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Antidifferentiation of Noncommutative Functions*

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***Abstract*— In [1], an algebraic construction is used to develop a generalized derivative operator, called the difference-differential operator, for free noncommutative functions. This paper will construct an inverse to the difference-differential operator and determine what conditions must be imposed on an nc function to allow it to have an antiderivative.**

We begin by providing the necessary background on noncommutative (nc) sets and free nc functions; we refer the reader to [1] for more detail. Let $\mathcal{R}$ be a unital, commutative ring and $\mathcal{M}, \mathcal{N}$ be $\mathcal{R}$-modules. Denote by $\mathcal{M}_{\text{nc}}$ the set of all square matrices of all sizes with entries from $\mathcal{M}$. Then a subset Ω of $\mathcal{M}_{\text{nc}}$ is called a nc set if it is closed under direct sums, i.e.,

$$X, Y \in \Omega \Longrightarrow X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega.$$

Define

1) Ω_n to be the set of all $n \times n$ matrices in Ω, and
2) $\Omega_{\text{d.s.e.}} = \{X \in \mathcal{M}_{\text{nc}} | X \oplus Y \in \Omega \text{ for some } Y \in \mathcal{M}_{\text{nc}}\}$.

Further, Ω is called right admissible if for all $X \in \Omega_n, Y \in \Omega_m$ and for all matrices $Z \in \mathcal{M}^{n \times m}$, there exists an invertible $r \in \mathcal{R}$ such that $\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$.

A mapping $f \colon \Omega \to \mathcal{N}_{\text{nc}}$ satisfying $f(\Omega_n) \subseteq \mathcal{N}^{n \times n}$ for all $n \in \mathbb{N}$ is called a (free) nc function if it respects direct sums and similarities, i.e., for all $X \in \Omega_n, Y \in \Omega_m$ and invertible $S \in \mathcal{R}^{n \times n}$ such that $SXS^{-1} \in \Omega$,

$$f(X \oplus Y) = f(X) \oplus f(Y)$$
$$f(SXS^{-1}) = Sf(X)S^{-1}$$

On nc sets, this pair of conditions is equivalent to the condition that f respects intertwinings. That is, given $X \in \Omega_n, Y \in \Omega_m$, and $S \in \mathcal{R}^{m \times n}$,

$$SX = YS \implies Sf(X) = f(Y)S.$$

When Ω is a right admissible nc set, a difference-differential operator $\Delta = \Delta_R$ acting on f can be defined by evaluating f on block upper triangular matrices,

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta f(X,Y)(Z) \\ 0 & f(Y) \end{bmatrix}.$$

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The new function, Δf, can be extended to a linear function of Z and is shown to have the following properties with respect to direct sums and similarities,

$$\Delta f(X^0 \oplus X^1, Y)\left(\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\right) = \begin{bmatrix} \Delta f(X^0, Y)(Z_1) \\ \Delta f(X^1, Y)(Z_2) \end{bmatrix},$$
$$\Delta f(X, Y^0 \oplus Y^1)(\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}) = \begin{bmatrix} \Delta f(X, Y^0)(Z_1) & \Delta f(X, Y^1)(Z_2) \end{bmatrix},$$
$$\Delta f(SXS^{-1}, Y)(SZ) = S\Delta f(X,Y)(Z),$$
$$\Delta f(X, SYS^{-1})(ZS^{-1}) = \Delta f(X,Y)(Z)S^{-1}.$$

Equivalently, Δf is said to respect intertwinings,

$$SX = WS \implies S\Delta f(X,Y)(Z) = \Delta f(W,Y)(SZ),$$
$$SY = WS \implies \Delta f(X,Y)(ZS) = \Delta f(X,W)(Z)S.$$

