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Basis adaptation for a max-plus eigenvector method arising in optimal control*

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***Abstract*— A standard max-plus eigenvector method arising in nonlinear optimal control is encapsulated within a basis function adaptation iteration. A level set corresponding to the target Hamiltonian back-substitution error is estimated at each step, using the value function approximation obtained by the eigenvector method. This estimate is used to construct and add new functions to the basis employed by the eigenvector method, with the objective of enlarging the target level set iteratively. The utility of the ensuing iteration is illustrated by example.**

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I. Introduction

The standard max-plus eigenvector method of [1] exploits attendant max-plus linearity, semiconvexity, and semigroup properties of dynamic programming to provide an algebraic representation for the value function associated with a continuous time nonlinear optimal control problem. This representation corresponds to the coordinate vector defined by the value function with respect to a countably infinite basis of an infinite dimensional vector space of semiconvex functions. The basis involved consists of quadratic functions, defined on the state space, whose locations form a dense set. A finite cardinality truncation of this basis yields an approximation of the value function, and it is this approximation that is generated by the standard numerical method of [1].

In truncating the basis as indicated, the user imposes an a priori choice of basis function locations. This yields a value function approximation that is defined with respect to that choice. In this paper, the process of choosing these basis function locations is addressed in an adaptive way by exploiting the dependence of the a posteriori value function approximation error, defined with respect to the associated Hamiltonian, on these locations. In particular, given an initial finite cardinality set of basis functions, the standard max-plus eigenvalue vector method is applied to yield a value function approximation. This value function approximation is used to identify a Voronoi tessellation of an approximation for the target level set of the Hamiltonian, in which each convex polytope corresponds to the set of states for which a basis function is active. Elements of this tessellation, and hence individual basis functions, are identified that correspond to the worst case Hamiltonian, and new basis functions added in directions that improve this worst case. Unused basis functions are pruned, and the basis updated. The algorithm proceeds iteratively, with the standard max-plus eigenvector method applied at each step to the adapted basis. Its operation is illustrated by example.

In terms of organization, the optimal control problem and corresponding max-plus eigenvector method developed for its solution [1] is recalled in Sections II and III. The proposed iterative basis adaptation is reported in Section IV, along with its application by simple example in Section V. A very brief summary of conclusions is provided in Section VI.

Throughout, $\mathbb{N}$ and $\mathbb{Z}$ denote the natural numbers and integers respectively, while $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ denote the real numbers, non-negative reals, and extended reals. $\mathbb{R}^n$ denotes Euclidean space of dimension $n \in \mathbb{N}$. The inner product and norm on $\mathbb{R}^n$ are denoted respectively by $\langle \, , \rangle$ and $| \cdot |$, and an open ball of radius $r \in \mathbb{R}_{>0}$, centred at $x \in \mathbb{R}^n$, is denoted by $\mathscr{B}(x; r)$. Matrices with real entries and dimensions $n, m \in \mathbb{N}$ are denoted by $\mathbb{R}^{n\times m}$. The set of self-adjoint matrices in $\mathbb{R}^{n\times n}$ is denoted by Σ (with the dependence on n suppressed), and the subset of matrices positive definite relative to $M \in \Sigma$ is

$$\Sigma_{>M} \doteq \{P \in \Sigma \,|\, P - M > 0\}.$$

An extended real-valued function is (lower) closed convex if it is either the constant function $-\infty$, or it is lower semicontinuous, convex, and maps into $\mathbb{R} \cup \{+\infty\}$. It is (upper) closed concave if its additive inverse is (lower) closed convex. $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is semiconvex (resp. semiconcave) if there exists a $K \in \Sigma$ such that $x \mapsto \psi(x) + \frac{1}{2}\langle x, K\, x\rangle$ is closed convex ($x \mapsto \psi(x) - \frac{1}{2}\langle x, K\, x\rangle$ is closed concave). The space of semiconvex (semiconcave) functions on $\mathbb{R}^n$ is denoted by $\mathscr{S}_+^K$ ($\mathscr{S}_-^K$).

The max-plus algebra [1], [2], [3] is an idempotent semifield over $\overline{\mathbb{R}}$ equipped with addition and multiplication operations defined by

$$a \oplus b \doteq \max(a, b), \quad a \otimes b \doteq a + b,$$

for all $a, b \in \overline{\mathbb{R}}$. A max-plus vector space is a vector space over the max-plus algebra, and is referred to as a moduloid or idempotent semimodule [2], [3]. Given any $K \in \Sigma$, the corresponding space $\mathscr{S}_+^K$ of semiconvex functions is a max-plus vector space. The max-plus integral of an extended real-valued function $f : \Omega \subset \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by $\int_\Omega^\oplus f(x)\, dx \doteq \sup_{x\in\Omega} f(x)$.

Given Banach spaces $\mathscr{X}$, $\mathscr{Y}$ with norms $| \cdot |_{\mathscr{X}}$, $| \cdot |_{\mathscr{Y}}$, a function $f : \mathscr{X} \to \mathscr{Y}$ is Fréchet differentiable at $x \in \mathscr{X}$, with Fréchet derivative $D_x f(x) \in \mathcal{L}(\mathscr{X}; \mathscr{Y})$, if

$$0 = \lim_{|h|_{\mathscr{X}}\to 0} \frac{|f(x + h) - f(x) - D_x f(x)\, h|_{\mathscr{Y}}}{|h|_{\mathscr{X}}}.$$

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The space of all k-times continuously Fréchet differentiable functions on $\Omega \subset \mathscr{X}$ is denoted by $C^k(\Omega;\mathscr{Y})$, $k \in \mathbb{N}$. The corresponding derivatives are denoted by $D_x^k f(x) \in \mathcal{L}((\mathscr{X})^k;\mathscr{Y})$, in which $(\mathscr{X})^k \doteq (\mathscr{X})^{k-1} \times \mathscr{X}$, $k \in \mathbb{N}$. Where $\mathscr{X}$ is a Hilbert space and $\mathscr{Y} \doteq \mathbb{R}$, the Fréchet derivative satisfies $D_x f(x)\, h = \langle h, \nabla_x f(x)\rangle_{\mathscr{X}}$, in which $\nabla_x f(x) \in \mathscr{X}$ denotes its Riesz representation.

