23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Robust output feedback regulation for infinite-dimensional systems

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***Abstract*— In this paper, a robust output regulation problem is considered for general infinite-dimensional systems. More precisely, for the design of the regulator, the controlled output $y(t)$ and the reference signal $y_r(t)$ are assumed to be available. One mode of the designed regulator is driven by the tracking error $e_y(t)$, and is used for the reference signal tracking and disturbance rejection, while another mode driven by $y(t)$ is used to address the plant stabilization. In particular, simpler and sufficient conditions are provided to injection and feedback gains. The robustness is achieved based on Internal Model Principle and verified for non-destabilizing system with model uncertainties and simultaneous satisfaction of $\mathscr{G}-$conditions is guaranteed. An uncertain hyperbolic PDE system is used to demonstrate the paper results.**
keyword: Infinite-dimensional systems, robust output regulation, model uncertainty, output feedback.

I. INTRODUCTION

This paper is concerned with the *robust output regulation* for linear infinite-dimensional systems which include distributed parameter and time-delay systems. In control theory, besides the stabilization of control systems, another important problem is to accomplish the reference trajectory tracking in the presence of exogenous disturbances. From the practical point of view, regulating the output of a fixed plant which is a central problem in control theory, should account for robustness due to uncertain system parameters. Robustness does not account for only uncertain parameters but also encompass the fault tolerant designs [11]. In this paper, perturbations in the considered plant are treated to be of relative small order so that the closed-loop stability is not effected or destroyed by perturbation magnitudes. This is a standard assumption in many classical works [2], [7], [20] for linear finite-dimensional systems and recent works [19], [14], [16], [10] and [4] for linear infinite-dimensional systems. The *robust output regulation* has to assure asymptotic tracking despite of the presence of disturbances and model uncertain parameters. To this end, *Internal Model Principle (IMP)* is an attainable solution. An internal model of the exosystem has to be taken into account when designing the regulator. The nominal extended system consisting of the plant and designed controller, needs to be stabilized. In this way, the parameters of the controller are determined and the output regulation is achieved as well. The most important advantage of the *IMP* is the fact that once stabilization of the closed-loop system is completed, then the disturbance rejection will be achieved automatically, and moreover the reference signal tracking is accomplished with only reference signal available.

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Compared to traditional geometric approaches, all designed parameters are independent from the design solutions of the extended regulator equations and thus robustness is ensured.

Although the general properties of robust output regulation problems for infinite-dimensional systems were studied in [18], [17], [8], [4], the development of geometric approaches for the corresponding regulator design is still a topic of interest. For distributed parameter system with bounded input and output operators, [1] generalized geometric approach by extending design from finite-dimensional systems to infinite-dimensional systems. Then, the state feedback regulator problem was considered in [13] for regular systems with unbounded input and output operators, and in particular for the first-order hyperbolic systems, the solution of Riccati equation was included to address the state and error feedback regulator problems, see [21]. Moreover, auxiliary Sylvester equations were used and novel error feedback regulator designed was easily developed in [24]. Along the same line, the combination of geometric approaches and backstepping transformations resolved the output feedback regulation problems of boundary controlled parabolic systems in [3] and of boundary controlled hyperbolic systems with integral terms in [25]. Recently, Works in [14], [16], [15] extended Internal Model Principle and designed robust output regulators for infinite-dimensional systems with infinite-dimensional exosystems. Along the same line, in [6], [5] boundary regulators are developed for both parabolic and 2×2 hyperbolic systems in backstepping coordinates, in which the design becomes rather simple.

In this paper, an output feedback regulator is designed for infinite-dimensional systems. The design novelty is shown by constructing the *robust output regulator*. Namely, along the design procedure, *extended regulator equations* are obtained. In particular, since the disturbance location is included the resulting *extended regulator equations*, the disturbance location does not need to be known to the regulator. Instead, only the internal model S, the reference signal $y_r(t)$ and the output to be controlled $y(t)$ are needed. The final results still satisfy $\mathscr{G}-$conditions proposed in [9]. However, more detailed conditions are provided in this paper to ensure the solvability of the *extended regulator equations* and the decoupling *Sylvester equation*.

After introducing the problem formulation in the next section, Section III demonstrates the design of the output feedback regulator based on a decoupled cascade system in the nominal case. Sufficient conditions for the solvability of the *extended regulator equations* and *decoupling Sylvester equation*, and observability are given in the same section. The robustness of the achieved output regulation is explained in Section IV. Finally, the robust reference trajectory tracking and disturbance rejection for coupled hyperbolic system with spatially-varying parameter uncertainties presents the developed regulator design.

II. Problem formulation

In this paper, we consider the following infinite-dimensional single input single output (SISO) system:

$$\dot{x}(t) = A_p x(t) + B_p u(t) + Dd(t), t > 0 \tag{1a}$$

$$y(t) = C_p x(t), t \geq 0 \tag{1b}$$

where $x(t) \in X$, $u(t) \in U$, $y(t) \in Y$ and $d(t) \in \mathbb{R}$ are state, input, output, and unmeasurable disturbance generated by an exosystem that will be shown shortly. A_p, B_p and C_p are operators A, B and C perturbed by model uncertainties ΔA, ΔB and ΔC, respectively. The spaces X, U, and Y are Hilbert spaces. $A : \mathscr{D}(A) \subset X \to X$ is the system operator that generates a C_0-semigroup $T(t)$. The other operators are bounded: $B \in \mathscr{L}(U,X)$, $C \in \mathscr{L}(X,Y)$, and D denotes the disturbance location.

