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Analysis of performance and state-feedback design for MJLS in polyhedral cones

Ying Chen[1], Paolo Bolzern[2], Patrizio Colaneri[2], Yuming Bo[1] and Baozhu Du[1]

Abstract— **This paper deals with Markov Jump Linear Systems (MJLS) with the state constrained in a polyhedral cone. This class of systems can be seen as the generalization of Positive MJLS (where the cone is the positive orthant of the state space). First, we provide results on the analysis of some relevant performance indices, including the expected $\mathcal{L}_1$ norm of the impulse response. The problem of state-feedback design preserving cone-invariance and stability while guaranteeing an upper bound of such a performance index is also tackled. Two different solutions are worked out. The first one is based on nonlinear programming but has the advantage of providing a full parametrization of all admissible gains. The second solution is based on linear programming but is more conservative. Some numerical examples are presented to show the effectiveness of these design procedures.**

Keywords: Cone-invariance; $\mathcal{L}_1$ performance; Markov jump linear systems; Polyhedral cones; State-feedback design

I. Introduction

Markov jump linear systems (MJLS) describe a class of dynamic systems randomly jumping among linear subsystems according to a stochastic switching signal generated by a Markov chain [1]–[3]. MJLS are usually adopted to handle random events like unexpected faults or changes in actuators, abrupt environmental disturbances, sudden changes of system modes, etc.

Recently, considerable attention has been focused on Positive MJLS (PMJLS), in which the subsystems are positive systems. By definition, the state variables of positive systems remain nonnegative whenever initialized in the nonnegative orthant and driven by nonnegative inputs [4]. Positive systems arise frequently in economics, biology and sociology applications. As a peculiar feature, the analysis of stability and $\mathcal{L}_1$ performance measures for positive systems can be tackled by means of linear programming. The reader interested in the study of $\mathcal{L}_1$ performance for deterministic linear positive systems is referred to [5]–[8].

Several results on the properties of PMJLS are reported in the survey paper [9]. Further recent contributions can be found in [10]–[14]. A common approach used in these papers is to construct an equivalent deterministic positive system describing the average behavior of the state and output variables. Then the mean stability of a PMJLS can be assessed in terms of stability of the deterministic positive linear system and input-output $\mathcal{L}_1$ performance measures of a PMJLS can be investigated through suitable norms of the deterministic system.

Positive systems can be regarded as a special case of cone-invariant systems with the nonnegative orthant being the positively invariant set. A generalization where the invariant set is a general proper cone has been recently considered in some papers, see e.g. [15]–[19]. Such systems have applications in various fields including rendezvous of multiple agents [20] and molecular biology [21]. When such systems are affected by random jumps, it becomes natural to introduce the class of cone-invariant MJLS. In this paper we will focus on MJLS in polyhedral cones defined by a set of linear inequalities in the state space, which apparently have not been considered before, apart from the related paper [22] dealing with stability and state-feedback stabilization of MJLS in polyhedral cones.

The main contributions of the present paper include:

1) A characterization of the expected $\mathcal{L}_1$ norm of the system impulse response.
2) A state-feedback design procedure aimed at preserving the cone-invariance property and minimizing the above $\mathcal{L}_1$ norm.

The derivation relies on the study of an equivalent deterministic system. The state-feedback design problems are solved following two different approaches. The first one provides necessary and sufficient conditions for having a performance index less than a prescribed value but is based on nonlinear programming. On the contrary, the second one is based on linear programming but the conditions are only sufficient, so leading to conservative results.

The rest of the paper is organized as follows. Section 2 gives basic notations and some necessary definitions and mathematical preliminaries. Section 3 addresses the performance analysis in a deterministic setting. The main results are given in Section 4. Numerical examples are presented in Section 5. Section 6 concludes the paper.

II. Notations and preliminaries

In this paper, the symbols $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ represent the set of real numbers, the space of n-vectors of real numbers and the space of $n \times n$ matrices with real number entries, respectively. The symbol $\mathbb{N}_+$ represents the set of positive integers. $\mathbb{S}$ denotes a finite set $\mathbb{S} := \{1, 2, \ldots, N\}, N \in \mathbb{N}_+$. I_n is the identity matrix with of dimension n. $\mathbf{1}_m$ stands for a column vector of dimension m and each of its entries is equal

[1]Ying Chen, Yuming Bo and Baozhu Du are with Department of Automation, Nanjing University of Science and Technology, P.R. China. Email: chnchenying@163.com(Y. Chen), byming@mail.njust.edu.cn(Y. Bo), dubaozhu@gmail.com(B. Du)

[2]Paolo Bolzern and Partizio Colaneri are with Dipartimento di Elettronica, Informazione e Bioingegneria of Politecnico di Milano, Italy. Email: paolo.bolzern@polimi.it(P. Bolzern), colaneri@elet.polimi.it(P. Colaneri)

to 1. The symbol e_k denotes the k-th column of the identity matrix. A^T is the transpose of matrix A. We use $\prec, \succ$ $(\preceq, \succeq)$ to denote component-wise inequalities. E$[\cdot]$ denotes the expectation and $Pr(\cdot)$ denotes probability. The symbol $\otimes$ represents the Kronecker product and (ξ, ζ) represents the inner product of vectors ξ and ζ. diag$\{A_i\}$ denotes the block diagonal matrix obtained by orderly putting A_i on the diagonal. col$\{\xi_i\}$ stands for a column vector obtained by putting vectors ξ_i orderly stacked in a single column.