II. Optimal control problem

Attention is restricted to finite dimensional continuous time nonlinear optimal control problems of the form considered in [1]. The infinite horizon value function $W : \mathbb{R}^n \to \overline{\mathbb{R}}$ of interest is defined with respect to its finite horizon counterpart $W_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ and the underlying dynamic programming evolution operator $\mathcal{S}_t$, for any $t \in \mathbb{R}_{\geqslant 0}$, by

$$W(x) \doteq \sup_{t\geqslant 0} W_t(x) = \lim_{t\to\infty} W_t(x), \quad W_t(x) \doteq [\mathcal{S}_t\, \Psi_0](x),$$
$$[\mathcal{S}_t\, \psi](x) \doteq \sup_{w\in\mathscr{W}[0,t]} \left\{ \int_0^t l(x_s) - \tfrac{\gamma^2}{2}\, |w_s|^2\, ds + \psi(x_t) \right\} \quad (1)$$

for all $x \in \mathbb{R}^n$. Here, $\Psi_0 : \mathbb{R}^n \to \mathbb{R}$ denotes the zero terminal payoff, i.e. $\Psi_0(x) \doteq 0$ for all $x \in \mathbb{R}^n$, while $\mathscr{W}[0,t] \doteq \mathscr{L}_2([0,t];\mathbb{R}^m)$ denotes the input space. Trajectories of the nonlinear dynamics underlying (1), each denoted with an abuse of notation by $s \mapsto x_s$, $s \in [0,t]$, are defined with respect to an initial state $x \in \mathbb{R}^n$ and input $w \in \mathscr{W}[0,t]$ by

$$x_s \doteq [\chi(x,w)]_s \doteq x + \int_0^s f(x_r) + \sigma\, w_r\, dr. \quad (2)$$

Standard assumptions [1, p.59] restrict the problem data $f : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma \in \mathbb{R}^{n\times m}$, $l : \mathbb{R}^n \to \mathbb{R}$ throughout. For simplicity, $Q \doteq \sigma\,\sigma'$ is assumed invertible, i.e. $Q \in \Sigma_{>0}$. The gain parameter $\gamma \in \mathbb{R}_{\geqslant 0}$ is assumed fixed a priori sufficiently large such that W is proper. The value functions W_t and W of (1) are semiconvex [1], and are unique viscosity solutions of the respective non-stationary and stationary Hamilton-Jacobi-Bellman (HJB) PDEs

$$0 = \frac{\partial W_t}{\partial t}(x) + H(x, \nabla_x W_t(x)), \quad W_0(x) = \Psi_0(x),$$
$$0 = H(x, \nabla_x W(x)), \qquad W(0) = 0, \quad (3)$$

for all $t \in \mathbb{R}_{\geqslant 0}$ and $x \in \mathbb{R}^n$, in which $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the (completed squares) Hamiltonian

$$H(x,p) \doteq -l(x) - \langle p, f(x)\rangle - \tfrac{1}{2\gamma^2}\, \langle p,\, \sigma\,\sigma'\, p\rangle \quad (4)$$

for all $x, p \in \mathbb{R}^n$.

III. Max-plus eigenvector method [1]

A max-plus eigenvector method for the infinite horizon problem (1) was developed in [1], and provides the foundation of the adaptive algorithm proposed in the sequel. It relies on the dynamic programming evolution operator preserving semiconvexity on short time horizons.

Assumption 3.1: There exists an invertible $M \in \Sigma$ and a $\tau_0^* \in \mathbb{R}_{>0}$ such that $\mathcal{S}_\tau : \mathscr{S}_+^{-M} \to \mathscr{S}_+^{-M}$ for all $\tau \in [0, \tau_0^*]$.

This property, combined with max-plus linearity of the dynamic programming evolution operator, admits the development of max-plus fundamental solution semigroups for optimal control problems of the form (1), see [4], [5], [6], [7], [8]. The max-plus *dual space* fundamental solution semigroup, denoted by $\{\mathcal{B}_t^\oplus\}_{t\in\mathbb{R}_{\geqslant 0}}$, is key to the max-plus eigenvector method of interest here, see for example [7]. An element $\mathcal{B}_t^\oplus$ of this semigroup is a max-plus linear max-plus integral operator, given by

$$\mathcal{B}_t^\oplus\, a \doteq \int_{\mathbb{R}^n}^\oplus B_t(\cdot, z) \otimes a(z)\, dz, \quad (5)$$

for all $a \in \operatorname{dom}(\mathcal{B}_t^\oplus)$. Specifically, the kernel B_t involved is defined with respect to the *semiconvex dual* $\mathcal{D}_\varphi : \mathscr{S}_+^{-M} \to \mathscr{S}_-^{-M}$ of an auxiliary value function $\mathcal{S}_t\, \varphi(\cdot, z)$, $z \in \mathbb{R}^n$, by

$$B_t(y,z) \doteq [\mathcal{D}_\varphi\, \mathcal{S}_t\, \varphi(\cdot, z)](y) \quad (6)$$

for all $y, z \in \mathbb{R}^n$, in which $\mathcal{S}_t$ is as per (1), and

$$\mathcal{D}_\varphi\, \psi \doteq -\int_{\mathbb{R}^n}^\oplus \varphi(x, \cdot) \otimes [-\psi(x)]\, dx, \quad (7)$$
$$\varphi(x,z) \doteq \tfrac{1}{2}\, \langle x - z,\, M(x-z)\rangle \quad (8)$$

for all $x, z \in \mathbb{R}^n$, $\psi \in \mathscr{S}_+^{-M}$. The set of quadratic support functions $\{\psi_i\}_{i\in\mathbb{N}}$, with elements $\psi_i : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi_i \doteq \varphi(\cdot, z_i), \quad (9)$$

with respect to a (countable) dense set $\{z_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$, forms a basis for $\mathscr{S}_+^{-M}$, see [1, Theorem 2.13, p.20]. Under Assumption 3.1, as the value function W_t of (1) is semiconvex, it may be shown [5], [7] for fixed $\tau \in (0, \tau_0^*]$ that

$$W_{k\tau} = \mathcal{D}_\varphi^{-1}\, a_k, \ a_k \doteq \mathcal{B}_\tau^\oplus\, a_{k-1}, \ a_0 \doteq \mathcal{D}_\varphi\, \Psi_0, \quad (10)$$

for all $k \in \mathbb{N} \cup \{0\}$, in which $\mathcal{D}_\varphi^{-1} : \mathscr{S}_-^{-M} \to \mathscr{S}_+^{-M}$ is the inverse semiconvex transform, defined for all $a \in \mathscr{S}_-^{-M}$ by

$$\mathcal{D}_\varphi^{-1}\, a \doteq \int_{\mathbb{R}^n}^\oplus \varphi(\cdot, z) \otimes a(z)\, dz. \quad (11)$$

