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Estimation of infinite-dimensional systems with unknown disturbances Extended Abstract

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***Abstract*— For many problems of estimation there are disturbances other than Gaussian noise. Also, for many applications only some aspect of the state needs to be estimated, not the whole state. In an H_∞ estimation approach the aim is to reduce the estimation error over all disturbances. A derivation of H_∞-output estimation in the infinite-dimensional setting is provided. A framework for calculation using finite-dimensional approximations is described. Output estimation is compared with a Kalman filter for an example with disturbances.**

Keywords: distributed parameter systems, infinite-dimensional systems, estimator, estimation, H_∞

I. Introduction

In most systems the full state is not measured, but must be estimated based on the available measurements. The most well known estimation technique is the Kalman filter, which minimizes the estimation error variance under assumptions on the process and sensor noise. The well-posedness of minimizing the error variance at a finite-time for distributed parameter systems was established in [1]; see also the book [2]. It was shown in [3] that the infinite-time Kalman filter, which is obtained by solving an algebraic Riccati equation, minimizes the steady-state error variance for distributed parameter systems.

In many systems the disturbance is unknown and may not be Gaussian noise. Then a reasonable objective is to find an estimate with error that is small over all disturbances. This is sometimes formulated as a minimax game problem: the designer is trying to minimize the error while some other player is choosing the disturbance to maximize the error.

Also, in many applications only some aspect of the state needs to be estimated, not the whole state. The problem is to estimate a linear combination of the states $C_1 z$ where z is the state and C_1 may be an operator other than the identity. This occurs when only some subset of the states is of interest. Another situation where C_1 may not be the identity is when C_1 is state feedback, and the estimator is to be combined with state feedback to construct an output feedback controller.

The relationship between design for what is known as the full control problem and the estimation problem has been used in the development of an output feedback controller for $\mathcal{H}_\infty$ control; see [4] for the finite-dimensional result and [5] for the extension to infinite-dimensions. The full control problem is used to find a good estimate of Kz where K is the state feedback that solves the full information problem. It is assumed in the development that $A-B_2K$ is exponentially stable, which is reasonable when K is designed as state feedback. However, estimation without control is sometimes of interest. Estimation in the $\mathcal{H}_\infty$ context without control was shown for finite-dimensional systems in [6]; see also the tutorial article [7].

In this talk a solution to estimation for infinite-dimensional systems with unknown disturbances is presented. A framework for calculation of a sub-optimal estimator using finite-dimensional approximation is described. The approach is illustrated with an example.

II. Problem Formulation

Let v indicate all external disturbances and write the system as

$$\begin{aligned} \dot{z}(t) &= Az(t) + B_1 v(t), \quad z(0) = 0 \\ y_1(t) &= C_1 z(t) + u(t) \\ y_2(t) &= C_2 z(t) + D_{21} v(t) \end{aligned} \tag{OE}$$

where z is the state, A with domain $D(A)$ generates a strongly continuous semigroup $S(t)$ on a Hilbert space $\mathcal{Z}$, and $B_1 \in \mathcal{L}(\mathcal{U}_1, \mathcal{Z})$, $C_1 \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}_1)$, $C_2 \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}_2)$ where $\mathcal{L}(\mathcal{U}_1, \mathcal{Z})$ indicates bounded linear operators from a separable Hilbert space $\mathcal{U}_1$ to $\mathcal{Z}$. The disturbance term v is generally due to uncontrolled inputs, such as process noise and sensor noise, but sometimes modelling errors are regarded as disturbances to the model. The goal is to find an estimate $\hat{z}(t)$ of the state $z(t)$, so that $C_1(z(t) - \tilde{z}(t))$ is small, based only on external signals y_2, v and the model (OE).

Finding an estimate for $C_1 z(t)$ using only the measurements $y_2(t)$ can be formulated as finding $u(t)$ so that $y_1(t)$ is as small as possible. Then $u(t)$ will be an estimate of $-C_1 z(t)$. The problem is thus to find a system H so that with $u(s) = H(s) y_2(s)$; the output y_1 of (OE) satisfies, for some desired error $\gamma > 0$,

$$\sup_{v \in \mathcal{L}_2(0,\infty;\mathcal{V})} \|y_1\|_2 < \gamma. \tag{1}$$

This corresponds to the $\mathcal{H}_\infty$-norm of the transfer function from v to y_1 being less than γ and so this approach is referrred to as *$\mathcal{H}_\infty$-estimation.*

The estimation problem is closely related to the full control

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problem

$$\begin{aligned}\dot{z}_1(t) &= Az(t) + B_1 v(t) + \begin{bmatrix} I & 0 \end{bmatrix} \tilde{v}(t), \quad z(0) = 0\\ y_1(t) &= C_1 z_1(t) + \begin{bmatrix} 0 & I \end{bmatrix} \tilde{v}(t) \\ y_2(t) &= C_2 z_1(t) + D_{21} v(t).\end{aligned} \tag{FC}$$

The two systems differ only in the effect of the controlled input u (or $\tilde{u}$) on the state and on the cost y_1.