The attention of this paper will be focused on systems described in the following forms:
(a) Continuous-time deterministic linear system

$$\Xi : \begin{array}{l} \dot{x}(t) = Ax(t) + B_w w(t) + Bu(t) \\ z(t) = Cx(t) + D_w w(t) + Du(t) \end{array}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state variable, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input and $z(t) \in \mathbb{R}^{n_z}$ is the performance output.
(b) Continuous-time MJLS

$$\Xi_M : \begin{array}{l} \dot{x}(t) = A(r_t)x(t) + B_w(r_t)w(t) + B(r_t)u(t) \\ z(t) = C(r_t)x(t) + D_w(r_t)w(t) + D(r_t)u(t) \end{array}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state variable, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the deterministic disturbance input, $z(t) \in \mathbb{R}^{n_z}$ is the performance output. The signal r_t represents the jumping process, which is a homogeneous finite state Markov process with right continuous trajectories and takes values in the set $\mathbb{S}$. The jumping process is characterized by the transition probability

$$Pr(r_{t+\Delta} = j \mid r_t = i) = \begin{cases} \lambda_{ij}\Delta + o(\Delta), & j \neq i \\ 1 + \lambda_{ii}\Delta + o(\Delta), & j = i \end{cases}$$

where $\Delta > 0, \lim_{\Delta \to 0}(o(\Delta)/\Delta) = 0, \lambda_{ij} \geq 0\,(i, j \in \mathbb{S}, j \neq i)$, and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$. We further define the probability distribution at time t being $\pi(t) := \begin{bmatrix}[\pi(t)]_1 & \dots & [\pi(t)]_N\end{bmatrix}^T$, where $[\pi(t)]_i = Pr(r_t = i)$. Given an initial probability distribution π_0, the probability distribution $\pi(t)$ obeys the differential equation

$$\dot{\pi}(t)^T = \pi(t)^T \Lambda,$$

where Λ denotes the transition rates matrix with λ_{ij} being its (i, j)th entry. We will assume that Λ is an irreducible matrix (see [2]), so that a unique stationary probability distribution $\bar{\pi}$ exists.

In this paper, we assume that the initial probability distribution π_0 coincides with $\bar{\pi} = \lim_{t\to\infty} \pi(t)$. Moreover, the symbols $A_i, B_{wi}, B_i, C_i, D_{wi}, D_i$ are used to denote the system matrices of the ith mode.

Definition 1: [23], [24]. Consider a set $\mathcal{K} \subseteq \mathbb{R}^n$, $\mathcal{K}^G$ denotes the set consisting of all finite nonnegative linear combinations of the elements of $\mathcal{K}$; $\mathcal{K}$ is said to be a cone if $\mathcal{K} = \mathcal{K}^G$; $\mathcal{K}$ is said to be convex if $\alpha\xi_1 + (1-\alpha)\xi_2 \in \mathcal{K}$ for any points $\xi_1, \xi_2 \in \mathcal{K}$ and $\alpha \in [0, 1]$; to be a closed set if every limit point of $\mathcal{K}$ is a point of $\mathcal{K}$; to be solid if its interior Int$(\mathcal{K})$ is not an empty set; and to be pointed if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$. A cone which is closed, convex, solid and pointed is called a proper cone. The dual of $\mathcal{K}$ is defined as the set $\mathcal{K}^* = \{\zeta \in \mathbb{R}^n : (\zeta, \xi) \geq 0 \text{ for all } \xi \in \mathcal{K}\}$. In addition, $\xi_2 \preceq_{\mathcal{K}} \xi_1$ indicates $\xi_1 - \xi_2 \in \mathcal{K}$, and $\xi_2 \prec_{\mathcal{K}} \xi_1$ means $\xi_1 - \xi_2 \in \text{Int}(\mathcal{K})$.

Definition 2: A polyhedral cone $\mathcal{P}[F]$ (referred to as $\mathcal{P}$) with $F \in \mathbb{R}^{m \times n}$, is defined by $\mathcal{P} = \{\xi \in \mathbb{R}^n : F\xi \succeq 0\}$, in which $\mathcal{P} \setminus \{0\} \neq \emptyset$ and rank$(F) = n$. F is called the constraint-matrix (c-matrix) of $\mathcal{P}$.

Definition 3: A nonzero vector v lying in a polyhedral cone $\mathcal{P} \subseteq \mathbb{R}^n$ is called an extreme ray if there are $n-1$ linearly independent constraints binding on v. Denote by V, and call it e-matrix of $\mathcal{P}$, the $n \times m$ matrix whose columns are the extreme rays of $\mathcal{P}$.

Any vector in the cone $\mathcal{P}$ can be expressed as a nonnegative linear combination of its extreme rays, namely $\xi \in \mathcal{P}$ if and only if $\xi = Vp, p \succeq 0$.

Definition 4: A square matrix $A \in \mathbb{R}^{n\times n}$ is cross-positive on a polyhedral cone $\mathcal{P}$ if for any $\xi \in \mathcal{P}$, $\zeta \in \mathcal{P}^*$ with $(\zeta, \xi) = 0$, one has $(\zeta, A\xi) \geq 0$. A is called $\mathcal{P}$-nonnegative if $A\mathcal{P} \subseteq \mathcal{P}$.

Hereafter, we will assume that the variables $x(t)$, $w(t)$, $z(t)$ of systems Ξ and Ξ_M are restricted in polyhedral cones, that will referred to as $\mathcal{P}_x, \mathcal{P}_w, \mathcal{P}_z$ respectively. With reference to $\mathcal{P}_x$, $\mathcal{P}_w, \mathcal{P}_z$, given matrices $F_x \in \mathbb{R}^{m_x\times n_x}$, $F_w \in \mathbb{R}^{m_w\times n_w}$, $F_z \in \mathbb{R}^{m_z\times n_z}$ will denote their c-matrices, and $V_x \in \mathbb{R}^{n_x\times m_x}$, $V_w \in \mathbb{R}^{n_w\times m_w}$, $V_z \in \mathbb{R}^{n_z\times m_z}$ will denote the corresponding e-matrices.

Definition 5: [19] Given $\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z$, system Ξ (or Ξ_M) is said to be cone-invariant with respect to $(\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z)$ if for any $w(t) \in \mathcal{P}_w$ and any initial state $x_0 \in \mathcal{P}_x$, we have $x(t) \in \mathcal{P}_x$ and $z(t) \in \mathcal{P}_z$. In particular, system Ξ (or Ξ_M) is said to be cone-invariant with respect to $\mathcal{P}_x$ if for any initial state $x_0 \in \mathcal{P}_x$ and null input, we have $x(t) \in \mathcal{P}_x, t \geq 0$.

