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Input-to-state stability of unbounded bilinear control systems

Birgit Jacob[1] Felix Schwenninger[2]

Abstract— **We study input-to-state stability of bilinear control system with possibly unbounded control operator and unbounded bilinearity. We show that every internally exponentially stable bilinear control system is integral input-to-state stable. An application to the bilinearly controlled Fokker-Planck equation is given.**

I. Introduction

The concept of *input-to-state stability*, introduced by E. Sontag in 1989 [1], is a well-studied stability notion of control systems with respect to external inputs. For a survey on input-to-state stability for finite-dimensional systems we refer the reader to [2]. A variant of classic input-to-state stability is the notion of *integral input-to-state stability*, see e.g., [3]. We note that for linear, finite-dimensional systems input-to-state stability, integral input-to-state stability *and (exponential) internal stability (i.e. without control)* are all equivalent. However, when it comes to bilinear, finite-dimensional systems integral input-to-state stability is a weaker notion than input-to-state stability, and the latter is rarely satisfied.

For infinite-dimensional systems, input-to-state stability and integral input-to-state stability have been less studied, but more intensively in the recent past, see [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]. In [9] infinite-dimensional bilinear control systems with *bounded* control operator and bilinearity are studied and the equivalence of integral input-to-state stability and exponential stability is shown. In this talk we aim to generalize this result to infinite-dimensional bilinear control systems with unbounded operators. This generalization enables us to show integral input-to-state estimates for a Fokker-Planck equation controlled through a bilinear control operator.

II. Input-to-state stabilty

We study infinite-dimensional bilinear control systems of the form

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + u_1(t)B_1x(t) + B_2u_2(t),\ t \geq 0\\ x(0) &= x_0,\end{aligned}\right\} \quad (1)$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, $B_1 \in L(X, X_{-1})$, $B_2 \in L(U, X_{-1})$ for some Banach space U, $u_1 \in L^\infty(0,\infty)$, $u_2 \in L^\infty(0,\infty;U)$ and $x_0 \in X$. Note that B_1 and B_2 are possibly unbounded from X to X and from U to X. Here X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X$ for some β in the resolvent set $\rho(A)$ of A. The semigroup $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} is an extension of A, see e.g. [16]. Thus we may consider Equation (1) on the Banach space X_{-1}. A continuous function $x : [0, t_0] \to X$ is called a *mild solution of* (1), if

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)[u_1(s)B_1x(s) + B_2u_2(s)]ds,$$

for every $t \in [0, t_0]$.

Definition 1: Let $q \in [1,\infty]$ and Y be a Banach space. An operator $B \in L(Y, X_{-1})$ is called an *q-admissible control operator* if

$$\int_0^t T_{-1}(t-s)By(s)ds \in X$$

for every $t \geq 0$ and $y \in L^q(0,\infty;Y)$.

Clearly, p-admissibility implies q-admissibility if $1 \leq p \leq q \leq \infty$. If X is reflexive, then 1-admissibility implies boundedness of the operator B, that is $B \in L(Y, X)$, [17, Thm. 4.8]. Moreover, the operator B is q-admissible if and only if for every $t \geq 0$ there exists a constant $K_t \geq 0$ such that

$$\left\| \int_0^t T_{-1}(t-s)By(s)ds \right\| \leq K_t \|y\|_{L^q} \quad (2)$$

for every $y \in L^q(0,\infty;Y)$. Clearly, the best constant K_t is non-decreasing in t.

Definition 2: Let $q \in [1,\infty]$. The operator $B \in L(Y, X_{-1})$ is called an *infinite-time q-admissible control operator*, if $B \in L(Y, X_{-1})$ is a q-admissible control operator and $\sup_{t\in[0,\infty)} K_t < \infty$.

We note that for exponentially stable semigroups $(T(t))_{t\geq 0}$ an operator $B \in L(Y, X_{-1})$ is infinite-time q-admissible if and only if B is q-admissible.

Proposition 3: If $B_1 \in L(X, X_{-1})$ and $B_2 \in L(U, X_{-1})$ are q-admissible control operators with $q \in [1,\infty)$, then for every $x_0 \in X$ and every $u_1 \in L^\infty(0,\infty)$, $u_2 \in L^\infty(0,\infty;U)$ the system (1) possesses a unique mild solution on $[0,\infty)$.