The exosystem that is assumed to generate the disturbance $d(t)$ and the reference signal $y_r(t)$ to be tracked is of the form:

$$\dot{v}(t) = Sv(t), t > 0, v(0) = v_0 \in W \tag{2a}$$

$$d(t) = Ev(t), t \geq 0 \tag{2b}$$

$$y_r(t) = Qv(t), t \geq 0 \tag{2c}$$

on a finite-dimensional space $W = \mathbb{C}^n$. For simplicity, the matrix $S \in \mathscr{L}(W) = \mathbb{C}^{n\times n}$ is configured as $S = diag(iw_1, \cdots, iw_n)$. Here $\{iw_k\}_{k=1}^n \subset \mathbb{R}$ are distinct and purely imaginary eigenvalues of S. E and Q are matrices with appropriate dimensions. Consequently, such exosystem generates e.g., steplike and sinusoidal functions with known frequency.

We make the following assumption on the plant:

Assumption 1:

(a1.) The pair (A,B) is exponentially stabilizable.

(a2.) The pair (C,A) is exponentially detectable.

(a3.) The transfer function of the plant $P(s) \in \mathscr{L}(U,Y)$ is nonzero for $s \in \sigma(S)$.

For the perturbed plant, we make the following assumption:

Assumption 2:

(b1.) The operator A_p remains an infinitesimal generator of a C_0-semigroup and B_p, C_p remian bounded linear operators.

(b2.) The stability of the closed-loop system is conserved.

In this paper, the *robust output regulation problem* is considered and addressed. This amounts to stabilizing *tracking error system* so that the *output tracking error* $e_y(t)$ satisfies:

$$\lim_{t\to\infty} e_y(t) = \lim_{t\to\infty} (y(t) - y_r(t)) = 0 \tag{3}$$

for all initial conditions of the plant (1), of the exosystem (2) and of the controller to be designed. The property (3) is said to be *robust* when it holds for all model uncertainties ΔA, ΔB and ΔC for the non-destabilizing closed-loop system.

III. Nominal output regulation

In this section, the controller is designed based on the nominal system with $\Delta A = 0$, $\Delta B = 0$, and $\Delta C = 0$ in (1). According to [17], to reach *robust output regulation*, the *Internal Model Principle* has to be satisfied. For this reason, a part of controller has the following form:

$$\dot{\hat{v}}(t) = S\hat{v}(t) + B_y e_y(t), t > 0, \hat{v}(0) \in \mathbb{C}^n \tag{4}$$

To satisfy the *Internal Model Principle*, S and B_y have to satisfy $\mathscr{G}-$conditions in [9]. Therefore, one choice is to set all elements of B_y to be nonzero. In other words, if we write $B_y = \begin{bmatrix} b_{y1} \\ \vdots \\ b_{yn} \end{bmatrix}$, then $b_{yk} \neq 0$ for all $k = 1, 2, \cdots, n$. On the other side, this configuration ensures the excitation of all modes in (4) by $e_y(t)$. Consequently, the following observer-based controller is used for the stabilization of the composite system consisting of the plant (1) and the system (4).

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\Delta y(t) \tag{5a}$$

$$u(t) = k_x \hat{x}(t) + k_v^T \hat{v}(t) \tag{5b}$$

The system (5a) is a state observer estimating the state x of the nominal plant and the observer injection gain L is to be determined. $\Delta y = y(t) - C\hat{x}(t)$. Moreover, (5b) shows formal feedback control law as well as feedback gains $k_x \in \mathscr{L}(X,U)$ and $k_v \in \mathscr{L}(\mathbb{C}^n, U)$. To solve the output regulation given in (3) for the nominal case, we need to solve the stabilization of the nominal plant (1) with internal model (4) by considering solving (5). To this end, the bounded change of coordinates is introduced:

$$v(t) = Iv(t) \tag{6a}$$

$$e_x(t) = x(t) - \pi_w^T v(t) \tag{6b}$$

$$e_v(t) = \hat{v}(t) - \Pi v(t) \tag{6c}$$

$$\hat{e}_x(t) = \hat{x}(t) - \hat{\pi}_w^T v(t) \tag{6d}$$

with $\pi_w^T \in \mathscr{L}(\mathbb{C}^n, X)$, $\hat{\pi}_w^T \in \mathscr{L}(\mathbb{C}^n, X)$, and $\Pi \in \mathbb{R}^{n\times n}$.