Definition 6: A given MJLS Ξ_M is mean stable if $E[x(t)] \to 0$ as $t \to \infty$, for any initial state x_0.

Remark 1: If system Ξ_M is cone-invariant with respect to $(\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z)$, mean stability implies that all the state trajectories converge to the origin almost surely, i.e., $Pr(\lim_{t\to\infty} x(t) = 0) = 1$. Note that this kind of stability, which is peculiar to cone-invariant systems, can be applied to positive systems as well since the nonnegative orthant can be viewed as a special proper cone. The related results can be found in [12].

Lemma 1: [19] Given $\mathcal{P}_x$, suppose $A \in \mathbb{R}^{n_x\times n_x}$ is Hurwitz and cross-positive on $\mathcal{P}_x$, then $-A^{-1}$ is $\mathcal{P}_x$-nonnegative.

Lemma 2: [19] Given $\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z$, the open-loop system Ξ is cone-invariant with respect to $(\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z)$ if and only if A is cross-positive on $\mathcal{P}_x$, $B_w\mathcal{P}_w \subseteq \mathcal{P}_x$, $C\mathcal{P}_x \subseteq \mathcal{P}_z$ and $D_w\mathcal{P}_w \subseteq \mathcal{P}_z$.

Lemma 3: [19] [25] Given $\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z$, the open-loop system Ξ is cone-invariant with respect to $(\mathcal{P}_w, \mathcal{P}_x, \mathcal{P}_z)$ if and only if there exist a Metzler matrix $\widetilde{A}$, nonnegative matrices $\widetilde{B}_w, \widetilde{C}, \widetilde{D}_w$ such that $F_xA = \widetilde{A}F_x, F_xB_w = \widetilde{B}_wF_w, F_zC = \widetilde{C}F_x, F_zD_w = \widetilde{D}_wF_w$.

Lemma 4: [22] $A \in \mathbb{R}^{n_x\times n_x}$ is cross-positive on $\mathcal{P}_x$ if and only if there exists a nonnegative scalar α such that

$$F_x(A+\alpha I_{n_x})V_x \succeq 0.$$

Lemma 5: [17]. Given $\mathcal{P}_x$, suppose that A is cross-positive on $\mathcal{P}_x$, then system $\dot{x}(t)=Ax(t)$ is Hurwitz stable if and only if there exists a vector $s \succ_{\mathcal{P}_x} 0$, such that $As \prec_{\mathcal{P}_x} 0$.

Lemma 6: [22] Given $\mathcal{P}_x$, suppose that $A_i, i\in\mathbb{S}$ are cross-positive on $\mathcal{P}_x$, then $\hat{A}=\Lambda^T\otimes I_{n_x}+\mathrm{diag}\{A_i\}$ is cross-positive on $\mathcal{P}_x^N$.

III. Performance measures in $\mathcal{L}_1$ for deterministic systems

In order to analyze the mean properties of a cone-invariant MJLS, it is convenient to transform it into an equivalent deterministic system. For this reason, in this section, we first study a $\mathcal{L}_1$ performance index associated with the impulse response of deterministic systems. A state-feedback controller ensuring stability, cone-invariance and upper bounded performance is designed afterwards.

We consider system Ξ with a zero initial state and $w(t)=\delta(t)e_k, k=1,2,\ldots,n_w$, $\mathcal{P}_w=\mathbb{R}_+^{n_w}$ (the nonnegative orthant of $\mathbb{R}^{n_w}$), where $\delta(t)$ denotes the Dirac unit impulse. Let $z^{[k]}(t)$ be the associated response and define the performance as

$$J_1=\sum_{k=1}^{n_w}\int_0^\infty \eta_z^T z^{[k]}(t)dt, \tag{1}$$

where $\eta_z\in\mathcal{P}_z^*$.

First, we characterize the index J_1 for an open-loop system, i.e. with $u(t)=0$.

Proposition 1: Given $\mathcal{P}_x,\mathcal{P}_z$ and system Ξ with $x_0=0, u(t)=0$, assume that system Ξ is stable and cone-invariant with respect to $(\mathbb{R}_+^{n_w},\mathcal{P}_x,\mathcal{P}_z)$. Then the performance index (1) can be computed as

$$J_1=-\eta_z^T CA^{-1}B_w\mathbf{1}_{n_w}+\eta_z^T D_w\mathbf{1}_{n_w}. \tag{2}$$

Moreover, given a scalar $\gamma>0$, $J_1<\gamma$ if and only if there exists a vector $s\succ_{\mathcal{P}_x}0$ such that

$$As+B_w\mathbf{1}_{n_w}\prec_{\mathcal{P}_x}0, \tag{3a}$$
$$\eta_z^T(Cs+D_w\mathbf{1}_{n_w})<\gamma. \tag{3b}$$

Proof: The performance index (1) can be expressed as (2) by observing that

$$z^{[k]}(t)=Ce^{At}B_we_k+D_we_k\delta(t),$$

and

$$\int_0^\infty z^{[k]}(t)dt=(-CA^{-1}B_w+D_w)e_k,$$

in view of Hurwitz stability of matrix A.

Next, it can be shown that condition (3) is sufficient and necessary for having $J_1<\gamma$.

Sufficiency. From (2) and (3b),

$$\begin{aligned}J_1&=-\eta_z^TCA^{-1}B_w\mathbf{1}_{n_w}+\eta_z^TD_w\mathbf{1}_{n_w}\\&<-\eta_z^TCA^{-1}B_w\mathbf{1}_{n_w}+\gamma-\eta_z^TCs\\&=\gamma-\eta_z^TCA^{-1}(B_w\mathbf{1}_{n_w}+As)\\&=\gamma-\eta_z^TC(-A^{-1})(-B_w\mathbf{1}_{n_w}-As).\end{aligned}$$

Notice that $C(-A^{-1})(-B_w\mathbf{1}_{n_w}-As)\succ_{\mathcal{P}_z}0$ from (3a) due to the fact that $-A^{-1}$ is $\mathcal{P}_x$-nonnegative and $C\mathcal{P}_x\subseteq\mathcal{P}_z$. It follows that $\eta_z^TC(-A^{-1})(-B_w\mathbf{1}_{n_w}-As)>0$. Hence $J_1<\gamma$.

