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Certificates of polynomial nonnegativity via hyperbolic optimization

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***Abstract*— We present a new approach to certifying the nonnegativity of homogeneous multivariate polynomials that is based on the theory of hyperbolic polynomials. Moreover, the search for certificates of nonnegativity can be automated by solving a hyperbolic optimization problem. The main technical fact that enables these nonnegativity certificates is a polynomial parameterization (up to closure) of the dual cone of a hyperbolicity cone, a construction essentially due to Kummer, Plaumann, and Vinzant. This extended abstract presents the basic idea of such hyperbolic certificates of nonnegativity, and discusses what is known about the relationship between sums of squares and polynomials with hyperbolic certificates of nonnegativity.**

I. EXTENDED ABSTRACT

The problem of deciding nonnegativity of a multivariate polynomial is a fundamental problem that arises, for instance, when searching for certificates of stability of dynamical systems, or as an approach to globally solving polynomial optimization problems (see, e.g., [1]).

Deciding nonnegativity of a multivariate polynomial is, in general, a computationally challenging problem. One way to make progress is to find sufficient conditions for polynomial nonnegativity that can be verified in a computationally efficient manner. One such condition is that if a polynomial can be written as a sum of squares, then it is nonnegative. The converse holds for homogeneous polynomials of degree $2d$ in n variables if and only if $2d = 2$, $n = 2$, or $(n, 2d) = (3, 4)$, a classical result of Hilbert [2].

While it is difficult to decide whether a polynomial is nonnegative, deciding whether a polynomial is a sum of squares of other polynomials is much more tractable. In fact, it can be recast as the problem of deciding whether an affine subspace intersects the cone of positive semidefinite matrices. This is a semidefinite feasibility problem that can typically be solved using numerical methods for semidefinite optimization.

Hyperbolic optimization is a natural generalization of semidefinite optimization. In hyperbolic optimization the role of the positive semidefinite cone is replaced by a convex cone constructed from a hyperbolic polynomial—a multivariate polynomial with certain real-rootedness properties (see Section I-A). One can recover the positive semidefinite cone by taking the hyperbolic polynomial to be the determinant restricted to symmetric matrices. Hyperbolic optimization problems can, in principle, be solved efficiently via interior point algorithms. Can we exploit this to obtain classes of certificates of the nonnegativity of multivariate polynomials that are different from sum-of-squares certificates?

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In this extended abstract, we introduce *hyperbolic certificates of nonnegativity*. Given a hyperbolic polynomial we construct a convex set of polynomials, the nonnegativity of which is a consequence of the hyperbolicity of p. Moreover we can search over this family of nonnegative polynomials via solving a hyperbolic optimization problem. The approach is based on the existence of a polynomial map, the image of which is (up to closure) the dual of the hyperbolicity cone (see (1) for the definition of a hyperbolicity cone). This map was identified by Kummer, Plaumann, and Vinzant [3], who also established a number of its fundamental properties.

Beyond introducing these families of certificates, and discussing some of their properties that can be easily deduced from prior work, our main contributions are:

- to show that the nonnegativity of any sum of squares can also be certified using a hyperbolic certificate of nonnegativity (Theorem 1.6);
- to show that for any $d \geq 4$ and any $n \geq d$ there exists a polynomial of degree $2d-2$ in n variables that is not a sum of squares, but the nonnegativity of which can be certified using an appropriate hyperbolic certificate of nonnegativity (Theorem 1.9).