In order to make the resulting $(e_x, e_v, \hat{e}_x)-$subsystem independent from v, one obtains $\pi_w^T, \Pi, \hat{\pi}_w^T$ as solutions to the extended regulator equations:

$$A\pi_w^T - \pi_w^T S + DE + Bk_v^T \Pi + Bk_x \hat{\pi}_w^T = 0 \tag{7a}$$

$$A\hat{\pi}_w^T - \hat{\pi}_w^T S + L\Delta\tilde{\pi}_y + Bk_v^T \Pi + Bk_x \hat{\pi}_w^T = 0 \tag{7b}$$

$$C\pi_w^T - Q = 0 \tag{7c}$$

$$S\Pi - \Pi S = 0 \tag{7d}$$

with $\Delta\tilde{\pi}_y = C\left[\pi_w^T - \hat{\pi}_w^T\right]$. Observe the above equation, since S is a diagonal matrix with distinct eigenvalues, then Π in (7d) is also a diagonal matrix with arbitrary elements on the diagonal line. This freedom is used to make (7a)-(7c) hold simultaneously.

Then, the closed-loop nominal system and $e_y(t) = y(t) - y_r(t)$ takes the form:

$$\dot{e}_x(t) = Ae_x(t) + Bk_x\hat{e}_x(t) + Bk_v^T e_v(t) \tag{8a}$$

$$\dot{\hat{e}}_x(t) = A\hat{e}_x(t) + Bk_x\hat{e}_x(t) + Bk_v^T e_v(t) + L\Delta\tilde{y}(t) \tag{8b}$$

$$\dot{e}_v(t) = Se_v(t) + B_y Ce_x(t) \tag{8c}$$

$$e_y(t) = Ce_x(t) \tag{8d}$$

in the new coordinates, with $\Delta\tilde{y}(t) = C[e_x(t) - \hat{e}_x(t)]$. Suppose that the extended regulator equations (7) have a solution π_w^T, Π, and $\hat{\pi}_w^T$. Then, a stabilization of the origin of $(e_x, e_v, \hat{e}_x)-$subsystem (8a)-(8c) directly indicates nominal output regulation (3) in view of (8d). This stabilization is independent from the disturbance location and the modelling of signals in (2b)-(2c) because the extended regulator equations (7) is assumed to be solvable. Therefore, knowledge of these quantities is not needed but only the solvability of the extended regulator equations (7) is required. Moreover, if no exogenous signals are present to affect the plant, then (6b)-(6d) means $e_x = x$, $e_v = v$ and $\hat{e}_x = \hat{x}$. As a result, the controller (8b)-(8c) also stabilizes the origin of the closed-loop plant in the original coordinates.

Equations (8a)-(8c) is a highly coupled system containing infinite and finite-dimensional subsystems. In order to simplify the presentation of the coupled system and thus to determine k_x, k_v^T and L easily, two new coordinates are introduced here:

$$\tilde{e}_x(t) = e_x(t) - \hat{e}_x(t) \tag{9}$$

$$\varepsilon_v(t) = e_v(t) - He_x(t) \tag{10}$$

in (6b) and (6c), where $H \in \mathscr{L}(X, \mathbb{C}^n)$ is to be determined.

In the above new coordinates, a cascade system is then obtained:

$$\begin{aligned}\dot{e}_x(t) = &\left[A + B\left(k_x + k_v^T H\right)\right] e_x(t) \\ &-Bk_x\tilde{e}_x(t) + Bk_v^T \varepsilon_v(t)\end{aligned} \tag{11a}$$

$$\dot{\varepsilon}_v(t) = \left(S - HBk_v^T\right)\varepsilon_v(t) + HBk_x\tilde{e}_x(t) \tag{11b}$$

$$\dot{\tilde{e}}_x(t) = (A - LC)\,\tilde{e}_x(t) \tag{11c}$$

if the operator $H \in \mathscr{L}(X, \mathbb{C}^n)$ satisfies the following Sylvester equation:

$$SH - H\left[A + B\left(k_x + k_v^T H\right)\right] + B_y C = 0 \tag{12}$$

on $\mathscr{D}(S)$.

For simplicity, we use $\tilde{k}_x = k_x + k_v^T H$ and $\tilde{B}_H = HB$. Then, (8) and (12) can be written as:

$$\dot{e}_x(t) = \left[A + B\tilde{k}_x\right] e_x(t) - Bk_x\tilde{e}_x(t) + Bk_v^T \varepsilon_v \tag{13a}$$

$$\dot{\varepsilon}_v(t) = \left(S - \tilde{B}_H k_v^T\right)\varepsilon_v + HBk_x\tilde{e}_x(t) \tag{13b}$$

$$\dot{\tilde{e}}_x(t) = (A - LC)\,\tilde{e}_x(t) \tag{13c}$$

if the operator satisfies:

$$SH - H(A + B\tilde{k}_x) + B_y C = 0 \tag{14}$$

Obviously, one can first find $\tilde{k}_x$ such that the operator $A + B\tilde{k}_x$ generates an exponentially stable C_0-semigroup $T_K(t)$ for $t > 0$, and thus $T_K(t)$ has property $\|T_K(t)\| \leq M_x e^{\beta_x t}$ for constants $M_x > 0$ and $\beta_x < 0$. Suppose that the equation (14) is solvable, then H can be obtained. Finally, $\tilde{B}_H = BH$ can be computed and the remaining part is to find k_v^T such that the matrix $S - \tilde{B}_H k_v^T$ is Hurwitz. Due to **(a2)** in Assumption 1, it is possible to find L so stabilize the operator $A - LC$ and the resulting C_0-semigroup $T_L(t)$ satisfies $\|T_L(t)\| \leq M_o e^{\beta_o t}$ for $M_o > 0$ and $\beta_o < 0$.