Necessity. Suppose $J_1<\gamma$. Choose $s=-(1+\epsilon)A^{-1}B_w\mathbf{1}_{n_w}$ with a sufficiently small $\epsilon>0$ such that

$$\begin{aligned}&\eta_z^T(Cs+D_w\mathbf{1}_{n_w})\\&=-\eta_z^TCA^{-1}B_w\mathbf{1}_{n_w}+\eta_z^TD_w\mathbf{1}_{n_w}-\epsilon\eta_z^TCA^{-1}B_w\mathbf{1}_{n_w}\\&=J_1-\epsilon\eta_z^TCA^{-1}B_w\mathbf{1}_{n_w}<\gamma.\end{aligned}$$

By considering that $-A^{-1}$ is $\mathcal{P}_x$-nonnegative, $s\succ_{\mathcal{P}_x}0$ holds. Finally

$$As+B_w\mathbf{1}_{n_w}=-\epsilon B_w\mathbf{1}_{n_w}\prec_{\mathcal{P}_x}0.$$

Hence (3) is verified. ∎

Remark 2: Note that condition (3) is made up of linear inequalities and can be checked through standard linear programming tools.

In the following, two algorithms are provided to design a state-feedback gain K such that, with $u(t)=Kx(t)$,

(i) the closed-loop system is stable and cone-invariant with respect to $(\mathbb{R}_+^{n_w},\mathcal{P}_x,\mathcal{P}_z)$.
(ii) $J_1<\gamma$ for a given $\gamma>0$.

Theorem 1 (Nonlinear programming): Given $\mathcal{P}_x,\mathcal{P}_z$, a scalar $\gamma>0$ and system Ξ with $x_0=0$, assume that the open-loop system Ξ is cone-invariant with respect to $(\mathbb{R}_+^{n_w},\mathcal{P}_x,\mathcal{P}_z)$. Then a state-feedback gain K ensuring (i) and (ii) exists if and only if there exist a strictly positive diagonal matrix $S\in\mathbb{R}^{m_x\times m_x}$ and a matrix $H\in\mathbb{R}^{n_u\times m_x}$ satisfying

$$F_x(AV_xS+BH)\mathbf{1}_{m_x}+F_xB_w\mathbf{1}_{n_w}\prec 0, \tag{4a}$$
$$F_x(AV_xS+BH+\alpha V_xS)\succeq 0, \tag{4b}$$
$$F_z(CV_xS+DH)\succeq 0, \tag{4c}$$
$$\eta_z^T(CV_xS+DH)\mathbf{1}_{m_x}+\eta_z^TD_w\mathbf{1}_{n_w}<\gamma, \tag{4d}$$

for some nonnegative scalar α, and such that

$$H=KV_xS \tag{5}$$

admits a feasible solution K. Such a solution provides an admissible state feedback gain K.

Proof: Sufficiency. Let $S\succ 0$ diagonal and H satisfy the constraint (4), K be such that (5) holds, and $s=V_xS\mathbf{1}_{m_x}\succ_{\mathcal{P}_x}0$. Then

$$F_x(A+BK)s+F_xB_w\mathbf{1}_{n_w}\prec 0, \tag{6a}$$
$$F_x(A+BK+\alpha I_{n_x})V_xS\succeq 0, \tag{6b}$$
$$F_z(C+DK)V_xS\succeq 0, \tag{6c}$$
$$\eta_z^T[(C+DK)s+D_w\mathbf{1}_{n_w}]<\gamma. \tag{6d}$$

According to Lemma 4 and (6b), $A+BK$ is cross-positive on $\mathcal{P}_x$. And (6c) indicates that $(C+DK)\mathcal{P}_x\subseteq\mathcal{P}_z$. Moreover, the cone-invariance of the open-loop system Ξ guarantees that $B_w\mathbf{1}_{n_w}\in\mathcal{P}_x$. Then from (6a), one can obtain that

$$(A+BK)s\prec_{\mathcal{P}_x}-B_w\mathbf{1}_{n_w}\prec_{\mathcal{P}_x}0,$$

which implies that $A+BK$ is Hurwitz, see Lemma 5. Thus $J_1 < \gamma$ by applying Proposition 1 and taking into account (6d).

Necessity. Let K be such that (i) and (ii) are satisfied. Then, in view of Lemmas 2, 4 and Proposition 1, there exists a vector $s \succ_{\mathcal{P}_x} 0$ such that

$$\begin{aligned} (A+BK)s + B_w \mathbf{1}_{n_w} &\prec_{\mathcal{P}_x} 0, \\ F_x(A+BK+\alpha I_{n_x})V_x &\succeq 0, \\ F_z(C+DK)V_x &\succeq 0, \\ \eta_z^T[(C+DK)s + D_w\mathbf{1}_{n_w}] &< \gamma. \end{aligned}$$

Since $s \succ_{\mathcal{P}_x} 0$, it can be written as $s = V_x S \mathbf{1}_{m_x}$ with S diagonal and strictly positive. By letting $H = KV_xS$, the proof is concluded. ■