Theorem 1: Consider the output estimation problem (OE) and the full control problem (FC).

1) The map from v to y_1 of the output estimation system with control u is equal to that of the full control system with control
$$\tilde{v} = \begin{bmatrix} 0 \\ I \end{bmatrix} u.$$
2) The map from v to y_1 of the full control problem (FC) is equal to that of the output estimation system connected to the following system
$$\begin{aligned}\dot{z}_e(t) &= Az_e(t) + \begin{bmatrix} I & 0 \end{bmatrix} \tilde{v}(t) \quad z_e(0) = 0,\\ u(t) &= C_1 z_e(t) + \begin{bmatrix} 0 & I \end{bmatrix} \tilde{v}(t) \\ \tilde{y}(t) &= C_2 z_e(t) + y_2(t).\end{aligned} \tag{P}$$

Proof: The approach is similar to the intermediate step of constructing an output feedback controller; see [4] or [8, chap. 8] for the finite-dimensional case and [5] for the infinite-dimensional development.

Statement (1) follows from simple substitution.

Consider then statement (2). Eliminating the intermediate variables $u(t)$ and $y_2(t)$ the system (OE) connected to (P) is

$$\begin{aligned}\begin{bmatrix} \dot{z}(t) \\ \dot{z}_e(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} z(t) \\ z_e(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} v(t) + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \tilde{v}(t)\\ y_1(t) &= \begin{bmatrix} C_1 & C_1 \end{bmatrix} \begin{bmatrix} z(t) \\ z_e(t) \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} \tilde{v}(t)\\ \tilde{y}(t) &= \begin{bmatrix} C_2 & C_2 \end{bmatrix} \begin{bmatrix} z(t) \\ z_e(t) \end{bmatrix} + D_{21} v(t).\end{aligned}$$

The state space for this system is $\mathcal{Z} \times \mathcal{Z}$. The generator is $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ with domain $D(A) \times D(A)$.

Only the sum $z + z_e$ affects y_1 and $\tilde{y}$. Define a new state $\tilde{z} = \begin{bmatrix} z + z_e \\ z_e \end{bmatrix}$, on the same state space. The system description has the same generator, but is now

$$\begin{aligned}\dot{\tilde{z}}(t) &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \tilde{z}(t) + \begin{bmatrix} B_1 \\ B_1 \end{bmatrix} v(t) + \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix} \tilde{v}(t)\\ y_1(t) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \tilde{z}(t) + \begin{bmatrix} 0 & I \end{bmatrix} \tilde{v}(t)\\ \tilde{y}(t) &= \begin{bmatrix} C_2 & 0 \end{bmatrix} \tilde{z}(t) + D_{21} v(t).\end{aligned}$$

Thus, the second component z_e is unobservable and does not affect y_1 or $\tilde{y}$. The map from $\tilde{v}$ to $\tilde{y}$ is the same as that of

$$\begin{aligned}\dot{z}_1(t) &= Az_1(t) + B_1 v(t) + \begin{bmatrix} I & 0 \end{bmatrix} \tilde{v}(t)\\ y_1(t) &= C_1 z_1(t) + \begin{bmatrix} 0 & I \end{bmatrix} \tilde{v}(t)\\ \tilde{y}(t) &= C_2 z_1(t) + D_{21} v(t).\end{aligned}$$

This is the full control system (FC). Thus, with zero initial conditions and the same disturbance v and control signal $\tilde{v}$, the full control system (FC) will have the same output y_1 as the connected systems (OE), (P). ∎

Theorem 1 implies that if a suitable controller is found for the full control problem (FC) a controller can be constructed with the same closed loop input-output map for the output estimation problem (OE). Letting A^* with domain $D(A^*)$ indicate the adjoint operator of A, and indicating similarly the adjoint operators of B_1,C_1, C_2, D_{21}, the full control problem is dual to the full information problem

$$\begin{aligned}\dot{z}(t) &= A^* z + C_1^* v(t) + C_2^* u(t), \quad z(0) = 0\\ y_1(t) &= B_1^* z(t) + D_{21}^* u(t)\\ y_2(t) &= \begin{bmatrix} I \\ 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} v(t).\end{aligned} \tag{FI}$$

Thus the solution to the full control problem follows immediately from that for full information. This duality and Theorem 1 can be used to construct an estimator u for the original problem OE.