Defining $[e_k]_i \doteq a_k(z_i)$ to be the i^{th} element of an infinite dimensional vector e_k, for all $i, k \in \mathbb{N}$, (10) yields

$$\begin{aligned} [e_k]_i = [\mathcal{B}_\tau^\oplus a_{k-1}](z_i) &= \int_{\mathbb{R}^n}^\oplus B_\tau(z_i, z) \otimes a_{k-1}(z)\, dz \\ &= \bigoplus_{j\in\mathbb{N}} B_\tau(z_i, z_j) \otimes a_{k-1}(z_j) \\ &= \bigoplus_{j\in\mathbb{N}} B_\tau(z_i, z_j) \otimes [e_{k-1}]_j = [B_\tau \otimes e_{k-1}]_j, \end{aligned} \quad (12)$$

in which B_τ denotes both the kernel of $\mathcal{B}_\tau^\oplus$ and its representation as a compatibly dimensioned square matrix, defined element-wise by $[B_\tau]_{ij} \doteq B_\tau(z_i, z_j)$ for all $i, j \in \mathbb{N}$. Combining (7), (10), (11), (12) yields an exact representation for the infinite horizon value function W of (1), with

$$\begin{aligned} W &= \bigoplus_{i\in\mathbb{N}} \psi_i(\cdot) \otimes [e_\infty]_i, \quad e_\infty \doteq \lim_{k\to\infty} e_k, \\ e_k &= B_\tau \otimes e_{k-1}, \quad [e_0]_i = [\mathcal{D}_\varphi\, \Psi_0](z_i), \ i \in \mathbb{N}. \end{aligned} \quad (13)$$

Truncating the basis $\{\psi_i\}_{i\in\mathbb{N}}$ to finite cardinality $\nu \in \mathbb{N}$ yields an approximation $\widehat{W}$ for the value W, given by

$$\begin{aligned} \widehat{W} &= \bigoplus_{i=1}^{\nu} \psi_i(\cdot) \otimes [\hat{e}_\infty]_i, \quad \hat{e}_\infty \doteq \lim_{k\to\infty} \hat{e}_k, \\ \hat{e}_k &= B_\tau \otimes \hat{e}_{k-1}, \quad [\hat{e}_0]_i = [\mathcal{D}_\varphi\, \Psi_0](z_i),\ i \in \mathbb{N}_{\leqslant \nu}, \end{aligned} \tag{14}$$

which is precisely the max-plus eigenvector method of [1].

Remark 3.2: As one basis function must be centred at the origin, see [1, Lemma 4.20, p.77], it is useful to label $z_1 \doteq 0$. Subsequently, the same lemma requires that $[B_\tau]_{11} = 0$.

IV. Basis adaptation

The max-plus eigenvector method (14), as reported in [1], assumes an a priori fixed basis $\mathscr{B} \doteq \{\psi_i\}_{i\in\mathbb{N}_{\leqslant\nu}}$ of $\nu \in \mathbb{N}$ elements centred at a priori fixed locations $\{z_i\}_{i\in\mathbb{N}_{\leqslant\nu}}$. The objective here is to identify a process by which such a basis can be adapted iteratively, by adding and pruning basis functions, so as to yield an improved approximation of the value function (1) via (14) at each step. A Hamiltonian back-substitution error inferred from (3), (4) is used as the means for evaluating the quality of these approximations point-wise. It is assumed that an initial basis $\mathscr{B}^0$ of cardinality $\nu^0 \in \mathbb{N}$ and a bounded convex polytope $\mathscr{Y}^0 \subset \mathbb{R}^n$ are given, subject to some minor assumptions to be detailed in the development that follows. For convenience, quantities associated with the k^{th} adapted basis are denoted with a superscript k. Initial values correspond to $k = 0$.

A. Value function approximation and Voronoi tessellation

Each iteration is initialized with a basis $\mathscr{B}^k$ of cardinality $\nu^k \in \mathbb{N}$, and a bounded convex polytope $\mathscr{Y}^k \subset \mathbb{R}^n$. The max-plus eigenvector method [1] applied to this basis yields an approximation (14) of the value function (1) given by

$$\widehat{W}^k(x) \doteq \bigoplus_{i=1}^{\nu^k} \psi_i^k(x) \otimes [\hat{e}_\infty^k]_i, \tag{15}$$

in which $\hat{e}_\infty^k$ is the limit of the corresponding idempotent iteration in (14). Evaluation of (15) via the $\oplus$ operation indicates that basis function $\psi_i^k \in \mathscr{B}^k$ is *active* at $x \in \mathscr{Y}^k$ if $\psi_i^k(x) + [\hat{e}_\infty^k]_i > \psi_j^k(x) + [\hat{e}_\infty^k]_j$ for all $j \in \mathbb{N}_{\leqslant\nu^k}\backslash\{i\}$. Consequently, recalling (8), (9), the set $\mathscr{Y}_i^k \subset \mathscr{Y}$ on which ψ_i^k is active in (14) is

$$\begin{aligned} \mathscr{Y}_i^k &\doteq \left\{ x \in \mathscr{Y}^k \,\middle|\, \Gamma_{ij}^k(x) > 0\ \forall\ j \in \mathbb{N}_{\leqslant\nu^k}\backslash\{i\} \right\}, \\ \Gamma_{ij}^k(x) &\doteq \langle x,\, M(z_j^k - z_i^k)\rangle + [\hat{e}_\infty^k]_i + \tfrac{1}{2}\langle z_i^k,\, M\, z_i^k\rangle \\ &\qquad - \left[[\hat{e}_\infty^k]_j + \tfrac{1}{2}\langle z_j^k,\, M\, z_j^k\rangle\right]. \end{aligned} \tag{16}$$

Given $i \in \mathbb{N}_{\leqslant\nu^k}$, the set $\mathscr{Y}_i^k$ is the interior of the intersection of a family of half-spaces. Its closure $\overline{\mathscr{Y}_i^k}$ is hence a convex polytope, with a finite set of vertices denoted for convenience by $\mathscr{V}(\mathscr{Y}_i^k) \subset \overline{\mathscr{Y}^k}$. The union of these convex polytopes over $i \in \mathbb{N}_{\leqslant\nu^k}$ is the whole set $\mathscr{Y}^k$, so that together they define a *Voronoi tessellation* $\mathscr{T}^k \doteq \{\overline{\mathscr{Y}_i^k}\}_{i\in\mathbb{N}_{\leqslant\nu^k}}$ of $\mathscr{Y}^k$. The (possibly empty) set $\mathscr{B}_-^k \subset \mathscr{B}^k$ of basis functions not used in the approximation (15) are identified by way of the volume of their associated convex polytopes. These are pruned from the basis in the subsequent considerations, via

$$\mathscr{B}_\sharp^k \doteq \mathscr{B}^k \setminus \mathscr{B}_-^k, \tag{17}$$

with the resulting cardinality being denoted by $\nu_\sharp^k \in \mathbb{N}_{\leqslant\nu^k}$.