The significant step to guarantee the successful design of the regulator is to ensure the existence of solution to the Sylvester equation (14). In what follows, solvability conditions will be provided.

Lemma 1: (Solvability) The decoupling Sylvester equation (14) has a unique solution $H \in \mathscr{L}(X, W)$ on the domain $\mathscr{D}(S)$, if the following operator is invertible:

$$\left[s - \left(A + B\tilde{k}_x\right)\right], \forall s \in \sigma(S) \tag{15}$$

Proof: We begin the proof with defining vectors $e_k^T = \begin{bmatrix} 0_{1,k-1} & 1 & 0_{1,n-k} \end{bmatrix}$ with $0 < k \leq n$. Premultiplying (14) by e_k^T, one obtains a set of scalar equations:

$$iw_k h_k^* - h_k^*\left(A + B\tilde{k}_x\right) + b_{yk}^* C = 0 \tag{16}$$

for $k = 1, 2, \cdots, n$, with $h_k^* = e_k^T H$ and $b_{yk}^* = e_k^T B_y$. Obviously, it is easy to write H and B_y as $Hx = \begin{bmatrix} h_1^* & \cdots & h_n^* \end{bmatrix}^T x$ and $B_y = \begin{bmatrix} b_{y1}^* & \cdots & b_{yn}^* \end{bmatrix}^T$. Therefore, the invertibility of $\left[iw_k - \left(A + B\tilde{k}_x\right)\right]$ directly indicates the existence and uniqueness of h_k^* for $k = 1, 2, \cdots, n$, and thus the existence and uniqueness of H is ensured. ■

From (16), h_k^* is computed as:

$$h_k^* = -b_{yk}^* CR\left(iw_k; A + B\tilde{k}_x\right)$$

Then, it is straightforward to write Hx as

$$Hx = -\begin{bmatrix} b_{y1}CR\left(iw_1 : A + B\tilde{k}_x\right) \\ \vdots \\ b_{yn}CR\left(iw_n : A + B\tilde{k}_x\right) \end{bmatrix} x$$

As a consequence, we compute the term $\tilde{B}_H = HB$:

$$HB = -\begin{bmatrix} b_{y1}\tilde{P}_K(iw_1) \\ \vdots \\ b_{yn}\tilde{P}_K(iw_n) \end{bmatrix}$$

with $\tilde{P}_K(iw_k) = CR\left(iw_k : A + B\tilde{k}_x\right)B$, $k = 1, 2, \cdots, n$.

Lemma 2: (Observability) If the pair (S, B_y) is controllable, then the pair $(S, \tilde{B}_H)$ is also controllable.

Proof: From Ch.2.4.1 of [12], the pair $(S, \tilde{B}_H)$ is controllable if $e_k^T \tilde{B}_H = \tilde{b}_{hk}^* \neq 0$ for all $k = 1, 2, \cdots, n$, since S has distinct eigenvalues. Here e_k^T is already defined in Lemma 1. From Lemma 1, the term $\tilde{b}_{hk}^*$ is rewritten as:

$$\begin{aligned}\tilde{b}_{hk}^* &= e_k^T HB \\ &= -e_k^T B_y C\left[iw_k - \left(A + B\tilde{k}_x\right)\right]^{-1} B \\ &= -e_k^T B_y \tilde{P}_K\left(iw_k\right)\end{aligned}$$

Due to **(a3)** in Assumption 1, it is easy to find $\tilde{k}_x$ such that the transfer function of stabilized plant $\tilde{P}_K(s)$ is nonzero for $s \in \sigma(S)$, i.e, $\tilde{P}_K(s) \neq 0$, $\forall s \in \sigma(S)$. Because the pair (S, B_y) is controllable, then $e_k^T B_y \neq 0$ for all $k = 1, 2, \cdots, n$, and thus $\tilde{b}_{hk}^* \neq 0$. ■

From Lemma 2, the controllability of the pair (S, B_y) directly means $b^*_{yk} \neq 0$, $\forall k = 1, 2, \cdots, n$. For simplicity, B_y can be chosen such that b^*_{yk} are nonzero constant for all $k = 1, 2, \cdots, n$. It is easy to check that this configuration always satisfy $\mathscr{G}-$conditions presented in [9], i.e, $\mathscr{N}(B_y) = \{0\}$ and $\mathscr{R}(B_y) \cap \mathscr{R}(S - iw_k I) = \{0\}$.