III. Estimator Design

A result for synthesis of an estimator with the desired estimation error over all disturbances is presented here. It will be assumed that

$$\begin{bmatrix} D_{21} \\ B_1 \end{bmatrix} D_{21}^* = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{2}$$

in order to simplify the formulae. As long as D_{21} has full row rank, a transformation can be used to put the problem into this form.

Theorem 2: Assume that (A, B_1) is stabilizable and $(A, C_2$ is detectable. There is an estimate of $C_1 z$ so that the error satisfies (1) if and only if there is a solution $\Pi \geq 0$ of

$$\Pi A^* + A\Pi + \Pi \left(\frac{1}{\gamma^2} C_1^* C_1 - C_2^* C_2 \right) \Pi + B_1 B_1^* = 0 \tag{3}$$

such that $A + \Pi \left(\frac{1}{\gamma^2} C_1^* C_1 - C_2^* C_2 \right)$ generates an exponentially stable semigroup. Defining $F = \Pi_2 C_2^*$, the optimal estimate u is

$$\begin{aligned}\dot{z}_e(t) &= (A - FC_2) z_e(t) - F y_2(t), \quad z_e(0) = 0,\\ u(t) &= C_1 z_e(t)\end{aligned} \tag{4}$$

Proof: From Theorem 1, there is an estimator that achieves $\mathcal{H}_\infty$-norm less than γ if and only if there is a controller for the full control problem (FC) with $\mathcal{H}_\infty$- attenuation less than γ. The full control problem is dual to the full information problem and so there is a control that achieves such attenuation if and only if there is a solution to (3) such that $A + \Pi \left(\frac{1}{\gamma^2} C_1^* C_1 - C_2^* C_2 \right)$ generates an exponentially stable C_0-semigroup [5, Thm. 4.4]. Moreover, again using duality, and defining $F = \Pi C_2^*$, $A - FC_2$ generates an exponentially stable C_0-semigroup and a suitable controller for the full control problem is the feedback $F = \Pi C_2^*$. From Theorem 1, connecting this controller to the auxiliary system

(P) will yield $\mathcal{H}_\infty$-error less than γ for output estimation. This leads to

$$\begin{aligned}\dot{z}_e(t) &= Az_e(t) + \begin{bmatrix} I & 0 \end{bmatrix} \tilde{v}(t) \\ u(t) &= C_1 z_e(t) + \begin{bmatrix} 0 & I \end{bmatrix} \tilde{v}(t) \\ \tilde{y}(t) &= C_2 z_e(t) + y_2(t) \\ \tilde{v}(t) &= \begin{bmatrix} -\Pi C_2^* \\ 0 \end{bmatrix} \tilde{y}(t).\end{aligned}$$

Eliminating $\tilde{v}$ from the first equation yields (4). ■

Unlike minimum variance estimation, the definition of the output $C_1 z$ affects the design of an $\mathcal{H}_\infty$-estimator. Also, a Kalman filter is designed to minimize the error variance for a single disturbance, white noise, while the $\mathcal{H}_\infty$-estimator is designed to minimize the error over all disturbances. In this respect, an $\mathcal{H}_\infty$-estimator is more robust.

As the desired error $\gamma \to \infty$, the Kalman filter is recovered.

IV. Computation

In practice a finite-dimensional system is used to estimate the original infinite-dimensional system. Provided that a suitable approximation scheme is used, estimation error arbitrarily close to that of the full infinite-dimensional system can be obtained.

Suppose the approximation lies in some finite-dimensional subspace $\mathcal{Z}_n$ of the state-space $\mathcal{Z}$, with an orthogonal projection $P_n : \mathcal{Z} \to \mathcal{Z}_n$ where for each $z \in \mathcal{Z}$, $\lim_{n\to\infty} \|P_n z - z\| = 0$. The space $\mathcal{Z}_n$ is equipped with the norm inherited from $\mathcal{Z}$. Define $B_n = P_n B$, $C_n = C|_{\mathcal{Z}_n}$ (the restriction of C_n to $\mathcal{Z}_n$) and define $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ using some method. This leads to a sequence of finite-dimensional approximations

$$\begin{aligned}\frac{dz}{dt} &= A_n z(t) + B_n u(t), \qquad z(0) = P_n z_0, \\ y(t) &= C_n z(t).\end{aligned}$$

Let S_n indicate the semigroup generated by A_n.