B. Hamiltonian back-substitution error

A Hamiltonian back-substitution error [9] may be defined by substitution of the value function approximation (15) back into the Hamiltonian (4). To this end, given any $i \in \mathbb{N}_{\leqslant\nu_\sharp^k}$ for which the corresponding convex polytope $\mathscr{Y}_i^k$ must be nonempty by (17), observe by (15) that

$$\begin{aligned} \widehat{W}^k(x) &= \psi_i^k(x) + [\hat{e}_\infty^k]_i \\ \nabla_x \widehat{W}^k(x) &= \nabla_x\, \psi_i^k(x) = M(x - z_i^k) \end{aligned} \tag{18}$$

for any $x \in \mathscr{Y}_i^k$. Consequently, the Hamiltonian back-substitution error applicable on $\mathscr{Y}_i^k$ may be defined on all of $\mathscr{Y}^k$ by $h(\cdot, z_i^k) : \mathscr{Y}^k \to \mathbb{R}$ via (4) by

$$h(x, z_i^k) \doteq H(x, \nabla_x \psi_i^k(x)) \tag{19}$$

for all $x \in \mathscr{Y}^k$. Observe by the boundedness of $\mathscr{Y}^k$ that there exists an $r \in \mathbb{R}_{>0}$ such that $\mathscr{Y}_i^k \subset \mathscr{Y}^k \subset \mathscr{B}_0(r)$. Hence, applying [1, Theorem 5.16, p.120], there exists a $\tau^* \in (0, \tau_0^*]$ such that the auxiliary finite horizon value function defined by $W_t^{ki}(x) \doteq (\mathcal{S}_t\, \psi_i^k)(x)$ for all $t \in \mathbb{R}_{\geqslant 0}$, $x \in \mathscr{Y}^k$, satisfies $W_t^{ki} \in C^2((0, \tau^*) \times \mathscr{Y}^k; \mathbb{R})$. Meanwhile, Assumption 3.1 requires that $W_t^{ki} \in \mathscr{S}_+^{-M}$ for all $t \in [0, \tau^*]$, so that (9) and the definition of $\mathscr{S}_+^{-M}$ imply that $W_t^{ki} - \psi_i = W_t^{ki} - W_0^{ki}$ is convex for all $t \in [0, \tau^*]$. Consequently, $\lim_{t\to 0^+}(W_t^{ki} - W_0^{ki})/t = \lim_{t\to 0^+} \frac{\partial W_t^{ki}}{\partial t} : \mathscr{Y}^k \to \mathbb{R}$ exists and is convex. Furthermore, W_t^{ki} satisfies a non-stationary HJB analogous to (3), everywhere on $\mathscr{Y}^k$, with

$$0 = \frac{\partial W_t^{ki}}{\partial t}(x) + H(x, \nabla_x W_t^{ki}(x))$$

for all $x \in \mathscr{Y}^k$, $t \in [0, \tau^*]$. In particular,

$$h(x, z_i^k) = H(x, \nabla_x \psi_i^k(x)) = -\lim_{t\to 0^+} \frac{\partial W_t^{ki}}{\partial t}(x).$$

for all $x \in \mathscr{Y}^k$, so that $h(\cdot, z_i^k)$ must be concave on $\mathscr{Y}^k$.

Theorem 4.1: Given Assumption 3.1, and any $i \in \mathbb{N}_{\leqslant\nu_\sharp^k}$, the Hamiltonian back-substitution error $h(\cdot, z_i^k) : \overline{\mathscr{Y}^k \to \mathbb{R}}$ of (19) is concave, and $h(\cdot, z_i^k)$ achieves its minimum on $\overline{\mathscr{Y}_i^k}$ at a vertex of $\overline{\mathscr{Y}_i^k}$.

Remark 4.2: For sufficiently smooth f, l in (2), (4), $h(\cdot, z_i^k)$ is concave on $\mathscr{Y}^k$ if and only if its second Fréchet derivative $D_x^2\, h(x, z_i^k) \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ is negative semidefinite for all $x \in \mathscr{Y}^k$, [10, Theorem 2.14, p. 47]. For example, with $l(x) \doteq \frac{1}{2}\langle x,\, C'C\, x\rangle$, $C'C \in \Sigma_{\geqslant 0}$, for all $x \in \mathbb{R}^n$,

$$D_x^2 h(x, z_i^k)\, h\, h = -\langle h, \Pi(x, z_i^k) h\rangle$$

with $\Pi(x, z_i^k) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \equiv \mathbb{R}^{n\times n}$ given by

$$\begin{aligned} \Pi(x, z_i^k) &\doteq C'C + M\, \nabla_x f(x) + \nabla_x f(x)'\, M \\ &\qquad + \tfrac{1}{\gamma^2}\, M\, \sigma\, \sigma'\, M + \sum_{j=1}^{n} [M(x - z_i^k)]_j\, \nabla_{xx} f_j(x), \end{aligned}$$

in which $f_j(x)$ denotes the j^{th} entry of $f(x) \in \mathbb{R}^n$,

$$\nabla_x f(x) \doteq [\nabla_x f_1(x) \ \cdots \ \nabla_x f_n(x)]' \in \mathbb{R}^{n\times n},$$
$$\nabla_{xx} f_j(x) \doteq D_x \nabla_x f_j(x) \in \Sigma \subset \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n),$$

and $\nabla_x f_j(x) \in \mathbb{R}^n$ denotes the Riesz representation of the Fréchet derivative of the j^{th} component of f, for all $x \in \mathbb{R}^n$. Non-negativity of $\Pi(\cdot, z_i^k)$ may be checked directly, given f, M, and z_i^k. In the linear case, i.e. $f(x) \doteq A\,x$, $A \in \mathbb{R}^{n\times n}$, $\Pi(x, z_i^k)$ is independent of $x, z_i^k \in \mathbb{R}^n$, with

$$\Pi(x, z_i^k) = \Gamma(M) \doteq A'M + M\,A + \tfrac{1}{\gamma^2} M\,\sigma\,\sigma'\,M + C'C.$$