Through above illustration, the output feedback regulator (4) and (5) is designed to resolve the output regulation problem for the plant (1) with the exosystem (2). Moreover, injection and feedback gains are found. In order to make the design procedure more clear, the main steps are listed in following:

Algorithm 1: Design of the output feedback regulator
step 1: Find feedback gain $\tilde{k}_x$ to stabilize the operator $A + B\tilde{k}_x$;
step 1: Given selected B_y, solve the Sylvester equation (14) for H;
step 2: Calculate $\tilde{B}_H = HB$ and find k_v^T such that $S - \tilde{B}_H k_v^T$ is Hurwitz;
step 3: Find L to stabilize the operator $A - LC$;
step 4: Given computed $\tilde{k}_x$, k_v^T and H, compute $k_x = \tilde{k}_x - k_v^T H$;
step 5: Construct the output feedback regulator (4)–(5).

Theorem 1: Consider the controller (4)-(5) with the feedback gains k_x and k_v^T and the observer gain L found in **Algorithm 1**. If we define a negative constant as $\tilde{\beta} = \max(\beta_x, \beta_v, \beta_o) < 0$ with $\beta_v = \max_{\lambda \in \sigma(S - \tilde{B}_H k_v^T)} \Re(\lambda)$, then the origin of the $(e_x, e_v, \hat{e}_x)-$subsystem (8a)-(8c) is exponentially stable in the norm $\|\cdot\| = \left(\|\cdot\|^2_X + \|\cdot\|^2_{\mathbb{C}^n} + \|\cdot\|^2_X \right)^{\frac{1}{2}}$. More, precisely, the closed-loop state $\tilde{\varepsilon}(t) = col(e_x, e_v, \hat{e}_x)$ satisfies $\|\tilde{\varepsilon}(t)\| \leq M_{\tilde{\alpha}} e^{\tilde{\alpha} t} \|\tilde{\varepsilon}(0)\|, t \geq 0$, for all $\tilde{\varepsilon}(t) \in X \oplus \mathbb{C}^n \oplus X$ and $M_{\tilde{\alpha}}$. Moreover, the output regulation is achieved, i.e., (3) holds.

Theorem 2: The extended regulator equations (7) has a solution $\pi_w \in \mathscr{L}(\mathbb{C}^n, X)$, $\hat{\pi}_w \in \mathscr{L}(\mathbb{C}^n, X)$, and $\Pi \in \mathscr{L}(\mathbb{C}^n)$, if the function $P_{LK}(s) = CR(s; A - LC + Bk_x)B$ exists and is nonzero for all $s \in \sigma(S)$, and all elements of k_v^T are nonzero.

Proof: We begin with rewriting the extended regulator in (7) as the following decoupled form:

$$\pi_w^T S - A\pi_w^T = DE + Bk_v^T \Pi + Bk_x \hat{\pi}_w^T \tag{17a}$$

$$\hat{\pi}_w^T S - (A + Bk_x - LC)\hat{\pi}_w^T = LQ + Bk_v^T \Pi \tag{17b}$$

$$C\pi_w^T - Q = 0 \tag{17c}$$

$$S\Pi - \Pi S = 0 \tag{17d}$$

Postmultiplying (17a), (17b) and (17c) with e_k, and premultiplying and postmultiplying (17d) with e_k^T and e_k for $k = 1, 2, \cdots, n$, where e_k^T, $\forall k = 1, 2, \cdots, n$ were already defined in the proof of Lemma 1, a set of sub-equations are obtained:

$$iw_k \pi^*_{wk} - A\pi^*_{wk} = DEe_k + Bk_v^T \Pi_k + Bk_x \hat{\pi}^*_{wk} \tag{18a}$$

$$iw_k \hat{\pi}^*_{wk} - (A + Bk_x - LC)\hat{\pi}^*_{wk} = Lq^*_k + Bk_v^T \Pi_k \tag{18b}$$

$$C\pi^*_{wk} = q^*_k \tag{18c}$$

$$iw_k \Pi^*_k - iw_k \Pi^*_k = 0 \tag{18d}$$

with $\pi_w^T = \begin{bmatrix} \pi^*_{w1} & \cdots & \pi^*_{wn} \end{bmatrix}$, $\hat{\pi}_w^T = \begin{bmatrix} \hat{\pi}^*_{w1} & \cdots & \hat{\pi}^*_{wn} \end{bmatrix}$, $\Pi = diag(\Pi^*_1, \cdots, \Pi^*_n)$, $\Pi_k = \begin{bmatrix} 0_{1,k-1} & \Pi^*_k & 0_{1,n-k} \end{bmatrix}^T$ and $Q = \begin{bmatrix} q^*_1 & \cdots & q^*_n \end{bmatrix}$. Observe (18d), Π^*_k is arbitrary and to be determined to assist (18a)-(18c). Observe (18b), given fixed but unknown Π^*_k, the invertibility of $[s - (A + Bk_x - LC)]$, $\forall s \in \sigma(S)$ uniquely determine the existence of $\hat{\pi}^*_{wk}$, i.e., $\hat{\pi}^*_{wk} = R(iw_k; A + Bk_x - LC)\left(Lq^*_k + Bk_v^T \Pi_k\right)$. The existence of the function $P_{LK}(s)$ directly means the invertibility of $[s - (A + Bk_x - LC)]$. Then, from (18a), π^*_{wk} are uniquely determined due to **(a3)** in Assumption 1 and $\pi^*_{wk} = R(iw_k; A)\left[DEe_k + Bk_v^T \Pi_k + Bk_x \hat{\pi}^*_{wk}\right]$ given fixed Π^*_k and calculated $\hat{\pi}^*_{wk}$. Now we insert the solutions π^*_{wk} and $\hat{\pi}^*_{wk}$ into (18c) to determine Π^*_k. Expanding (18c) yields:

$$\begin{aligned} &-CR(iw_k; A)[I + Bk_x R(iw_k; A + Bk_x - LC)]Bk_v^T \Pi_k \\ &= CR(iw_k; A)DEe_k - q^*_k \\ &\quad + CR(iw_k; A)Bk_x R(iw_k; A + Bk_x - LC)Lq^*_k \end{aligned} \tag{19}$$