Definition 3: The control systems (A_n, B_n) are *uniformly stabilizable* if there exists a sequence of feedback operators $\{K_n\}$ with $\|K_n\| \le M_1$ for some constant M_1 such that $A_n - B_n K_n$ generate $S_{Kn}(t)$, $\|S_{Kn}(t)\| \le M_2 e^{-\alpha_2 t}$, $M_2 \ge 1$, $\alpha_2 > 0$.

Definition 4: The control systems (A_n, C_n) are *uniformly detectable* if there exists a sequence of operators $\{F_n\}$ with $\|F_n\| \le M_1$ for some constant M_1 such that $A_n - F_n C_n$ generate $S_{Kn}(t)$, $\|S_{Kn}(t)\| \le M_2 e^{-\alpha_2 t}$, $M_2 \ge 1$, $\alpha_2 > 0$.

It is easy to show that if the original problem is exponentially stabilizable (detectable), and the eigenfunctions of A form an orthonormal basis for $\mathcal{Z}$, then an approximation scheme formed using the first n eigenfunctions is uniformly stabilizable (detectable). However, in practice other approximation methods, such as finite-elements are typically used. Many such approximations, such as linear splines for the diffusion equation and cubic splines for damped beam vibrations are uniformly stabilizable (detectable), provided that the original system is stabilizable (detectable) [9, Thm. 5.2,5.3].

Theorem 5: Assume that

- For each $z \in \mathcal{Z}$, and all intervals of time $[t_1, t_2]$
$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} \|S_n(t)P_n z - S(t)z\| = 0,$$
$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} \|S_n^*(t)P_n z - S^*(t)z\| \to 0;$$
- For all $u \in U$, $y \in Y$, $\|C_{1n}^* y - C_1^* y\| \to 0$ and $\|B_n^* P_n z - B^* z\| \to 0$;
- C_1 and C_2 are compact operators.
- (A_n, B_{1n}) are uniformly stabilizable and (A_n, C_{2n}) are uniformly detectable;

If there is an estimator for the original problem (OE) that achieves estimation error less than γ, then for sufficiently large n the finite-dimensional Riccati equation

$$\Pi_n A_n^* + A_n \Pi_n + \Pi_n \left(\frac{1}{\gamma^2} C_{1n}^* C_{1n} - C_{2n}^* C_{2n} \right) \Pi_n + B_{1n} B_{1n}^* = 0$$

has a nonnegative, self-adjoint solution Π_n such that

1) the semigroup $S_{n2}(t)$ generated by $A_n + \Pi_n(\frac{1}{\gamma^2} C_{1n}^* C_{1n} - C_{2n}^* C_{2n})$ is uniformly exponentially stable; that is, there exist positive constants M_1 and ω_1 with $\|S_{n2}(t)\| \le M_1 e^{-\omega_1 t}$;
2) Defining $F_n = \Pi_n C_{2n}^*$, the semigroups $S_{nK}(t)$ generated by $A_n + F_n C_{2n}$ are uniformly exponentially stable; that is, there exists $M_2, \omega_2 > 0$ wiith $\|S_{nK}(t)\| \le M_2 e^{-\omega_2 t}$.
3) As $n \to \infty$, for all $z \in \mathcal{Z}$, $\Pi_n P_n z \to \Pi z$ where Π solves (3) and also F_n converges to $F = \Pi C_2^*$ in norm.
4) The the optimal $\mathcal{H}_\infty$-estimation error $\hat{\gamma}_n$ for the approximating system converges to the optimal estimation error $\hat{\gamma}$ for (OE); that is,
$$\lim_{n\to\infty} \hat{\gamma}_n = \hat{\gamma}.$$
5) For sufficiently large n, F_n provides estimation error less than γ when used in the estimator (4).

Proof: Convergence of the optimal feedback and performance for the full information problem is in [10, Thm. 2.5,2.8]. The conclusions then follow by duality with the full control problem (FC) and Theorem 1. ■

In the talk, this approach will be illustrated with an example and comparison with a Kalman filter.

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