More generally, let $\mathcal{M}_0, \ldots, \mathcal{M}_k, \mathcal{N}_0, \ldots, \mathcal{N}_k$ be $\mathcal{R}$-modules and let $\Omega^{(0)}, \ldots, \Omega^{(k)}$ be nc sets in $\mathcal{M}_{0,\text{nc}}, \ldots, \mathcal{M}_{k,\text{nc}}$, respectively. Then, a function f defined on $\Omega^{(0)} \times \ldots \times \Omega^{(k)}$ with values k-linear mappings from $\mathcal{N}_1^{n_0 \times n_1} \times \ldots \times \mathcal{N}_k^{n_{k-1} \times n_k}$ to $\mathcal{N}_0^{n_0 \times n_k}$, or equivalently,

$$f(X^0, \ldots, X^k) \in \hom_{\mathcal{R}}(\mathcal{N}_1^{n_0 \times n_1} \otimes \ldots \otimes \mathcal{N}_k^{n_{k-1} \times n_k}, \mathcal{N}_0^{n_0 \times n_k})$$

for all $(X^0, \ldots, X^k) \in \Omega_{n_0}^{(0)} \times \ldots \times \Omega_{n_k}^{(k)}$ and all $n_0, \ldots, n_k \in \mathbb{N}$, is called an order k (free) nc function if, given matrices of appropriate sizes, it respects direct sums and similarities, i.e.,

$$f(X_1^0 \oplus X_2^0, X^1, \dots, X^k)\left(\begin{bmatrix} Z_1^1 \\ Z_2^1 \end{bmatrix}, Z^2, \dots, Z^k\right) = \begin{bmatrix} f(X_1^0, X^1, \dots, X^k)(Z_1^1, Z^2, \dots, Z^k) \\ f(X_2^0, X^1, \dots, X^k)(Z_2^1, Z^2, \dots, Z^k) \end{bmatrix},$$

$$f(X^0, \dots, X^{j-1}, X_1^j \oplus X_2^j, X^{j+1}, \dots, X^k)\left(Z^1, \dots, Z^{j-1}, \begin{bmatrix} Z_1^j & Z_2^j \end{bmatrix}, \begin{bmatrix} Z_1^{j+1} \\ Z_2^{j+1} \end{bmatrix}, Z^{j+2}, \dots, Z^k\right)$$
$$= f(X^0, \dots, X^{j-1}, X_1^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z_1^j, Z_1^{j+1}, Z^{j+2}, \dots, Z^k)$$
$$+ f(X^0, \dots, X^{j-1}, X_2^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z_2^j, Z_2^{j+1}, Z^{j+2}, \dots, Z^k),$$

$$f(X^0, \dots, X^{k-1}, X_1^k \oplus X_2^k)(Z^1, \dots, Z^{k-1}, \begin{bmatrix} Z_1^k & Z_2^k \end{bmatrix}) = \operatorname{row}\left[f(X^0, \dots, X^{k-1}, X_1^k)(Z^1, \dots, Z^{k-1}, Z_1^k) \quad f(X^0, \dots, X^{k-1}, X_2^k)(Z^1, \dots, Z^{k-1}, Z_2^k)\right],$$

$$f(S_0 X^0 S_0^{-1}, X^1, \dots, X^k)(S_0 Z^1, Z^2, \dots, Z^k) = S_0 f(X^0, \dots, X^k)(Z^1, \dots, Z^k),$$

$$f(X^0, \dots, X^{j-1}, S_j X^j S_j^{-1}, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z^j S_j^{-1}, S_j Z^{j+1}, Z^{j+2}, \dots, Z^k) = f(X^0, \dots, X^k)(Z^1, \dots, Z^k),$$

$$f(X^0, \dots, X^{k-1}, S_k X^k S_k^{-1})(Z^1, \dots, Z^{k-1}, Z^k S_k^{-1}) = f(X^0, \dots, X^k)(Z^1, \dots, Z^k) S_k^{-1}.$$

Equivalently, for appropriately sized matrices, it respects intertwinings as follows:

$$T_0 X_1^0 = X_2^0 T_0 \implies T_0 f(X_1^0, X^1, \dots, X^k)(Z^1, \dots, Z^k) = f(X_2^0, X^1, \dots, X^k)(T_0 Z^1, Z^2 \dots, Z^k),$$

$$T_j X_1^j = X_2^j T_j \implies f(X^0, \dots, X^{j-1}, X_1^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z^j T_j, Z^{j+1}, \dots, Z^k) = f(X^0, \dots, X^{j-1}, X_2^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^j, T_j Z^{j+1}, Z^{j+2}, \dots, Z^k),$$

$$T_k X_1^k = X_2^k T_k \implies f(X^0, \dots, X^{k-1}, X_1^k)(Z^1, \dots, Z^{k-1}, Z^k T_k) = f(X^0, \dots, X^{k-1}, X_2^k)(Z^1, \dots, Z^k) T_k.$$

In this case, we write $f \in \mathcal{T}^k(\Omega^{(0)}, \dots, \Omega^{(k)}; \mathcal{N}_{0,\text{nc}}, \dots, \mathcal{N}_{k,\text{nc}})$.