From Woodbury formula and the identity $R(iw_k; A) = R(iw_k; A - LC)[I - LCR(iw_k; A - LC)]^{-1}$, we have

$$\begin{aligned} CR(iw_k; A) &= CR(iw_k; A - LC)[I - LCR(iw_k; A - LC)]^{-1} \\ &= [I - R(iw_k; A - LC)L]^{-1} CR(iw_k; A - LC) \end{aligned}$$

$$\begin{aligned} &CR(iw_k; A)[I + Bk_x R(iw_k; A + Bk_x - LC)] \\ &= [I - R(iw_k; A - LC)L]^{-1} \\ &\quad \times CR(iw_k; A - LC)[I + Bk_x R(iw_k; A - LC + Bk_x)] \\ &= [I - R(iw_k; A - LC)L]^{-1} CR(iw_k; A - LC + Bk_x) \end{aligned}$$

As a result, the equation (19) becomes to:

$$\begin{aligned} &[R(iw_k; A - LC)L - I]^{-1} CR(iw_k; A - LC + Bk_x)Bk_v^T \Pi_k \\ &= CR(iw_k; A)DEe_k - q^*_k \\ &\quad + CR(iw_k; A)Bk_x R(iw_k; A + Bk_x - LC)Lq^*_k \end{aligned}$$

Due to condition that $P_{LK}(s) \neq 0$, the terms $\forall s \in \sigma(S)$, $k_v^T \Pi_K$ are uniquely determined. If rewrite k_v^T as $k_v^T = \begin{bmatrix} k_{v1} & \cdots & k_{vn} \end{bmatrix}$, one immediately has $k_v^T \Pi_k = k_{vk} \Pi^*_k$. Apparently, Π^*_k can be determined uniquely only if $k_{vk} \neq 0$ for all $k = 1, 2, \cdots, n$. ■

In Theorem 3, k_v^T can be set up to satisfy an alternative condition that the pair (k_v^T, S) is observable. This requires $k_v^T e_k \neq 0$, $\forall k = 1, 2, \cdots, n$, since S is a diagonal matrix.

IV. ROBUST OUTPUT REGULATION

In this section, the robustness of the output regulation for the uncertain plant (1) with the output feedback regulator (4) and (5) is considered and discussed. Assume that for uncertain parameters $\tilde{A} = A + \Delta A$, $\tilde{B} = B + \Delta B$ and $\tilde{C} = C + \Delta C$, the extended regulator equations (7) still have unique solutions π_w^T, Π and $\hat{\pi}_w^T$. It is straightforward to see the closed-loop system (1), (2a) and (4)-(5) with $e_y(t) = y(t) - y_r(t)$ is given by (8) with $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. Considering the nominal case with $\Delta A = 0$, $\Delta B = 0$ and $\Delta C = 0$, the close-loop system (8a)-(8c) is exponentially stable due to Theorem 1. Hence, *robust output regulation* is achieved if the model uncertainties ΔA, ΔB and ΔC do not destroy the exponential stability of the closed-loop system (8a)-(8c). This result is shown in the next theorem.

Theorem 3: The output regulator (4)-(5) solves the output regulation problem for all model uncertainties $\Delta A = 0$, $\Delta B =$

0 and $\Delta C = 0$. If an exponentially stable origin of the closed-loop system (8a)-(8c) with $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ is ensured, then $\lim_{t\to\infty} e_y(t) = 0$.

Proof: Because the exponential stability of the closed-loop system (8a)-(8c) is not destabilized by model uncertainties, then the conditions in Theorem 2 are independent from the the model uncertainties and the observer design in (5). Therefore, the conditions in Theorem 2 are satisfied, which implies the solvability of the extended regulator equations. ■

V. Numerical simulation

In this subsection, an illustrative coupled hyperbolic PDE system is studied to demonstrate the construction and performance of the proposed regulator (4)-(5).