Theorem 2 (Linear programming): Given $\mathcal{P}_x, \mathcal{P}_z$, a scalar $\gamma > 0$ and system Ξ with $x_0 = 0$, assume that the open-loop system is cone-invariant with respect to $(\mathbb{R}_+^{n_w}, \mathcal{P}_x, \mathcal{P}_z)$. A state-feedback gain K ensuring (i) and (ii) exists if there exist a strictly positive diagonal matrix $\widetilde{S} \in \mathbb{R}^{m_x \times m_x}$ and a matrix $\widetilde{H} \in \mathbb{R}^{n_u \times m_x}$ satisfying

$$(\widetilde{A}\widetilde{S} + F_x B \widetilde{H})\mathbf{1}_{m_x} + F_x B_w \mathbf{1}_{n_w} \prec 0, \tag{7a}$$
$$\widetilde{A}\widetilde{S} + F_x B\widetilde{H} + \alpha \widetilde{S} \succeq 0, \tag{7b}$$
$$\widetilde{C}\widetilde{S} + F_z D\widetilde{H} \succeq 0, \tag{7c}$$
$$\tilde{\eta}_z^T(\widetilde{C}\widetilde{S} + F_z D\widetilde{H})\mathbf{1}_{m_x} + \tilde{\eta}_z^T F_z D_w \mathbf{1}_{n_w} < \gamma, \tag{7d}$$

for some nonnegative scalar α, where the matrix $\widetilde{A}$ is Metzler and such that $F_x A = \widetilde{A} F_x$, matrix $\widetilde{C}$ and vector $\tilde{\eta}_z$ are nonnegative and such that $F_z C = \widetilde{C} F_x$ and $\eta_z = F_z^T \tilde{\eta}_z$. An admissible state-feedback gain K is then obtained from $K = \widetilde{H}\widetilde{S}^{-1}F_x$.

Proof: Define $\tilde{x}(t) = F_x x(t)$, $\tilde{z}(t) = F_z z(t)$ and consider the system

$$\widetilde{\Xi} : \begin{aligned} \dot{\tilde{x}}(t) &= \widetilde{A}\tilde{x}(t) + F_x B u(t) + F_x B_w w(t) \\ \tilde{z}(t) &= \widetilde{C}\tilde{x}(t) + F_z D u(t) + F_z D_w w(t) \end{aligned}$$

Denote by $\tilde{z}^{[k]}(t)$ the associated impulse response. It follows that $\tilde{z}^{[k]}(t) = F_z z^{[k]}(t)$. Then

$$J_1 = \sum_{k=1}^{n_w}\int_0^\infty \eta_z^T z^{[k]}(t)dt = \sum_{k=1}^{n_w}\int_0^\infty \tilde{\eta}_z^T \tilde{z}^{[k]}(t)dt.$$

By letting $\widetilde{K} = \widetilde{H}\widetilde{S}^{-1}$ be the state-feedback gain of system $\widetilde{\Xi}$, condition (7) is equivalent to

$$\begin{aligned} (\widetilde{A} + F_x B\widetilde{K})\widetilde{S}\mathbf{1}_{m_x} + F_x B_w \mathbf{1}_{n_w} &\prec 0, \\ (\widetilde{A} + F_x B\widetilde{K} + \alpha I_{m_x})\widetilde{S} &\succeq 0, \\ (\widetilde{C} + F_z D\widetilde{K})\widetilde{S} &\succeq 0, \\ \tilde{\eta}_z^T(\widetilde{C} + F_z D\widetilde{K})\widetilde{S}\mathbf{1}_{m_x} + \tilde{\eta}_z^T F_z D_w \mathbf{1}_{n_w} &< \gamma, \end{aligned}$$

which indicates that $\widetilde{A} + F_x B\widetilde{K}$ is Metzler and Hurwitz. Notice that $K = \widetilde{K}F_x$. Then $\widetilde{A} + F_x B\widetilde{K}$ being Metzler and Hurwitz implies $A + BK$ being cross-positive and Hurwitz. Moreover, $F_z(C+DK) = (\widetilde{C} + F_z D\widetilde{K})F_x$ and $\widetilde{C} + F_z D\widetilde{K} \succeq 0$, so that the closed-loop system is cone-invariant with respect to $(\mathbb{R}_+^{n_w}, \mathcal{P}_x, \mathcal{P}_z)$. Finally, $J_1 < \gamma$ follows immediately from the last inequality. ■

Remark 3: Note that due to the cone-invariance of the open-loop system Ξ and $\eta_z \in \mathcal{P}_z^*$, $\widetilde{A}$, $\widetilde{C}$ and $\tilde{\eta}_z$ must exist and are usually not unique. Linear programming tools are efficient in computing them.

Remark 4: It is remarkable that the results on the impulse response expressed by Proposition 1 and Theorems 1 and 2 can be also used to analyze the free response performance index

$$J_1(x_0) = \int_0^\infty \eta_x^T x(t)dt,$$

where $\eta_x \in \mathcal{P}_x^*$, $x_0 \in \mathcal{P}_x$ is an arbitrary initial state and disturbance $w(t) = 0$.

Indeed, the statements regarding $J_1(x_0)$ can be derived straightforwardly from those of Proposition 1 and Theorems 1 and 2 by letting $n_w = 1$, $C = I_{n_x}$, $D_w = D = 0$ and replacing the following symbols: $B_w = x_0$, $\eta_z = \eta_x$, $F_z = F_x$.

IV. Mean performance in $\mathcal{L}_1$ for MJLS

In this section, we extend the results obtained for a deterministic system Ξ to a cone-invariant MJLS Ξ_M by introducing an equivalent average deterministic system in the following form:

$$\hat{\Xi}_M : \begin{aligned} \dot{\hat{X}}(t) &= \hat{A}\hat{X}(t) + \hat{B}_w w(t) \\ E\left[z(t)\right] &= \hat{C}\hat{X}(t) + \hat{D}_w w(t) \end{aligned}$$

where

$$\begin{aligned} &\mathrm{x}_i(t) = E\left[x(t)\mathcal{I}_{r_t=i}\right], \\ &\hat{X}(t) = \mathrm{col}\{\mathrm{x}_i(t)\},\ \hat{X}(0) = \bar{\pi} \otimes x_0, \\ &\hat{A} = \Lambda^T \otimes I_{n_x} + \mathrm{diag}\{A_i\},\ \hat{B}_w = \begin{bmatrix} B_{w1}[\bar{\pi}]_1 \\ B_{w2}[\bar{\pi}]_2 \\ \cdots \\ B_{wN}[\bar{\pi}]_N \end{bmatrix} \\ &\hat{C} = [C_1\, C_2\, \ldots\, C_N],\ \hat{D}_w = \sum_{i=1}^N D_{wi}[\bar{\pi}]_i, \end{aligned} \tag{8}$$

The symbol $\mathcal{I}_{\mathcal{E}}$ stands for the Dirac measure of the event $\mathcal{E}$ (see [2]), and all expectations are taken with respect to the stationary distribution $\bar{\pi}$. System $\hat{\Xi}_M$ describes the time evolution of the expected output when the disturbance is deterministic. Further discussion on this system can be found in [12] and [22].