Under this definition, we say our original nc functions are of order 0.

The difference-differential operator can be extended to order k nc functions as follows:

$$f\left(\begin{bmatrix} X_1^0 & Z \\ 0 & X_2^0 \end{bmatrix}, X^1, \dots, X^k\right)\left(\begin{bmatrix} Z_1^1 \\ Z_2^1 \end{bmatrix}, Z^2, \dots, Z^k\right) = \operatorname{col}\left[f(X_1^0, X^1, \dots, X^k)(Z_1^1, Z^2, \dots, Z^k) + {}_0\Delta f(X_1^0, X_2^0, X^1, \dots, X^k)(Z, Z_2^1, Z^2, \dots, Z^k),\ f(X_2^0, X^1, \dots, X^k)(Z_2^1, Z^2, \dots, Z^k)\right],$$

$$f\left(X^0, \dots, X^{j-1}, \begin{bmatrix} X_1^j & Z \\ 0 & X_2^j \end{bmatrix}, X^{j+1}, \dots, X^k\right)\left(Z^1, \dots, Z^{j-1}, \begin{bmatrix} Z_1^j & Z_2^j \end{bmatrix}, \begin{bmatrix} Z_1^{j+1} \\ Z_2^{j+1} \end{bmatrix}, Z^{j+2}, \dots, Z^k\right)$$
$$= f(X^0, \dots, X^{j-1}, X_1^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z_1^j, Z_1^{j+1}, Z^{j+2}, \dots, Z^k)$$
$$+ {}_j\Delta f(X^0, \dots, X^{j-1}, X_1^j, X_2^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z_1^j, Z, Z_2^{j+1}, Z^{j+2}, \dots, Z^k)$$
$$+ f(X^0, \dots, X^{j-1}, X_2^j, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z_2^j, Z_2^{j+1}, Z^{j+2}, \dots, Z^k),$$

$$f\left(X^0, \dots, X^{k-1}, \begin{bmatrix} X_1^k & Z \\ 0 & X_2^k \end{bmatrix}\right)\left(Z^1, \dots, Z^{k-1}, \begin{bmatrix} Z_1^k & Z_2^k \end{bmatrix}\right) = \operatorname{row}\left[f(X^0, \dots, X^{k-1}, X_1^k)(Z^1, \dots, Z^{k-1}, Z_1^k),\ {}_k\Delta f(X^0, \dots, X^{k-1}, X_1^k, X_2^k)(Z^1, \dots, Z^{k-1}, Z_1^k, Z) + f(X^0, \dots, X^{k-1}, X_2^k)(Z^1, \dots, Z^{k-1}, Z_2^k)\right].$$

In each case ${}_j\Delta f$, $j = 0, \dots, k$, yields an nc function of order $k+1$, so that

$${}_j\Delta\colon \mathcal{T}^k(\Omega^{(0)}, \dots, \Omega^{(k)}; \mathcal{N}_{0,\text{nc}}, \dots, \mathcal{N}_{k,\text{nc}}) \to \mathcal{T}^k(\Omega^{(0)}, \dots, \Omega^{(j-1)}, \Omega^{(j)}, \Omega^{(j)}, \Omega^{(j+1)}, \dots, \Omega^{(k)}; \mathcal{N}_{0,\text{nc}}, \dots, \mathcal{N}_{j,\text{nc}}, \mathcal{M}_{j,\text{nc}}, \mathcal{N}_{j+1,\text{nc}}, \dots, \mathcal{N}_{k,\text{nc}}).$$

This paper considers the process of undoing the operators ${}_j\Delta$. When $k = 0$, this means we are given a nc function, F, of order 1. It is proved that there exists an nc function, f, of order 0 such that $\Delta f = F$ if and only if ${}_0\Delta F = {}_1\Delta F$. This is done in the following steps.