A. Hyperbolic polynomials and hyperbolic optimization

A homogeneous polynomial p in n variables of degree d is *hyperbolic with respect to* $e \in \mathbb{R}^n$ if $p(e) \geq 0$ and, for all $x \in \mathbb{R}^n$, the univariate polynomial $t \mapsto p(x + te)$ has only real roots. Associated with any hyperbolic polynomial p, and direction of hyperbolicity e, is a closed cone

$$\begin{aligned}\Lambda_+(p, e) = \{x \in \mathbb{R}^n :\\ \text{all roots of } t \mapsto p(te - x) \text{ are nonnegative}\}. \quad (1)\end{aligned}$$

That this is always a convex cone is a classical result due to Gårding [4]. These convex cones, called *hyperbolicity cones*, are of interest in optimization because the convex function $-\log_e(p(x))$ is a self-concordant barrier function for the cone $\Lambda_+(p, e)$. As such, as long as p (and its gradient and Hessian) can be computed efficiently, convex optimization problems of the form

$$\min_x \langle c, x\rangle \;\text{ subject to }\; \begin{cases} Ax = b, \\ x \in \Lambda_+(p, e)\end{cases}$$

can be solved using interior point methods [5], [6]. Such problems are called *hyperbolic programs*, or *hyperbolic optimization problems*.

B. Conic certificates of nonnegativity

There is a natural general strategy for certifying nonnegativity of polynomials (and other types of functions, see [7] for interesting examples) by solving convex feasibility problems. We briefly outline this strategy before describing how to obtain instances of the general approach via hyperbolic polynomials.

Suppose $K \subseteq \mathbb{R}^n$ is a convex cone and

$$K^* = \{\ell \in (\mathbb{R}^n)^* \ : \ \ell[x] \geq 0 \ \text{ for all } x \in K\}$$

is its *dual cone*. (We use the notation $\ell[x]$ for the image of x under a linear functional ℓ.) Let $\psi : \mathbb{R}^m \to (\mathbb{R}^n)^*$ be a homogeneous polynomial mapping such that $\psi(x) \in K^*$ for all $x \in \mathbb{R}^m$. Given a homogeneous polynomial f of degree d, we can try to *certify* the nonnegativity of f by solving the convex feasibility problem:

$$\text{find } \ v \in K \ \text{ such that } \ f(x) = \psi(x)[v] \ \text{ for all } x \in \mathbb{R}^m.$$

If we can find such a $v \in K$, then f is necessarily nonnegative. This is because $v \in K$ and $\psi(x) \in K^*$ for all x, so (by the definition of the dual cone) $\psi(x)[v] \geq 0$ for all x. Another view of this construction is that it produces a convex cone of nonnegative polynomials $\{\psi(x)[v] \ : \ v \in K\}$, over which optimization problems may be formulated.

To make this formalism computational, we need

- a convex cone K with respect to which we can (efficiently) solve the associated conic feasibility problem;
- a homogeneous polynomial map ψ such that K^* is the closure of the convex hull of the image of ψ.

Certificates of polynomial nonnegativity via sums of squares have exactly this form, via the following well-known reformulation. The map $\psi(x)$ has the form $\psi(x) = m_d(x)m_d(x)^T$ where $m_d(x)$ is the vector consisting of all monomials of degree d in n variables. The cone K is the (self-dual) positive semidefinite cone. An expression of the form

$$f(x) = \operatorname{tr}\left(m_d(x)m_d(x)^T G\right)$$

where G is positive semidefinite, gives rise to an expression for f as a sum of squares. Indeed, if we expand G as $G = \sum_i v_i v_i^T$, then

$$f(x) = \operatorname{tr}\left(m_d(x)m_d(x)^T \sum_i v_i v_i^T\right) = \sum_i (v_i^T m_d(x))^2$$

is a sum of squares of polynomials.

C. Hyperbolic certificates of nonnegativity

We now outline how to use hyperbolic polynomials and hyperbolic programming to obtain maps ψ and cones K (beyond the positive semidefinite cone) to use in the framework of the previous section.

Let p be a homogeneous polynomial of degree d, hyperbolic with respect to $e \in \mathbb{R}^n$. Let $\Lambda_+(p,e)$ be the associated hyperbolicity cone. Next we describe a homogeneous polynomial map $\phi_{p,e} : \mathbb{R}^n \to \mathbb{R}^n$ that takes values in $\Lambda_+(p,e)^*$, the dual of the hyperbolicity cone.