According to Lemma 6, $\hat{A}$ is cross-positive on $\mathcal{P}_x^N$ if and only if A_i is cross-positive on $\mathcal{P}_x$. Moreover, $\hat{B}_w\mathcal{P}_w \subseteq \mathcal{P}_x^N$, $\hat{C}\mathcal{P}_x^N \subseteq \mathcal{P}_z$, $\hat{D}_w\mathcal{P}_w \subseteq \mathcal{P}_z$ if and only if $B_w\mathcal{P}_w \subseteq \mathcal{P}_x$, $C\mathcal{P}_x \subseteq \mathcal{P}_z$, $D_w\mathcal{P}_w \subseteq \mathcal{P}_z$. Thus systems Ξ_M and $\hat{\Xi}_M$ are equivalent in regard to mean stability and cone-invariance.

Consider system Ξ_M with a zero initial state and let $w(t) = \delta(t)e_k, k = 1, 2, \ldots, n_w$, $\mathcal{P}_w = \mathbb{R}_+^{n_w}$. Denote the associated response with $z^{[k]}(t)$, and define the expected performance index as

$$J_{1E} = E\left[\sum_{k=1}^{n_w}\int_0^\infty \eta_z^T z^{[k]}(t)dt\right], \tag{9}$$

where $\eta_z \in \mathcal{P}_z^*$.

First we characterize the index J_{1E} for an open-loop system, i.e. with $u(t)=0$.

Proposition 2: Given $\mathcal{P}_x, \mathcal{P}_z$ and MJLS Ξ_M with $x_0 = 0, u(t) = 0$, assume that Ξ_M is mean stable and cone-invariant with respect to $(\mathbb{R}_+^{n_w}, \mathcal{P}_x, \mathcal{P}_z)$. Then the performance index (9) can be computed as

$$J_{1E} = -\eta_z^T \hat{C}\hat{A}^{-1}\hat{B}_w \mathbf{1}_{n_w} + \eta_z^T \hat{D}_w \mathbf{1}_{n_w}, \tag{10}$$

where $\hat{A}$, $\hat{B}_w$, $\hat{C}$ and $\hat{D}_w$ are defined in (8).

Moreover, given a scalar $\gamma > 0$, $J_{1E} < \gamma$ if and only if there exist vectors $s_i \succ_{\mathcal{P}_x} 0, i \in \mathbb{S}$ such that

$$A_i s_i + \sum_{j=1}^N \lambda_{ji} s_j + B_{wi}[\bar{\pi}]_i \mathbf{1}_{n_w} \prec_{\mathcal{P}_x} 0, \tag{11a}$$

$$\sum_{i=1}^N \eta_z^T (C_i s_i + D_{wi}[\bar{\pi}]_i \mathbf{1}_{n_w}) < \gamma. \tag{11b}$$

Proof: By considering the equivalent system $\hat{\Xi}_M$, (10) holds according to Proposition 1.

Moreover, $J_{1E} < \gamma$ if and only if there exists a vector $s \succ_{\mathcal{P}_x^N} 0$ such that

$$\hat{A}s + \hat{B}_w \mathbf{1}_{n_w} \prec_{\mathcal{P}_x^N} 0,$$
$$\eta_z^T(\hat{C}s + \hat{D}_w \mathbf{1}_{n_w}) < \gamma.$$

The proof can be concluded by letting $s = \mathrm{col}\{s_i\}$ with $s_i \succ_{\mathcal{P}_x} 0$ for each $i \in \mathbb{S}$. ∎

Note that condition (11) is expressed as linear inequalities and can be checked through standard linear programming tools.

In the following, two algorithms are provided to design state-feedback gains K_i such that, with $u(t) = K_i x(t)$,

(i) the closed-loop system Ξ_M is mean stable and cone-invariant with respect to $(\mathbb{R}_+^{n_w}, \mathcal{P}_x, \mathcal{P}_z)$.
(ii) $J_{1E} < \gamma$ for a given $\gamma > 0$.

The proofs are similar to those of Theorems 1 and 2 and are therefore omitted.

Theorem 3 (Nonlinear programming): Given $\mathcal{P}_x, \mathcal{P}_z$, a scalar $\gamma > 0$ and MJLS Ξ_M with $x_0 = 0$, assume that the open-loop system is cone-invariant with respect to $(\mathbb{R}_+^{n_w}, \mathcal{P}_x, \mathcal{P}_z)$. State-feedback gains K_i ensuring (i) and (ii) exist if and only if there exist strictly positive diagonal matrices $S_i \in \mathbb{R}^{m_x \times m_x}$ and matrices $H_i \in \mathbb{R}^{n_u \times m_x}, i \in \mathbb{S}$ satisfying

$$F_x(A_i V_x S_i + B_i H_i + \sum_{j=1}^N \lambda_{ji} V_x S_j)\mathbf{1}_{m_x} + F_x B_{wi}[\bar{\pi}]_i \mathbf{1}_{n_w} \prec 0,$$
$$F_x(A_i V_x S_i + B_i H_i + \alpha V_x S_i) \succeq 0,$$
$$F_z(C_i V_x S_i + D_i H_i) \succeq 0,$$
$$\sum_{i=1}^N \eta_z^T [(C_i V_x S_i + D_i H_i)\mathbf{1}_{m_x} + D_{wi}[\bar{\pi}]_i \mathbf{1}_{n_w}] < \gamma,$$

for some nonnegative scalar α, and such that, for each $i \in \mathbb{S}$,

$$H_i = K_i V_x S_i$$

admit feasible solutions $K_i, i \in \mathbb{S}$. Such solutions provide admissible state-feedback gains $K_i, i \in \mathbb{S}$.