The bounded real lemma subsequently implies existence of an $M \in \Sigma$ such that $\Gamma(M) \in \Sigma_{>0}$, see for example [11]. □

The definition of Hamiltonian back-substitution error (19) extends in an obvious way to $h^k : \bigcup_{i\in\mathbb{N}_{\leqslant \nu_\sharp^k}} \mathscr{Y}_i^k \to \mathbb{R}$, with

$$h^k(x) \doteq \liminf_{\xi\to x} \sum_{i=1}^{\nu_\sharp^k} h(\xi, z_i^k)\, \iota_i^k(\xi), \tag{20}$$

in which h is as per (19), and $\iota_i^k : \mathscr{Y}^k \to \{0,1\}$ denotes the indicator function defined with respect to $\mathscr{Y}_i^k \subset \mathscr{Y}^k$.

C. Hamiltonian back-substitution error level sets

Let $\delta_H \in \mathbb{R}_{>0}$ denote the target Hamiltonian back-substitution error (20) for the iteration. Recall that basis $\mathscr{B}^k$ contains a basis function ψ_1^k centred at $z_1^k \doteq 0$, see Remark 3.2. It is straightforward to show that the corresponding polytope $\mathscr{Y}_1^k \subset \mathscr{Y}^0$ must contain z_1^k, and that $h(0, z_1^k) = 0$. By the asserted smoothness of $h(\cdot, z_1^k)$, note further that $\nabla_x h(0, z_1^k) = 0$, see Remark 4.2. Hence, the asserted concavity in Theorem 4.1 implies that $h(\cdot, z_1^k)$ achieves its maximum at $0 \in \mathscr{Y}_1^k$, and $h(x, z_1^k) \leqslant 0$ for all $x \in \mathscr{Y}^k \supset \mathscr{Y}_1^k$. Consequently, there exists an $r_1^k \in \mathbb{R}_{>0}$ such that

$$-\delta_H \leqslant h(x, z_1^k) \leqslant 0 \tag{21}$$

for all $x \in \mathscr{B}(0; r_1^k) \subset \mathscr{Y}_1^k$. If the left-hand inequality in (21) holds at the vertices of $\mathscr{Y}_1^k$, then it holds for all $x \in \mathscr{Y}_1^k$.

In view of (21), it is useful to consider Hamiltonian back-substitution level sets defined with respect to all basis functions. To this end, define

$$\Lambda^k \doteq \overline{\bigcup_{i\in\mathbb{N}_{\leqslant \nu_\sharp^k}} \Lambda_i^k}, \tag{22}$$

$$\Lambda_i^k \doteq \{x \in \mathscr{Y}_i^k \mid -\delta_H \leqslant h(x, z_i^k)\}, \tag{23}$$

for $i \in \mathbb{N}_{\leqslant \nu_\sharp^k}$. As $h(\cdot, z_i^k)$ is concave, Λ_i^k is convex, and so

$$\Lambda_i^k \equiv \mathscr{Y}_i^k \iff -\delta_H \leqslant h(v, z_i^k)\ \forall\ v \in \mathscr{V}(\mathscr{Y}_i^k). \tag{24}$$

By inspection of (15), (18), (19), (20), (23), Λ^k corresponds to the back-substitution error level set defined with respect to the value function approximation $\widehat{W}^k$ and δ_H. That is,

$$\Lambda^k \equiv \overline{\{x \in \mathscr{Y}^k \mid -\delta_H \leqslant H(x, \nabla_x \widehat{W}^k(x))\}}. \tag{25}$$

Although Λ^k is the closure of a union of convex sets, it need not be convex, or indeed even connected. However, (21) implies that there always exists an $r^k \in \mathbb{R}_{>0}$ satisfying $r^k \geqslant r_1^k$ such that $\mathscr{B}(0; r^k) \subset \Lambda^k$. Upper and lower bounds R^k and r^k for the maximal such ball radius inside Λ^k can be estimated via (24) and a partitioning of the tessellation $\mathscr{T}^k$. To this end, define the set Ω_i^k, $i \in \mathbb{N}_{\leqslant \nu_\sharp^k}$, by

$$\Omega_i^k \doteq \left\{ (r,\,h) \in \mathbb{R}_{>0} \times \mathbb{R} \ \middle|\ \begin{array}{c} r \doteq |v|, h \doteq h(v, z_i^k), \\ v \in \mathscr{V}(\mathscr{Y}_i^k) \end{array} \right\},$$

with a lexicographical ordering $(r_1, h_1) \preceq (r_2, h_2)$ for all $(r_1, h_1), (r_2, h_2) \in \Omega_i^k$ satisfying $r_1 \leqslant r_2$. Using the ensuing definitions of min and max on Ω_i^k, define the subsequent per-polytope radius bounds r_i^k and R_i^k by

$$\begin{aligned} r_i^k &\doteq \operatorname*{arg\,max}_{r\in\mathbb{R}_{>0}} \{(r,h) \in \Omega_i^k \mid -\delta_H \leqslant h\}, \\ R_i^k &\doteq \operatorname*{arg\,min}_{r\in\mathbb{R}_{>0}} \{(r,h) \in \Omega_i^k \mid -\delta_H > h\}, \end{aligned} \tag{26}$$

in which conventions $\arg\max \varnothing \doteq -\infty$ and $\arg\min \varnothing \doteq +\infty$ are adopted. Subsequently, a vertex v is referred to as either *within tolerance* if $-\delta_H \leqslant h(v, z_i^k)$, or *outside tolerance* if $-\delta_H > h(v, z_i^k)$. The bounds (26) define a disjoint partition of tessellation $\mathscr{T}^k$, according to this classification, with

$$\begin{gathered} \mathscr{T}^k = \widecheck{\mathscr{T}}^k \cup \widetilde{\mathscr{T}}^k \cup \widehat{\mathscr{T}}^k, \\ \widecheck{\mathscr{T}}^k \doteq \{\mathscr{Y}_i^k\}_{i\in\check{\mathcal{I}}^k},\ \widetilde{\mathscr{T}}^k \doteq \{\mathscr{Y}_i^k\}_{i\in\tilde{\mathcal{I}}^k},\ \widehat{\mathscr{T}}^k \doteq \{\mathscr{Y}_i^k\}_{i\in\hat{\mathcal{I}}^k}, \end{gathered} \tag{27}$$