If $y \in \mathbb{R}^n$ then let

$$Dp(x)[y] = \left.\frac{d}{dt}p(x+ty)\right|_{t=0}$$

be the directional derivative of p in the direction y. Let

$$D^2p(x)[y,y] = \left.\frac{d^2}{dt^2}p(x+ty)\right|_{t=0}$$

and extend $D^2p(x)[\cdot,\cdot]$ to a bilinear form by polarization. For each $x \in \mathbb{R}^n$, define $\phi_{p,e}(x) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ by

$$\phi_{p,e}(x)[v] = Dp(x)[e]Dp(x)[v] - p(x)D^2p(x)[e,v]. \quad (2)$$

Note that $\phi_{p,e}(x)[\cdot]$ is a polynomial mapping, homogeneous of degree $2d-2$. The following result of Kummer, Plaumann, and Vinzant [3, Theorem 3.1] establishes a close link between the hyperbolicity cone $\Lambda_+(p,e)$ and the nonnegativity of $\phi_{p,e}(x)[\cdot]$.

Theorem 1.1 (Kummer, Plaumann, Vinzant): If p is squarefree, homogeneous of degree d, and hyperbolic with respect to $e \in \mathbb{R}^n$ then

$$\phi_{p,e}(x)[v] \geq 0 \ \text{ for all } x \in \mathbb{R}^n \quad \iff \quad v \in \Lambda_+(p,e).$$

On the one hand, one can interpret this result as a description of any hyperbolicity cone as a slice of the cone of nonnegative polynomials of degree $2d-2$ in n variables. This is the interpretation that was the focus of Kummer, Plaumann, and Vinzant's work. On the other hand, we could also interpret this result as providing a family of non-negative polynomials that we can search over efficiently using hyperbolic optimization. This is the interpretation we focus on in this extended abstract.

We can use this basic result in a more elaborate way in the formalism introduced in Section I-B, as follows.

Definition 1.2 (Hyperbolic certificates of nonnegativity): Let $F(x) : \mathbb{R}^m \to \mathbb{R}^n$ be a homogeneous polynomial map. Let p be homogeneous of degree d and hyperbolic with respect to $e \in \mathbb{R}^n$. Define $\psi : \mathbb{R}^m \to (\mathbb{R}^n)^*$ by

$$\psi(x)[\cdot] := \phi_{p,e}(F(x))[\cdot] \quad \text{for all } x \in \mathbb{R}^m.$$

Then any polynomial of the form $\psi(x)[v]$ where $v \in \Lambda_+(p,e)$, is nonnegative, and we call it a *hyperbolic certificate of nonnegativity*.

Given a multivariate polynomial f, we can search for a hyperbolic certificate of nonnegativity of f by choosing F, p, and e in Definition 1.2, and solving the convex decision problem

$$\text{find } \ v \in \Lambda_+(p,e) \ \text{ s.t. } \ f(x) = \psi(x)[v] \ \text{ for all } x \in \mathbb{R}^m.$$

D. The map $\phi_{p,e}$ almost parameterizes the dual cone

A dual interpretation of Kummer, Plaumann, and Vinzant's result (Theorem 1.1) is that the closure of the conic hull of the image of $\phi_{p,e}$ is the dual cone $\Lambda_+(p,e)^*$. The next result tells us that the image of $\phi_{p,e}$ almost fills up all of $\Lambda_+(p,e)^*$.

Proposition 1.3: If p is homogeneous of degree d and hyperbolic with respect to $e \in \mathbb{R}^n$ then

$$\text{interior}(\Lambda_+(p,e)^*) \subseteq \phi_{p,e}(\mathbb{R}^n) \subseteq \Lambda_+(p,e)^*.$$

Here the right-hand inclusion is from Theorem 1.1, in the case where p is squarefree, and can easily be extended to the case in which p has repeated irreducible factors. The left-hand inclusion is essentially due to Güler (see [5, Theorem 6.2]).