$$\frac{\partial x_1}{\partial t}(\zeta,t) = -\frac{\partial x_1}{\partial \zeta}(\zeta,t) + \gamma_1(\zeta)x_1(\zeta,t) + \gamma_2(\zeta)x_2(\zeta,t) + u(t) \tag{20a}$$

$$\frac{\partial x_2}{\partial t}(\zeta,t) = -\frac{\partial x_2}{\partial \zeta}(\zeta,t) + \gamma_3(\zeta)x_1(\zeta,t) + \gamma_4(\zeta)x_2(\zeta,t) + 0.8e^{\zeta}d(t) \tag{20b}$$

$$x_1(0,t) = 0,\ x_2(0,t) = 0 \tag{20c}$$

$$y(t) = x_2(1,t) \tag{20d}$$

with $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in L^2(0,1)^2$ and $\zeta \in (0,1), t \in R^+$. The output to be controlled $y(t)$ is obtained through boundary point evaluation of the state $x_2(\zeta,t)$, respectively. The nominal spatial varying coefficients are $\gamma_1(\zeta) = 2\zeta^2(\zeta+2)$, $\gamma_2(\zeta) = -5e^{\zeta}$, $\gamma_3(\zeta) = 0.75(1+\zeta)$ and $\gamma_4(\zeta) = -3(1+e^{\zeta})$. We are interested in designing an output feedback regulator to achieve the robust asymptotic tracking of sinusoid reference signal $y_r(t) = M_0\sin(w_0 + \phi_0)$ for $M_0, \phi_0 \in \mathbb{R}$ and $w_0 = 5$, and the robust steplike disturbance rejection $d(t) = D_0$ for $D_0 \in \mathbb{R}$. These signals can be modelled by (2) with $S = diag(0, iw_0, -iw_0)$, $E = \begin{bmatrix} E_1 & 0 & 0 \end{bmatrix}$, $E_1 \in \mathbb{R}$ and $Q = \begin{bmatrix} 0 & -2 & 2 \end{bmatrix}$. The initial condition v_0 is $v_0 = \begin{bmatrix} 1 & 0.5i & 0.5i \end{bmatrix}$.

Writing (20) as the algebraic form (1), then operators in (1) are denoted by:

$$A = \begin{bmatrix} -\frac{\partial}{\partial\zeta} + \gamma_1(\zeta) & \gamma_2(\zeta) \\ \gamma_3(\zeta) & -\frac{\partial}{\partial\zeta} + \gamma_4(\zeta) \end{bmatrix},$$

on the domain $\mathscr{D}(A) = \left\{ h \in H^1(0,1)^2, h_1(0) = 0 = h_2(0) \right\}$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0.8e^{\zeta} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & \int_0^1 c(\zeta)(\cdot)d\zeta \end{bmatrix}$$

where $c(\zeta) = \frac{1}{2\upsilon}\mathbf{1}_{[\zeta_1-\upsilon,\zeta_1+\upsilon]}(\zeta)$ with $\zeta_1 = 0.995$ and $\upsilon = 0.05$, and $\mathbf{1}_{[a,b]}(\zeta) = \begin{cases} 1, \zeta \in [a,b] \\ 0, \text{ otherwise} \end{cases}$.

For the sake of simplicity, the injection gain B_y is chosen as: $B_y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ to ensure the controllability of pair (S, B_y). The operator A in the considered system generates an exponentially stable C_0-semigroup and thus k_x can be chosen as 0. Therefore, $\tilde{B}_H = -\begin{bmatrix} P(0) & P(iw_0) & P(-iw_0) \end{bmatrix}^T$ with the transfer function $P(s) = CR(s;A)B$. Using the numerical method in [22], $P(s)$ and $CR(s;A)$ are computed with $s = 0, i*5, -i*5$. Then, $\tilde{B}_H = \begin{bmatrix} -0.2095 & 0.1054+0.6771i & 0.1054-0.6771i \end{bmatrix}^T$. Moreover, H is calculated:

$$Hx = \begin{bmatrix} -0.2095 & 0.1054+0.6771i & 0.1054-0.6771i \\ -0.0962 & 0.1886+0.6266i & 0.1886-0.6266i \end{bmatrix}^T \times \begin{bmatrix} \int_0^1 c(\zeta)x_1(\zeta,t)d\zeta \\ \int_0^1 c(\zeta)x_2(\zeta,t)d\zeta \end{bmatrix}$$

Consequently, k_v^T is chosen as $k_v^T = \begin{bmatrix} -6 & 2 & 2 \end{bmatrix}$ to assign the eigenvalue set $\{-1.2101, -2.5439 \pm 8.9489i\}$ to $S - \tilde{B}_H k_v^T$. The corresponding feedback gain k_x is then $k_x x = -k_v^T Hx = 1.8894\int_0^1 c(\zeta)x_1(\zeta,t)d\zeta + 0.6904\int_0^1 c(\zeta)x_2(\zeta,t)d\zeta$. Based on results in [23], the injection gain L is chosen as $L = kB$ with $k \in [-k_*, k_*]$ for some positive constants $k_* > 0$. Figures 1-3 present the resulting disturbance rejection behaviour for parameter uncertainties $\Delta\gamma_1(\zeta) = -0.2\gamma_1(\zeta)$, $\Delta\gamma_2(\zeta) = -0.3\gamma_2(\zeta)$, $\Delta\gamma_3(\zeta) = 0.2\gamma_3(\zeta)$ and $\Delta\gamma_4(\zeta) = -0.3\gamma_4(\zeta)$. Thereby, the steplike disturbance is $D_0 = 2$. A sinusoid reference signal $y_r(t) = 2sin(5t)$ is applied and the resulting resulting reference trajectory tracking behaviour is shown in Figure 4. The result verifies robust output regulation in the reference tracking behaviour for the same uncertain model.