We will need the following well-known function classes from Lyapunov theory.

$$\begin{aligned}
\mathcal{K} &= \{\mu \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \mid \mu(0) = 0, \mu \text{ strictly increasing}\},\\
\mathcal{K}_\infty &= \{\theta \in \mathcal{K} \mid \lim_{x\to\infty} \theta(x) = \infty\},\\
\mathcal{L} &= \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \mid \gamma \text{ str. decreas.}, \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &= \{\beta : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot, t) \in \mathcal{K}\ \forall t, \beta(s, \cdot) \in \mathcal{L}\ \forall s\}.
\end{aligned}$$

[1] B. Jacob is with Functional Analysis group, School of Mathematics and Natural Sciences, University of Wuppertal, 42119 Wuppertal, Germany jacob@math.uni-wuppertal.de,

[2] F. Schwenninger is with Department of Mathematics, University of Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany, felix.schwenninger@uni-hamburg.de

Definition 4: (i) System (1) is called *input-to-state stable (ISS)*, if there exist functions $\beta \in \mathcal{KL}$ and $\mu_1, \mu_2 \in \mathcal{K}_\infty$ such that for every $x_0 \in X$, $u_1 \in L^\infty(0,t)$ and $u_2 \in L^\infty(0,t;U)$ there exists a unique mild solution x of (1) and

$$\begin{aligned} \|x(t)\| \quad &\leq \quad \beta(\|x_0\|, t) + \mu_1(\|u_1\|_{L^\infty(0,t)}) \\ &\quad + \mu_2(\|u_2\|_{L^\infty(0,t;U)}), \end{aligned}$$

for every $t \geq 0$.

(ii) System (1) is called *integral input-to-state stable (iISS)*, if there exist functions $\beta \in \mathcal{KL}$, $\theta_1, \theta_2 \in \mathcal{K}_\infty$ and $\mu_1, \mu_2 \in \mathcal{K}$ such that for every $x_0 \in X$ and $u_1 \in L^\infty(0,t)$ and $u_2 \in L^\infty(0,t;U)$ there exists a unique mild solution x of (1) and

$$\begin{aligned} \|x(t)\| \quad &\leq \quad \beta(\|x_0\|, t) + \theta_1 \left(\int_0^t \mu_1(\|u_1(s)\| ds \right) \\ &\quad + \theta_2 \left(\int_0^t \mu_2(\|u_2(s)\| ds \right), \end{aligned}$$

for every $t \geq 0$.

For finite-dimensional bilinear systems, Sontag [18] showed that exponentially stable systems are in general not ISS, but are always iISS. In [19] it is shown that system (1) is iISS if and only if the semigroup $(T(t))_{t\geq 0}$ is exponentially stable in the case of bounded control operators B_1 and B_2. In this talk we generalize this result to bilinear systems with unbounded control operators. Our main result is as follows.

Theorem 5: Assume that B is q-admissible for some $q \in [1, \infty)$. Then the system (1) is iISS if and only if the semigroup $(T(t))_{t\geq 0}$ is exponentially stable. Moreover, in this case $\mu_1, \mu_2 \in \mathcal{K}$ can be chosen as $\mu_1(s) = \mu_2(s) := s^q$.

III. Controlled Fokker-Planck equation

Following [20], [21] we consider a Fokker-Planck equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ of the form

$$\begin{aligned} \tfrac{\partial \rho}{\partial t}(x,t) &= \nu \Delta \rho(x,t) + \nabla \cdot \Big(\rho(x,t) \nabla (W(x) + \alpha(x) u(t)) \Big) \\ \rho(x, 0) \ &= \rho_0(x), \end{aligned}$$

where $x \in \Omega, t > 0$, with reflective boundary condition

$$0 = (\nu \nabla \rho + \rho \nabla W + \rho \nabla \alpha u) \cdot \vec{n}$$

on $\partial\Omega \times (0, \infty)$ where $\vec{n}$ refers to the unit normal vector on the boundary. Here ρ_0 denotes the initial probability distribution with $\int_\Omega \rho_0(x) dx = 1$ and $\nu > 0$. Furthermore, W, α are sufficiently smooth and the control thus enters through the potential $V(x,t) = W(x) + \alpha(x) u(t)$. Under the structural assumption that $\nabla \alpha \cdot \vec{n} = 0$ on the boundary, this system can be written as in (1) with $B_2 = 0$ The uncontrolled equation is not exponentially stable, as there exists a non-zero stationary point ρ_∞. Therefore, the system is considered on the orthogonal complement of $\operatorname{span} \rho_\infty$ in $X = L^2(\Omega)$, on which the system is in fact exponentially stable and the bilinearity is q-admissible for some $q \in [1, \infty)$.

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