E. Examples

We now present two of the most basic examples of hyperbolic polynomials, their hyperbolicity cones, and the associated map $\phi_{p,e}$.

Example 1.4 (Non-negative orthant): Let

$$p(x) = \prod_{i=1}^{n} x_i \quad \text{and} \quad e = 1_n.$$

In this case $\Lambda_+(p,e) = \Lambda_+(p,e)^* = \mathbb{R}^n_+$, the non-negative orthant. The mapping $\phi_{p,e}$ is given by

$$\phi_{p,e}(x)[u] = \sum_{j=1}^{n} u_j \prod_{i \neq j} x_i^2.$$

Clearly $\phi(x)[\cdot] \in (\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ for all x. Moreover, the image of $\phi_{p,e}$ contains all of the extreme rays of the cone. Indeed let $x^{(i)} \in \mathbb{R}^n$ be such that $x^{(i)}_j = 1$ if $j \neq i$ and $x^{(i)}_i = 0$. Then $\phi_{p,e}(x^{(i)})$ is the linear functional such that $\phi_{p,e}(x^{(i)})[u] = u_i$.

We note that the image of $\phi_{p,e}$ is not all of the orthant. For example, there is no $x \in \mathbb{R}^n$ such that

$$\phi_{p,e}(x) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Example 1.5 (Positive semidefinite cone): Let $p(X) = \det(X)$ where X is a symmetric matrix of indeterminates and let $e = I_n$ be the $n \times n$ identity matrix. In this case $\Lambda_+(p,e) = \Lambda_+(p,e)^* = \mathcal{S}^n_+$, the positive semidefinite cone. The mapping $\phi_{p,e}$ is given by

$$\phi_{p,e}(X)[U] = \text{tr}(U \text{adj}(X)^2)$$

where $\text{adj}(X)$ is the classical adjoint of X, given by $\det(X)X^{-1}$ for non-singular matrices X. Clearly $\text{adj}(X)^2 \in \mathcal{S}^n_+$ for all X because it is the square of a symmetric matrix. Moreover, the image of $\phi_{p,e}$ contains all of the extreme rays of the cone. Indeed if w is a unit vector in $\mathbb{R}^n$ then

$$\text{adj}(I - ww^T)^2 = ww^T ww^T = ww^T.$$

F. Relationship with sums of squares

In section I-C we have seen how to construct families of nonnegative polynomials that have hyperbolic certificates of nonnegativity from a choice of hyperbolic polynomial p, a direction of hyperbolicity e, and a polynomial map F. In this section, we discuss how these families of polynomials relate to sums of squares.

The following result shows that by an appropriate choice of hyperbolic polynomial p, direction of hyperbolicity e, and map F, we can establish nonnegativity of all homogeneous polynomials that are sums of squares.

Theorem 1.6: Suppose q is a homogeneous polynomial of degree $2d$ in n variables that is a sum of squares. Let $p(X) = \det(X)$ where X is $\binom{n+d-1}{d} \times \binom{n+d-1}{d}$. Let $m_d(x)$ denote the vector of monomials homogeneous of degree d in n variables. Let $F(x) = m_d(x)^T m_d(x) I - m_d(x) m_d(x)^T$. Then there exists $Q \in \Lambda_+(p, I) = \mathcal{S}_+^{\binom{n+d-1}{d}}$ such that

$$(m_d(x)^T m_d(x))^{2\binom{n+d-1}{d}-3} q(x) = \phi_{\det, I}(F(x))[Q].$$

Note that with this choice of $(p,e) = (\det, I)$, the hyperbolic optimization problem we would solve to certify nonnegativity of q would be exactly the usual semidefinite optimization problem we would solve to decide whether q is a sum of squares. (Note that, in order to make the degrees match, we needed to multiply on the left by a power of the known polynomial $m_d(x)^T m_d(x)$. We could, equivalently, ask for equality just for points x on the sphere. At these points $m_d(x)^T m_d(x) = 1$.)