Theorem 4 (Linear programming): Given $\mathcal{P}_x, \mathcal{P}_z$, a scalar $\gamma > 0$ and MJLS Ξ_M with $x_0 = 0$, assume that the open-loop system is cone-invariant with respect to $(\mathbb{R}_+^{n_w}, \mathcal{P}_x, \mathcal{P}_z)$. State-feedback gains K_i ensuring (i) and (ii) exist if there exist strictly positive diagonal matrices $\widetilde{S}_i \in \mathbb{R}^{m_x \times m_x}$ and matrices $\widetilde{H}_i \in \mathbb{R}^{n_u \times m_x}, i \in \mathbb{S}$ such that

$$(\widetilde{A}_i \widetilde{S}_i + F_x B_i \widetilde{H}_i + \sum_{j=1}^N \lambda_{ji} \widetilde{S}_j)\mathbf{1}_{m_x} + F_x B_{wi}[\bar{\pi}]_i \mathbf{1}_{n_w} \prec 0,$$
$$\widetilde{A}_i \widetilde{S}_i + F_x B_i \widetilde{H}_i + \alpha \widetilde{S}_i \succeq 0,$$
$$\widetilde{C}_i \widetilde{S}_i + F_z D_i \widetilde{H}_i \succeq 0,$$
$$\sum_{i=1}^N \tilde{\eta}_z^T [(\widetilde{C}_i \widetilde{S}_i + F_z D_i \widetilde{H}_i)\mathbf{1}_{m_x} + F_z D_{wi}[\bar{\pi}]_i \mathbf{1}_{n_w}] < \gamma,$$

for some nonnegative scalar α, where matrices $\widetilde{A}_i$ are Metzler and such that $F_x A_i = \widetilde{A}_i F_x$, matrices $\widetilde{C}_i$ and vector $\tilde{\eta}_z$ are nonnegative and such that $F_z C_i = \widetilde{C}_i F_x$, $\eta_z = \tilde{\eta}_z F_z^T$. Admissible state feedback gains K_i are then obtained from $K_i = \widetilde{H}_i \widetilde{S}_i^{-1} F_x$.

Remark 5: The performance analysis results on impulse response of MJLS can be also used to analyze the free response performance index

$$J_{1E}(x_0) = E\left[\int_0^\infty \eta_x^T x(t)dt\right],$$

where $\eta_x \in \mathcal{P}_x^*$, $x_0 \in \mathcal{P}_x$ is an arbitrary initial state and the disturbance $w(t) = 0$, by letting $n_w = 1$, $C_i = I_{n_x}$, $D_{wi} = D_i = 0$ and replacing the following symbols in Proposition 2 and Theorems 3 and 4:

$$B_{wi} = x_0,\ \eta_z = \eta_x,\ F_z = F_x.$$

Remark 6: In this paper, the state-feedback design problem is solved by two methods. The first one gives a full parameterization of all admissible gains but is a nonlinear programming problem. While the second one is expressed in the form of linear programming, possessing a numerical advantage, but is more conservative.

Remark 7: It is worth emphasizing that if the c-matrix F_x is square and invertible (equivalently, V_x is square and invertible), the nonlinear constraints in Theorems 1 and 3 are always satisfied and the two algorithms (nonlinear one and linear one) will be equivalent. In particular, if $F_x = I_{n_x}, F_z = I_{n_z}$, then $\mathcal{P}_x = \mathbb{R}_+^{n_x}, \mathcal{P}_z = \mathbb{R}_+^{n_z}$, thus leading to the results obtained in this paper applicable to PMJLS (see e.g. [12]); in other words, the proposed results can be viewed as a generalized form of those valid for PMJLS.

V. Numerical Examples

In this section, we consider polyhedral cones $\mathcal{P}_x, \mathcal{P}_w, \mathcal{P}_z$ with the following parameters:

$$F_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad F_z = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$$

and $\mathcal{P}_w = \mathbb{R}^2_+$. In the following two subsections, the state-feedback controllers for deterministic system Ξ and MJLS Ξ_M will be designed by using nonlinear programming and linear programming. The Matlab function "fmincon" and "linprog" are used to solve the nonlinear optimization problem and linear optimization problem, respectively.

A. Deterministic system

Consider a system Ξ described by:

$$A = \begin{bmatrix} -0.7 & 0.2 & 0.1 \\ 0.3 & -0.8 & 0 \\ -0.1 & 0.4 & -0.9 \end{bmatrix}, B = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & -0.5 \\ -0.2 & 0.7 \end{bmatrix},$$
$$B_w = \begin{bmatrix} 1.2 & 1.3 \\ 0.7 & 0.1 \\ -0.1 & -0.5 \end{bmatrix}, C = \begin{bmatrix} 1.7 & 1.5 & 0.2 \\ 0.3 & -0.5 & -0.3 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.1 \end{bmatrix}, D_w = \begin{bmatrix} 1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}.$$

Example 1 (Impulse response): Consider a zero initial state and $\eta_z = [3\ 1]^T$ for system Ξ with impulse disturbance. Our aim is to design a state-feedback controller $u(t) = Kx(t)$ such that the closed-loop system Ξ is stable, cone-invariant with respect to $(\mathbb{R}^2_+, \mathcal{P}_x, \mathcal{P}_z)$ and the performance J_1 is minimized. In order to minimize J_1, we minimize its upper bound γ instead. Table I gives the optimization results regarding two methods.