in which the cited polytope index sets are given by

$$\begin{aligned} \check{\mathcal{I}}^k &\doteq \{i \in \mathbb{N}_{\leqslant \nu_\sharp^k} \mid r_i^k > -\infty,\ R_i^k = \infty\}, && \textit{[inside tol.]} \\ \tilde{\mathcal{I}}^k &\doteq \{i \in \mathbb{N}_{\leqslant \nu_\sharp^k} \mid r_i^k > -\infty,\ R_i^k < \infty\}, && \textit{[mixed]} \\ \hat{\mathcal{I}}^k &\doteq \{i \in \mathbb{N}_{\leqslant \nu_\sharp^k} \mid r_i^k = -\infty,\ R_i^k < \infty\}. && \textit{[outside]} \end{aligned} \tag{28}$$

Note that as the sets defining the bounds (26) cannot both be empty, a fourth case in (28) is unnecessary. As indicated, the three cases listed correspond respectively to those polytopes with all vertices inside tolerance, those with some vertices inside and some outside tolerance (mixed), and those with all vertices outside tolerance. As $h(\cdot, z_i^k)$ achieves its minimum on $\mathscr{Y}_i^k$ at a vertex, polytopes may also be referred to as inside, mixed, or outside tolerance. It is convenient to assume that the polytope $\mathscr{Y}_1^k$ associated with the basis function ψ_1 centred at k is always inside tolerance.

Assumption 4.3: $\mathscr{Y}_1^k \subset \widecheck{\mathscr{T}}^k$.

In view of (26) and (27), upper and lower bounds R^k and r^k on the maximal such ball radius inside Λ^k are thus

$$\begin{aligned} R^k &\doteq \min\{R_i^k \mid i \in \tilde{\mathcal{I}}^k \cup \hat{\mathcal{I}}^k\}, \\ r^k &\doteq \max\{r_i^k \mid i \in \check{\mathcal{I}}^k,\ r_i^k < R^k\}. \end{aligned} \tag{29}$$

(Assumption 4.3 implies that $r^k \in \mathbb{R}_{>0}$.) Together, these radii define an annulus $\mathscr{A}^k \doteq \mathscr{B}(0; R^k) \setminus \mathscr{B}(0; r^k) \neq \varnothing$. Consideration of polytopes with vertices within $\mathscr{A}^k$, and in particular how their associated Hamiltonian back-substitution error can be improved, forms the foundation of the basis adaptation iteration to follow.

D. Basis adaptation iteration

By definition, the annulus $\mathscr{A}^k$ contains vertices belonging to polytopes that are either inside tolerance or mixed tolerance. It does not contain vertices of polytopes that are outside tolerance. That is, every polytope with a vertex in $\mathscr{A}^k$ is either within tolerance, or has at least one vertex that is outside tolerance. The set of all such outside tolerance vertices can be ordered according to their Hamiltonian back-substitution error, and a vertex $\hat{v}_i^k \in \mathscr{Y}_i^k$ furthest from tolerance identified. A parent polytope $\mathscr{Y}_i^k$, and corresponding existing basis function ψ_i^k, may subsequently be identified. As the auxiliary back-substitution error function $z \mapsto h(\hat{v}_i^k, z)$ defined via (19) is concave and depends smoothly on z, the location $z = \hat{z}_i^k$ of a prospective new basis function can be propagated from z_i^k to improve the expected back-substitution error. If this expected error is within tolerance, the prospective basis function is reserved for subsequent inclusion in the basis. This is repeated for an a priori fixed number of worst case vertices, yielding a set of new basis functions $\mathscr{B}_+^k$ to be added to $\mathscr{B}_\sharp^k$. A replacement for the bounded convex polytope $\mathscr{Y}^k$ involved is also identified as the largest convex polytope, with a priori fixed number of vertices, that is within the outer level set approximation $\mathscr{B}(0; R^k)$. The adaptation thus proceeds via the basis and polytope update steps

$$\begin{aligned} \mathscr{B}^{k+1} &\doteq \mathscr{B}_\sharp^k \cup \mathscr{B}_+^k = (\mathscr{B}^k \setminus \mathscr{B}_-^k) \cup \mathscr{B}_+^k, \\ \mathscr{Y}^{k+1} &\doteq \operatorname{co} \widehat{\mathscr{B}}_q(0; R^k), \end{aligned} \tag{30}$$

in which co denotes the convex hull, and $\widehat{\mathscr{B}}_q$ is a star shaped neighbourhood defined with respect to $q \in \mathbb{N}$ by

$$\begin{aligned} \widehat{\mathscr{B}}(0; r) &\doteq \left\{ \rho\, \hat{v} \,\middle|\, \hat{v} \in \widehat{\mathscr{V}}_q,\ \rho \in [0, r] \right\}, \\ \widehat{\mathscr{V}}_q &\doteq \{ \hat{v}_j \in \mathbb{R}^n \,|\, |\hat{v}_j| = 1,\ j \in \mathbb{N}_{\leqslant q} \}. \end{aligned}$$

Within the basis update step in (30), propagation of a basis function location from $z = \hat{z}_i^k$ to a new location $\tilde{z}_i^k$ is achieved for each $i \in \mathbb{N}_{\leqslant \nu_\sharp^k}$ via integration of the ODE

$$\tilde{z}_i^k \doteq \zeta_{\eta^+}, \qquad \begin{cases} \dot{\zeta}_\eta = F_i^k(\zeta_\eta), & \eta \in \mathbb{R}_{\geqslant 0}, \\ \zeta_0 = z_i^k, \end{cases} \tag{31}$$

in which F and $\eta^+ \in \mathbb{R}_{>0}$ remain to be specified. With the intention of applying steepest ascent, recall by (8), (9), (19) that $h(\hat{v}^k, z) = H(\hat{v}^k, p(\hat{v}^k, z))$, in which $p(v, z) \doteq M(v - z)$. Hence, applying the chain rule via (4) yields

$$\nabla_z h(\hat{v}_i^k, z) = M\, [f(\hat{v}_i^k) + \tfrac{1}{\gamma^2}\, \sigma\, \sigma\, M(\hat{v}_i^k - z)] \tag{32}$$

