T(I+M & I-M) \end{bmatrix}\\ &= \text{Ker}\,(-M^TS^{-1}W_B) = \text{Ker}\,W_B,\end{aligned}$$

which entails that the kernel of W_B is equal to the kernel of $\begin{bmatrix} -(I+M^T) & (I-M^T) \end{bmatrix}$. The vibrating string or the Timoshenko beam fall within the port-Hamiltonian framework, see [JZ12, Chapter 7]. Moreover, consider that the left extremity is clamped and that the right extremity is let free for control purpose. One can easily check that for these examples, we have the generation of a unitary group and then $D(A) = D(A^*)$. However, these examples are at most strongly stabilizable. Therefore, to satisfy the exponential stabilizability assumptions, dissipative elements would need to be added to get the exp. stabilizability of the system.

Usually, the Riccati equations (23) and (24) cannot be exactly solved for practical problems. Therefore, a suitable finite-dimensional approximation of (3) must be found. In recent years, efficient numerical methods, including Newton-Kleinman algorithms, were proposed in [MN10] and [GM96] to solve the Riccati equation for finite-dimensional approximations.

Eventually, the authors would like to point out that the method of spectral factorization by symmetric extraction was considered and investigated in [CW92]. It leads to a general methodology which allows to conserve the distributed nature of the system to derive the optimal feedback law. The latter method would probably require to solve challenging numerical problems for the class of SPHSs, due to the lack of knowledge on the eigenvalues and eigenfunctions for port-Hamiltonian systems.

V. Example: Stochastic vibrating string

Let us consider the case of an inhomogeneous vibrating string in random media governed by the following SPDE

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right) + \frac{1}{\rho(\zeta)}\eta(\zeta,t).$$

The boundary conditions are set as follows:

$$\frac{\partial w}{\partial t}(a,t) = 0, \qquad T(b)\frac{\partial w}{\partial \zeta}(b,t) + k\frac{\partial w}{\partial t}(b,t) = 0, \tag{43}$$

where $k \geq 0$ and the initial condition $w(\zeta, 0) = w_0(\zeta)$ is a real-valued random variable with a given Gaussian distribution. For further details on the deterministic case, interested readers are referred to [JZ12]. In addition the string is assumed to be actuated at the right extremity by distributed forces $b(\zeta)u(t)$ on $[b-\epsilon, b]$ where

$$b(\zeta) = \begin{cases} 1, & \zeta \in [b-\epsilon, b], \\ 0, & \text{elsewhere} \end{cases}.$$

Moreover, the observation process is given by the noisy mean value velocity, i.e.

$$y(t) = \int_0^t \int_a^b b(\zeta)\frac{\partial w}{\partial s}(\zeta,s)d\zeta ds + \sigma v(t), \tag{44}$$

where $\sigma \in \mathbb{R}$ and $v(t)$ is a real scalar valued Wiener process with unit incremental variance. The SPDE falls within the class of SPHSs and can be written as (3) by setting

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}. \tag{45}$$

The control operator $B : \mathbb{R} \to \mathcal{X}$ is given by

$$Bu := \begin{pmatrix} b(\zeta) \\ 0 \end{pmatrix} u(t) \tag{46}$$

and the noise intensity is given by $H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The boundary conditions yield a boundary matrix W_B given by

$$W_B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & k & k & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \tag{47}$$

which satisfies $W_B \Sigma W_B^T \geq 0$ and is full rank. Thus, A generates a contraction C_0-semigroup.
The functional cost is taken as

$$J(u) = \lim_{T\to\infty} \mathbb{E} \int_0^T \int_a^b b^2(\zeta) \left(\frac{\partial w}{\partial t}(\zeta, t) \right)^2 d\zeta + u^2(t) dt. \tag{48}$$

From [JZ12, Example 9.2.1], the system is exp. stable with small stability margin whenever the damping parameter k is small. The purpose would be to reach better stability margins at least for dominant modes of the system. Under the assumptions of Theorem 2, the LQG optimal control law is given by (28) and (29).

VI. Conclusion

In this paper a first investigation of the LQG control problem for infinite-dimensional stochastic port-Hamiltonian systems is presented. Based on [CI77] and [BV75], the separation principle is stated for SPHSs. Besides, the adjoint operator with domain of the dynamic operator is made explicit for port-Hamiltonian systems.
In particular, the preservation of the infinite-dimensional stochastic port-Hamiltonian framework in the LQG controller dynamic is investigated. As an infinite-dimensional extension of [WHGM14], it is proved that, under some conditions, the LQG control problem can be interpreted as the interconnection of infinite-dimensional stochastic port-Hamiltonian systems.

VII. Perspectives

Since most port-Hamiltonian systems are only strongly stabilizable and in order to set up the LQG control problem for SPHSs completely, the theory presented in this paper needs to be extended.
A numerical method for solving LQG control problems of stochastic port-Hamiltonian systems by the resolution of the Riccati equations (23) and (24) still has to be developed based on [MN10] and [GM96] for instance. Moreover, while most approximation are based on the spatial discretization of the geometric Dirac structure, a Galerkin method preserving the port-Hamiltonian framework is introduced in [HD12]. It has to be adapted to the class of SPHSs presented here. Indeed, the operator H acts on the Z-valued Wiener process $(w(t))_{t\geq 0}$, which leads to major difficulties in a numerical method. Hence, $(w(t))_{t\geq 0}$ has to be approximated. To do so, the Karhunen-Loève expansion (see [DPZ08, Proposition 4.3]) could be applied.
Furtermore, the authors would like to stress that in order to solve the stochastic LQ optimal control problem with complete observation, a new promising approach based on the Wiener chaos expansion via Wick-Hermite polynomials (see [Wie38] and [CM47]) was proposed in [TL16]. This allows to covert the stochastic LQ problem to an infinitely many deterministic control problem so that the randomness property is captured in Wick-Hermite basis.
Finally, the LQG control problem with boundary control and observation for SPHSs is not considered in this paper. The resulting unboundedness of the control and observation operators leads to mathematical difficulties. It is worth noticing that the case of control and observation applied at the boundary is a natural extension of this work and it still being investigated by the authors.

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