Example 2 (Free response): Consider a given initial state $x_0 = [2\ 4\ 1]^T, \eta_x = [2\ 3\ 2]^T$, and assume the disturbance $w(t) = 0$. Recalling Remark 4 and replacing the related symbols in Theorems 1 and 2, a state-feedback controller $u(t) = Kx(t)$ can be designed such that the closed-loop system Ξ is stable, cone-invariant with respect to $\mathcal{P}_x$ and the free response performance $J_1(x_0)$ is minimized. The optimization results are given in Table I.

It can be concluded that, in both cases, the nonlinear programming method provided a better performance index than the linear programming method.

B. MJLS

In this subsection, we consider a two-mode MJLS Ξ_M with the following parameters:

mode 1:

$$A_1 = \begin{bmatrix} -0.7 & 0.2 & 0.1 \\ 0.3 & -0.8 & 0 \\ -0.1 & 0.4 & -0.9 \end{bmatrix}, B_{w1} = \begin{bmatrix} 1.2 & 1.3 \\ 0.7 & 0.1 \\ -0.1 & -0.5 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 1.7 & 1.5 & 0.2 \\ 0.3 & -0.5 & -0.3 \end{bmatrix},$$
$$D_{w1} = \begin{bmatrix} 1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix};$$

mode 2:

$$A_2 = \begin{bmatrix} -0.8 & 0.3 & 0.1 \\ 0.3 & -0.5 & 0 \\ -0.3 & 0.5 & -1 \end{bmatrix}, B_{w2} = \begin{bmatrix} 1 & 1.3 \\ 0.7 & 0.5 \\ -0.1 & -0.6 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.2 \end{bmatrix}, C_2 = \begin{bmatrix} 1.5 & 1.5 & 0.2 \\ 0.3 & -0.8 & -0.3 \end{bmatrix},$$
$$D_{w2} = \begin{bmatrix} 0.5 & 0.6 \\ 0.2 & 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}.$$

The transition rate matrix is $\Lambda = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$.

Example 3 (Impulse response): Consider system Ξ_M with a zero initial state and $\eta_z = [4\ 1]^T$. A state-feedback controller $u(t) = K_i x(t)$ is designed by using nonlinear and linear programming so that the closed-loop system is mean stable, cone-invariant with respect to $(\mathbb{R}^2_+, \mathcal{P}_x, \mathcal{P}_z)$ and the upper bound of J_{1E}, i.e. γ, is minimized. Table II gives the optimization results by using nonlinear and linear algorithms.

Example 4 (Free response): Consider an initial state $x_0 = [2\ 3\ 1]^T$, $\eta_x = [2\ 3\ 2]^T$ for system Ξ_M and assume that $w(t) = 0$. Recalling Remark 5 and replacing the related symbols in Theorems 3 and 4, we can design a state-feedback controller $u(t) = K_i x(t)$ such that the closed-loop system is mean stable, cone-invariant with respect to $\mathcal{P}_x$ and the upper bound of performance $J_{1E}(x_0)$ is minimized. Table II gives the optimization results.

For the optimal solutions $J_{1E}(x_0)$ and J_{1E} obtained by two different methods, it is apparent that nonlinear programming provided a more desirable solution than linear programming.

VI. Conclusion

In this paper we have analyzed the $\mathcal{L}_1$ performance measures of the impulse response for MJLS with cone-invariance. A necessary and sufficient condition to characterize the performance index for deterministic systems is worked out first, and then extended to MJLS by constructing an equivalent deterministic system. Two viable methods (nonlinear programming and linear programming) in designing a state-feedback controller are derived to ensure the cone-invariance, mean stability and performance being upper bounded for the closed-loop MJLS. Both of them have advantages and disadvantages: the nonlinear one provides a full parametrization of all admissible gains and a preferable performance index while the linear one possesses a numerical

TABLE I

OPTIMIZATION RESULTS WITH FREE RESPONSE AND IMPULSE RESPONSE FOR DETERMINISTIC SYSTEM

		Nonlinear programming	Linear programming
Solver		fmincon	linprog
Impulse response	K	$\begin{bmatrix} -2.4498 & -0.1127 & -0.3873 \\ -0.5327 & 0.3452 & -0.7202 \end{bmatrix}$	$\begin{bmatrix} -0.6974 & -0.6974 & 0.1974 \\ 0.0230 & 0.0230 & -0.3980 \end{bmatrix}$
	$\gamma_{\min}$	11.0235	20.5065
	J_1	11.0235	20.5065
Free response	K	$\begin{bmatrix} 5.2708 & -7.8333 & 7.3333 \\ -5.5625 & 5.3750 & -5.7500 \end{bmatrix}$	$\begin{bmatrix} -0.6974 & -0.6974 & 0.1974 \\ 0.0230 & 0.0230 & -0.3980 \end{bmatrix}$
	$\gamma_{\min}$	27.2654	29.7578
	$J_1(x_0)$	27.2654	29.7578

TABLE II

OPTIMIZATION RESULTS WITH FREE RESPONSE AND IMPULSE RESPONSE FOR MJLS

		Nonlinear programming	Linear programming
Solver		fmincon	linprog
Impulse response	K_1	$[-1.75 \quad 1.25 \quad -1.25]$	$[-0.1667 \quad -0.1667 \quad 0.1667]$
	K_2	$[-0.2917 \quad -0.8333 \quad -0.1667]$	$[-0.5625 \quad -0.5625 \quad -0.4375]$
	$\gamma_{\min}$	31.6859	35.7489
	J_{1E}	31.6856	34.1768
Free response	K_1	$[-0.3334 \quad -0.1666 \quad 0.1666]$	$[-0.1667 \quad -0.1667 \quad 0.1667]$
	K_2	$[-0.2917 \quad -0.8333 \quad -0.1667]$	$[-0.5625 \quad -0.5625 \quad -0.4375]$
	$\gamma_{\min}$	29.8578	30.6610
	$J_{1E}(x_0)$	29.8577	30.6610

advantage but is more conservative. In perspective, further investigation on $\mathcal{L}_\infty$-induced norm is foreseen.

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