We have seen that hyperbolic certificates of nonnegativity can, with careful selection of the map F and the hyperbolic polynomial p, establish nonnegativity of all polynomials that are sums of squares. We now consider the reverse situation. Is it the case that $\phi_{p,e}(x)[u]$ is always a sum of squares when $u \in \Lambda_+(p,e)$?

We first review two results from [3]. The first shows that if a polynomial has a definite determinantal representation, then any corresponding hyperbolic certificate of nonnegativity is a sum of squares.

Theorem 1.7 (Kummer, Plaumann, Vinzant): Suppose there are symmetric matrices $A_1, \ldots, A_n$ such that $\sum_{i=1}^n A_i e_i$ is positive definite, and there exists a positive integer α such that $p(x)^\alpha = \det\left(\sum_{i=1}^n A_i x_i\right)$. Then, for any homogeneous polynomial map $F : \mathbb{R}^\ell \to \mathbb{R}^n$, and any $u \in \Lambda_+(p,e)$, $\phi_{p,e}(F(x))[u]$ is a sum of squares.

The next result follows directly from Theorem 1.7 and the Helton-Vinnikov Theorem [8], [9], which implies that any hyperbolic polynomial in three variables has a definite determinantal representation.

Corollary 1.8 (Kummer, Plaumann, Vinzant): If p is a hyperbolic polynomial in three variables, $F : \mathbb{R}^\ell \to \mathbb{R}^3$ is any polynomial map, and $u \in \Lambda_+(p,e)$, then $\phi_{p.e}(F(x))[u]$ is a sum of squares.

One may wonder whether $\phi_{p,e}(x)[u]$ is always a sum of squares when $u \in \Lambda_+(p,e)$. We have seen that this is the case for polynomials in three variables. It is also the case for hyperbolic polynomials p of degree 2, because, in this case, for each $u \in \Lambda_+(p,e)$ the polynomial $\phi_{p,e}(x)[u]$ is nonnegative and of degree two, and so is a sum of squares. In [3, Example 5.11] it is shown (but not explicitly) that there exists a hyperbolic polynomial p of degree four in eight variables, and an element $u \in \Lambda_+(p,e)$, such that $\phi_{p,e}(x)[u]$ is not a sum of squares. The following result, the main result of this extended abstract, asserts that many more such examples can be found.

Theorem 1.9: For any $d \geq 4$ and any $n \geq d$ there exists a homogeneous polynomial p of degree d, hyperbolic with respect to $e \in \mathbb{R}^n$, and some $u \in \Lambda_+(p, e)$, such that $\phi_{p,e}(x)[u]$ is not a sum of squares.

In the case $n = d = 4$, the choice of p is exactly the specialized Vámos polynomial $h_4(w, x, y, z)$, considered by Kummer [10], in the direction $e = (0, 1, 1, 0)$ and $u = (0, 1, 0, 0)$. Examples with degree d and number of variables n satisfying $4 \leq d \leq n$ can be constructed based on this example.

It is natural to conjecture that such examples exist whenever the degree (of the hyperbolic polynomial) is at least three, and the number of variables is at least four.

Conjecture 1.10: If $d \geq 3$ and $n \geq 4$, there exists a polynomial p homogeneous of degree d and hyperbolic with respect to $e \in \mathbb{R}^n$ and a point $u \in \Lambda_+(p, e)$ such that $\phi_{p,e}(x)[u]$ is not a sum of squares.

G. Discussion

One practical issue with experimenting with hyperbolic certificates of nonnegativity is the lack of widely available numerical methods for solving hyperbolic optimization problems. In some cases (see, e.g., [11]), explicit semidefinite programming descriptions of hyperbolicity cones are known, and could be used. One motivation for this work is to highlight possible systematic ways to use hyperbolic optimization, in order to spur the development of tools to solve this interesting class of optimization problems.

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