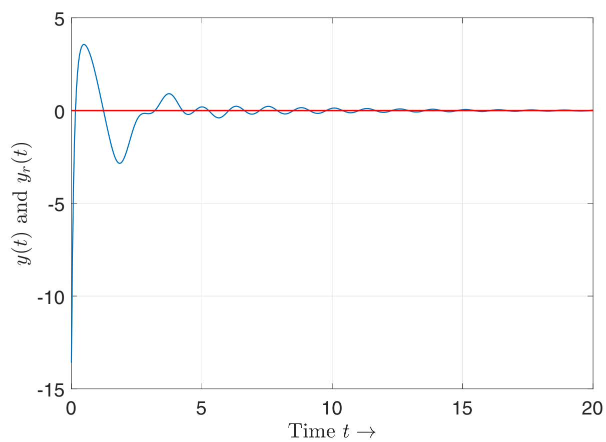

Fig. 1. Closed-loop disturbance rejection behaviour of closed-loop system with model uncertainties $\Delta\gamma_1(\zeta) = -0.2\gamma_1(\zeta)$, $\Delta\gamma_2(\zeta) = -0.3\gamma_2(\zeta)$, $\Delta\gamma_3(\zeta) = 0.2\gamma_3(\zeta)$ and $\Delta\gamma_4(\zeta) = -0.3\gamma_4(\zeta)$ for a steplike disturbance $D_0 = 2$.

VI. Conclusions

In this paper, due to the model uncertainties, a robust output regulator is considered and designed. Since the designed regulator has to satisfy Internal Model Principle and $\mathscr{G}-$conditions introduced in [17], the results in this paper are similar to the work in [17]. However, the design procedure and analysis are different. In particular, an easier interpretation is presented. Moreover, more details and additional conditions are provided for injection and feedback gains

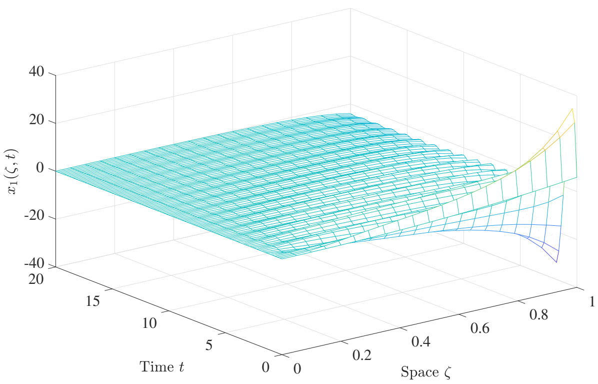

Fig. 2. The evolution of state $x_1(\zeta,t)$ under disturbance rejection control for uncertain closed-loop system.

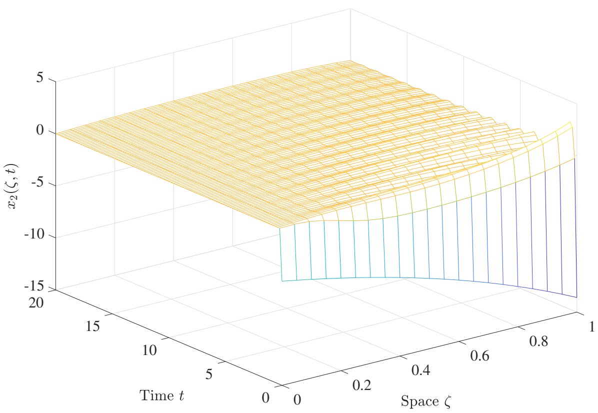

Fig. 3. The evolution of state $x_2(\zeta,t)$ under disturbance rejection control for uncertain closed-loop system.

design such that the solvability of the extended regulator equations is ensured and $\mathscr{G}-$condtions are easily satisfied. More precisely, results in Theorem 3 are new. Future work will extend the present approach to boundary controlled infinite-dimensional systems and high order systems.

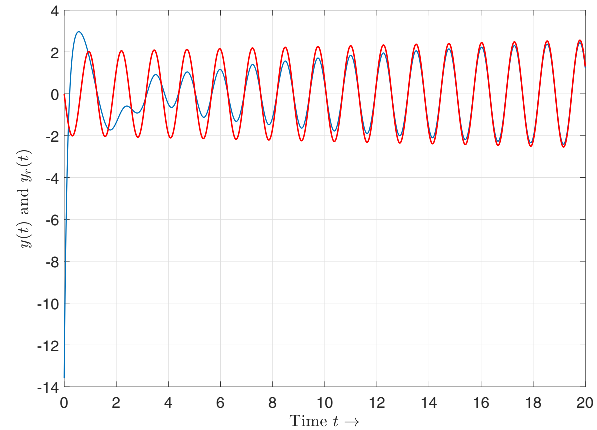

Fig. 4. Tracking behaviour for a sinusoidal reference signal $y_r(t) = 2sin(5t)$ of the closed-loop system with model uncertainties given in Figure 1. Here $y(t)$ is the output to be controlled.

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