which is in the steepest ascent direction for $h(\hat{v}^k, \cdot)$. As $h(\hat{v}^k, \cdot)$ is concave, and $Q \doteq \sigma\, \sigma' \in \Sigma_{>0}$ is invertible, a maximizer for $h(\hat{v}^k, \cdot)$ always exists. Indeed, by completion of squares and invertibility of $M \in \Sigma$,

$$\max_{z \in \mathbb{R}^n} h(\hat{v}_i^k, z) = -\tfrac{1}{2}\, |C\, \hat{v}_i^k|^2 + \tfrac{\gamma^2}{2}\, |Q^{-\frac{1}{2}} f(\hat{v}_i^k)|^2,$$

which may be positive, particularly for vertices located away from the origin. Hence, (32) is modified by the sign of the back-substitution error, and approximately normalized, to yield a candidate F_i^k for use in (31), given by

$$F_i^k(\zeta) \doteq \operatorname{sgn}(h(\hat{v}_i^k, \zeta))\, \frac{\nabla_z h(\hat{v}_i^k, \zeta)}{|\nabla_z h(\hat{v}_i^k, \zeta)| + \epsilon}, \tag{33}$$

for some a priori fixed $\epsilon \in \mathbb{R}_{>0}$. The stopping time η^+ is selected so as to ensure that the expected back-substitution error is within tolerance, and the "ripple" induced by a switch to a new basis function being active in (15) is likewise limited. In particular,

$$\eta^+ \doteq \sup \left\{ \eta \in (0, \bar{\eta}] \,\middle|\, \begin{array}{c} |h(\hat{v}_i^k, \zeta_\eta)| \leqslant (1 - \mu)\, \delta_H, \\ |h(\hat{v}_i^k, \zeta_\eta) - h(\hat{v}_i^k, z_i^k)| \leqslant \mu\, \delta_H \end{array} \right\} \tag{34}$$

in which $\mu \in (0, 1)$ and $\bar{\eta} \in \mathbb{R}_{>0}$ are fixed. A scheme related to (33), but based on the characteristics associated with the Hamiltonian (4), is also viable.

V. Example

A nonlinear system is considered with problem data [1, p.127] given by

$$\begin{aligned} f(\xi) &\doteq \begin{pmatrix} -2\,\xi_1\, [1 + \frac{1}{2}\, \tan^{-1}(3\, \xi_2^2/2)] \\ \frac{1}{2}\, \xi_1 - 3\, \xi_2\, \exp(-\xi_1/3) \end{pmatrix},\ \sigma \doteq \mathcal{I}_2, \\ l(\xi) &\doteq \tfrac{1}{2}\, |\xi|^2,\ \gamma^2 \doteq 1,\ \mathcal{M} \doteq -0.1\, \mathcal{I}_2, \end{aligned} \tag{35}$$

with $\xi \doteq (\xi_1, \xi_2) \in \mathbb{R}^2$, in which $\mathcal{I}_2 \in \mathbb{R}^{2\times 2}$ is the identity. Value function approximation and basis adaptation proceeds as per (15) and (30), using the basis propagation defined by (31), (33), and (34). The target Hamiltonian back-substitution error selected is $\delta_H \doteq 0.1$. Other parameters selected include $q \doteq 36$, $\mu \doteq 0.5$, $\bar{\eta} \doteq 1$. The standard max-plus eigenvector method [1] summarized by (14) is applied with $\tau \doteq 0.05$. Figures 1, 2, 3, and 4 illustrate evolution of the basis $\mathscr{B}^k$, tessellation $\mathscr{T}^k$, value function W^k, and associated Hamiltonian back-substitution error respectively. The filled polytopes in Figure 2 indicate those with a vertex furthest from tolerance, with the vertices involved identified by black circles. Hamiltonian back-substitution error reduction is confirmed via Figure 4.

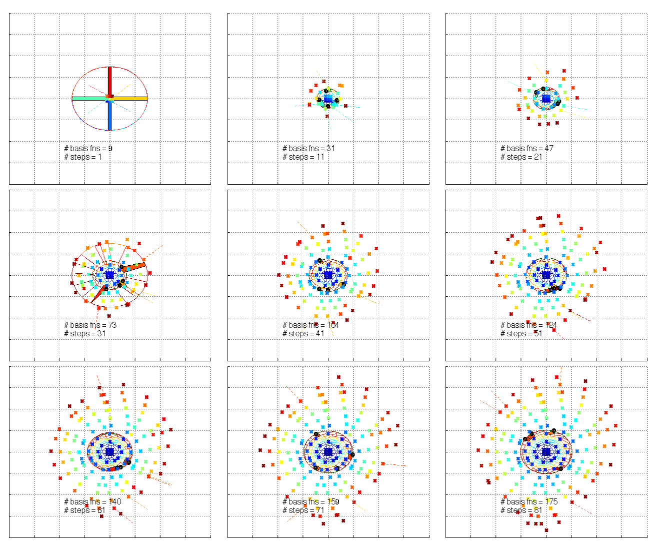

Fig. 1. Basis evolution.