First, an order 0 nc function f is defined up to the selection of the value of f at some arbitrary point $Y \in \Omega_s$. This definition is inspired by formula (2.19) in [1]: given $X \in \Omega_{sm}$,

$$f(X) = I_m \otimes f(Y) + \Delta f(I_m \otimes Y, X)(X - (I_m \otimes Y)).$$

It is then shown that this will yield an nc function f for which $\Delta f = F$ if and only if there exists a value $f(Y)$ such that

$$Tf(Y) - f(Y)T = F(Y, Y)(TY - YT) \tag{1}$$

for all matrices $T \in \mathcal{R}^{s\times s}$. Next, we define $D_Y\colon \mathcal{R}^{s\times s} \to \mathcal{N}^{s\times s}$ by $D_Y(S) = F(Y, Y)(SY - YS)$, and show that D_Y is a Lie-algebra derivation:

$$D_Y(S)T - TD_Y(S) + SD_Y(T) - D_Y(T)S = D_Y(ST - TS). \tag{2}$$

Let E_{ij} be the matrix with 1 in the i,j position and 0 elsewhere and let $F_{ij} := D_Y(E_{ij})$. Then, for some fixed $c \in \mathcal{N}$,

$$f(Y) = \sum_{i=1}^{s} (E_{ii}F_{ii} + E_{i1}F_{1i}E_{ii}) + cI_s$$

is a value for $f(Y)$ that satisfies equation (1), that is, D_Y is an inner derivation. This is proven by plugging matrices of the form E_{rs} and E_{uv} into (2) for S and T. This gives a large set of equalities which provide enough information to show that (1) holds for all matrices T of the form E_{pq}. It is then a simple matter to linearly extend this result to show that it holds for all matrices T.

For higher order nc functions, we turn to undoing the ${}_j\Delta$ operators. That is, given $k+1$ nc functions, $F_0, \ldots, F_k$, each of order $k+1$, it is proved that there exists an nc function, f, of order k such that ${}_j\Delta f = F_j$ for $0 \leq j \leq k$ if and only if ${}_i\Delta F_j = {}_{j+1}\Delta F_i$ for $0 \leq i \leq j \leq k$.

First, an order k nc function f is defined up to a selection of the value of f at some arbitrary point $(Y^0, \ldots, Y^k) \in \Omega_{s_0}^{(0)} \times \ldots \times \Omega_{s_k}^{(k)}$. For $Z^j \in \mathcal{N}_j^{s_{j-1}m_{j-1} \times s_j m_j}$ where $j = 0, \ldots, k$, f is defined at the "amplified" points $(I_{m_0} \otimes Y^0, \ldots, I_{m_k} \otimes Y^k)$ as

$$\begin{aligned} &f(I_{m_0} \otimes Y^0, \ldots, I_{m_k} \otimes Y^k)(Z^1, \ldots, Z^k) \\ &\quad = \left[\sum_{\substack{i_j=1 \\ j=1,\ldots,k-1}}^{m_j} f(Y^0, \ldots, Y^k)(Z^1_{i_0,i_1}, \ldots, Z^k_{i_{k-1},i_k}) \right]. \\ &\qquad\qquad i_0 = 1, \ldots, m_0, \quad i_k = 1, \ldots, m_k \end{aligned}$$

Then, given $X^j \in \Omega_{s_j m_j}^{(j)}$ and $Z^j \in \mathcal{N}_j^{s_{j-1}m_{j-1} \times s_j m_j}$ for $j = 0, \ldots, k$,

$$\begin{aligned} &f(X^0, \ldots, X^k)(Z^1, \ldots, Z^k) \\ &\quad = f(I_{m_0} \otimes Y^0, \ldots, I_{m_k} \otimes Y^k)(Z^1, \ldots, Z^k) \\ &\quad + \sum_{j=0}^{k} {}_j\Delta f(I_{m_0} \otimes Y^0, \ldots, I_{m_j} \otimes Y^j, X^j, \ldots, X^k) \\ &\qquad (Z^1, \ldots, Z^j, X^j - I_{m_j} \otimes Y^j, Z^{j+1}, \ldots, Z^k). \end{aligned}$$