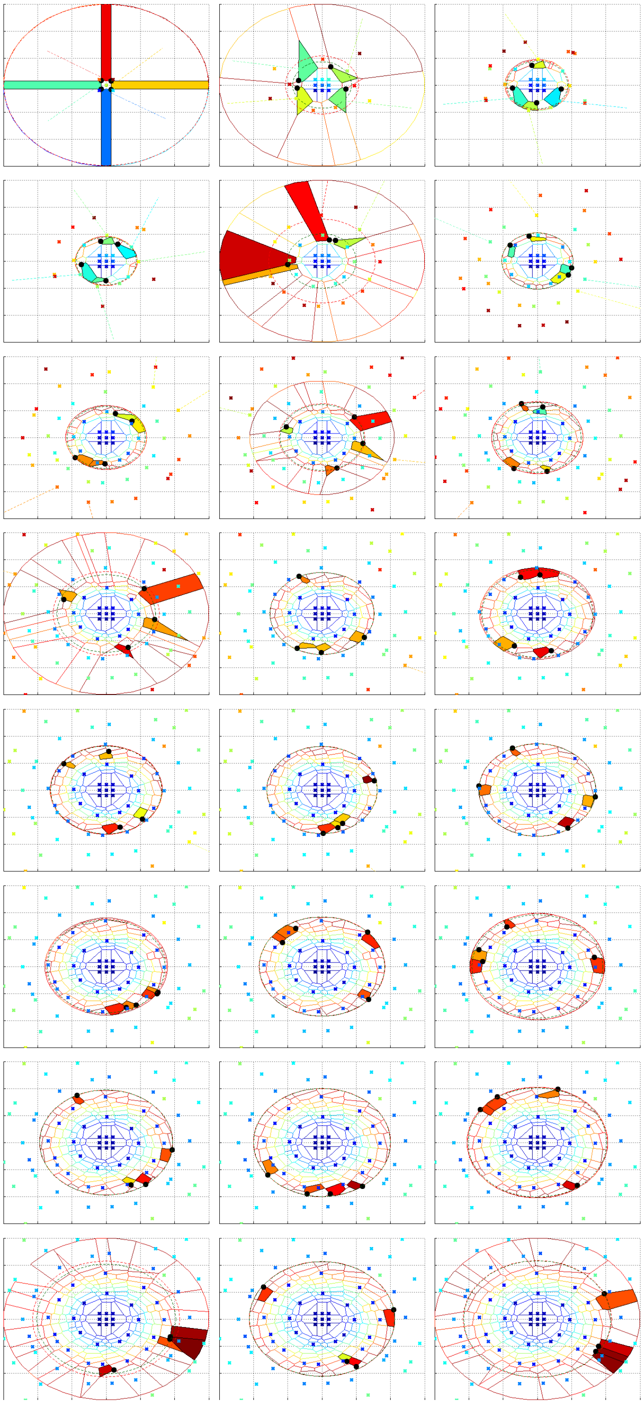

Fig. 2. Tessellation evolution.

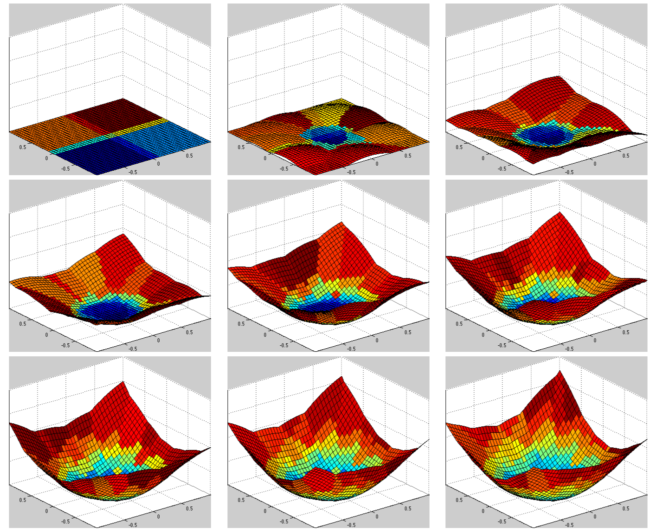

Fig. 3. Value function approximation evolution.

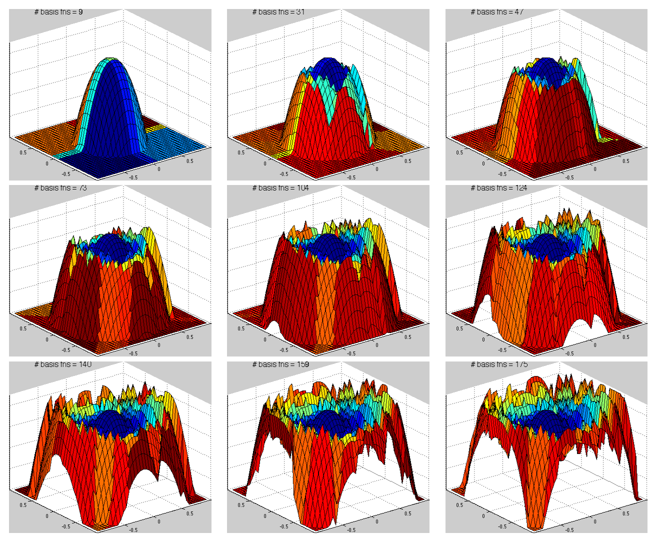

Fig. 4. Hamiltonian back-substitution error evolution.

VI. Conclusions

A basis adaptation iteration encapsulating a standard max-plus eigenvector method for optimal control is developed, with the objective of improving the Hamiltonian back-substitution error associated with the value function approximation obtained at each iteration. An example is included.

References

[1] W. McEneaney, *Max-plus methods for nonlinear control and estimation*, ser. Systems & Control: Foundations & Application, T. Basar, Ed. Birkhauser, 2006.

[2] V. Kolokoltsov and V. Maslov, *Idempotent analysis and applications*. Kluwer Publishing House, 1997.

[3] G. Litvinov, V. Maslov, and G. Shpiz, "Idempotent functional analysis: An algebraic approach," *Mathematical Notes*, vol. 69, no. 5, pp. 696–729, 2001.

[4] P. Dower, W. McEneaney, and H. Zhang, "Max-plus fundamental solution semigroups for optimal control problems," in *proc. SIAM Conference on Control Theory and Its Applications (Paris, France), 2015*, 2015, pp. 368–375.

[5] W. McEneaney, "A new fundamental solution for differential Riccati equations arising in control," *Automatica*, vol. 44, pp. 920–936, 2008.

[6] W. McEneaney and P. Dower, "The principle of least action and fundamental solutions of mass-spring and n-body two-point boundary value problems," *SIAM J. Control & Optimization*, vol. 53, no. 5, pp. 2898–2933, 2015.

[7] P. Dower and W. McEneaney, "A max-plus dual space fundamental solution for a class of operator differential Riccati equations," *SIAM J. Control & Optimization*, vol. 53, no. 2, pp. 969–1002, 2015.

[8] P. Dower and H. Zhang, "A max-plus primal space fundamental solution for a class of differential Riccati equations," *Mathematics of Control, Signals, and Systems*, vol. 29, no. 3, pp. 1–33, 2017.

[9] L. Grune and P. Dower, "Hamiltonian based a posteriori error estimation for Hamilton-Jacobi-Bellman equations," Universitat Bayreuth, https://epub.uni-bayreuth.de/id/eprint/3578, Tech. Rep., 2018.

[10] R. Rockafellar and R. Wets, *Variational Analysis*. Springer-Verlag, 1997.

[11] M. James and I. Petersen, "A non-smooth strict bounded real lemma," *Systems & Control Letters*, vol. 54, pp. 83–94, 2005.