It is then shown that this yields an nc function f such that ${}_j\Delta f = F_j$, $j = 0, \ldots, k$, if and only if there exists a value $f(Y^0, \ldots, Y^k)$ which, for appropriately sized matrices $T_0, \ldots, T_k$ over $\mathcal{R}$, satisfies

$$\begin{aligned} &T_0 f(Y^0, \ldots, Y^k)(Z^1, \ldots, Z^k) \\ &\qquad - f(Y^0, \ldots, Y^k)(T_0 Z^1, Z^2, \ldots, Z^k) \\ &\quad = F_0(Y^0, Y^0, Y^1, \ldots, Y^k) \\ &\qquad\qquad (T_0 Y^0 - Y^0 T_0, Z^1, \ldots, Z^k), \end{aligned} \tag{3}$$

$$\begin{aligned} &f(Y^0, \ldots, Y^k)(Z^1, \ldots, Z^{j-1}, Z^j T_j, Z^{j+1}, \ldots, Z^k) \\ &- f(Y^0, \ldots, Y^k)(Z^1, \ldots, Z^j, T_j Z^{j+1}, Z^{j+2}, \ldots, Z^k) \\ &\quad = F_j(Y^0, \ldots, Y^{j-1}, Y^j, Y^j, Y^{j+1}, \ldots, Y^k) \\ &\qquad (Z^1, \ldots, Z^j, T_j Y^j - Y^j T_j, Z^{j+1}, \ldots, Z^k), \end{aligned} \tag{4}$$

$$\begin{aligned} &f(Y^0, \ldots, Y^k)(Z^1, \ldots, Z^{k-1}, Z^k T_k) \\ &\qquad - f(Y^0, \ldots, Y^k)(Z^1, \ldots, Z^k) T_k \\ &\quad = F_k(Y^0, \ldots, Y^{k-1}, Y^k, Y^k) \\ &\qquad\qquad (Z^1, \ldots, Z^k, T_k Y^k - Y^k T_k). \end{aligned} \tag{5}$$

We let

$$\begin{aligned} &{}_j D(S) = ({}_j D_Y(S)) \\ &= F_j(Y^0, \ldots, Y^{j-1}, Y^j, Y^j, Y^{j+1}, \ldots, Y^k)(SY^j - Y^j S) \end{aligned}$$

for $j = 0, \ldots, k$; this is viewed as a k-linear function on $Z^1, \ldots, Z^k$ by setting

$$\begin{aligned} &F_j(Y^0, \ldots, Y^{j-1}, Y^j, Y^j, Y^{j+1}, \ldots, Y^k) \\ &\qquad\qquad (SY^j - Y^j S)(Z^1, \ldots, Z^k) \\ &\quad = F_j(Y^0, \ldots, Y^{j-1}, Y^j, Y^j, Y^{j+1}, \ldots, Y^k) \\ &\qquad\qquad (Z^1, \ldots, Z^j, SY^j - Y^j S, Z^{j+1}, \ldots, Z^k). \end{aligned}$$

It is then shown that for each F_j, $j = 0, \ldots, k$, there exists a value $f_j(Y^0, \ldots, Y^k)$ that satisfies the corresponding difference formula, (3), (4), or (5), by using

$$\begin{aligned} &({}_j D(S))\, T - T\, ({}_j D(S)) + S\, ({}_j D(T)) - ({}_j D(T))\, S \\ &\qquad = {}_j D(ST - TS). \end{aligned} \tag{6}$$

Lastly, utilising the fact that ${}_i\Delta F_j = {}_{j+1}\Delta F_i$, $i \leq j$, it is shown that $f_0(Y^0, \ldots, Y^k)$, $\ldots$, $f_k(Y^0, \ldots, Y^k)$ can be chosen equal to one another so that we have a single value $f(Y^0, \ldots, Y^k)$ with which to define the function.

The following is a brief outline of the paper. Section 1 goes through the details of the process of defining an antiderivative for a first order nc function. In Section 2, the details for antidifferentiating sets of higher order nc functions are provided. Finally, Section 3 specializes the antidifferentiability results in the following three important cases:

1) The modules have the form $\mathcal{R}^d$.
2) The nc functions being antidifferentiated are nc polynomials.
3) $\mathcal{R} = \mathbb{R}$ or $\mathbb{C}$, and the nc functions considered are analytic.

References

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