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Asymptotic Stability of a Class of Nonlinear Systems Admitting Homogeneity with Strictly Decreasing Degrees

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***Abstract*— Asymptotic stability is investigated for a class of inherently nonlinear systems admitting homogeneity with strictly decreasing degrees (HSDD). By using the technique of homogeneous domination and the concept of HSDD, a concrete Lyapunov/Chetaev function is constructively obtained. Then, by Lyapunov Stability Theorem and Chetaev Instability Theorem, a necessary and sufficient condition for the asymptotic stability is proposed. Finally, simulations are given to validate the theoretical result.**

I. INTRODUCTION

Asymptotic stability is one of the most important issues in nonlinear system theory, which is usually a basic requirement for mechanical systems in engineering practice. There are many methods for stability analysis such as Lyapunov first method, Lyapunov second method [1], center manifold theorem [2], [3], small gain theorem [4] and homogenous approximation [5]. Lyapunov first method shows that when the linearized system of a nonlinear system has no poles at the origin, the asymptotic stability of the nonlinear system is exactly determined by its linearization. If the linearized system has a pole at the origin, we usually call the nonlinear system as an inherently nonlinear system. The asymptotic stability problem of inherently nonlinear systems is more challenging. Lyapunov second method is a powerful tool to analyze asymptotic stability of inherently nonlinear systems. But, generally speaking, it is usually difficult to design a suitable Lyapunov function. For homogeneous systems, it is well-known that an asymptotically stable homogeneous system admits a homogeneous Lyapunov function. A natural idea is to generalize the concept of homogeneity and apply the generalized homogeneity to the stability analysis of a class of nonlinear systems. Actually, in [6]-[10], the concept homogeneity with monotone degrees (HMD) is proposed and has been successfully applied to the design of stabilizing controllers for inherently nonlinear systems. Note that even if a nonlinear system admits HMD, it may be not homogeneous. So it is interesting to analyze the asymptotic stability for an inherently nonlinear system with HMD.

Using the idea of HMD, in our recent paper [11], we have proved the asymptotic stability of a kind of power integral systems controlled by linear feedback in the form as follows:

$$\begin{aligned} \dot{x}_i &= x_{i+1}^{p_i}, \quad i=1,2,\cdots,n-1, \\ \dot{x}_n &= (-k_1x_1-k_2x_2-\cdots-k_nx_n)^{p_n}, \end{aligned} \tag{1}$$

where p_i's are ratios of positive odd integers satisfying $p_1 > p_2 > \cdots > p_n \geq 1$ and the feedback gains k_i's are positive. Actually, system (1) has the homogeneity with strictly decreasing degrees (HSDD) $[p_1-1, p_2-1, \cdots, p_n-1]$ according to the dilation weights $[1, 1, \cdots, 1]$. Motivated by the main result of [11], we try to consider the case of the general dilation weights $[r_1, r_2, \cdots, r_n]$. How to design stabilizing controller for a power integral chain system without the monotone condition on the powers is an interesting problem. For example, consider the nonlinear control system as follows:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3^{9/7}, \\ \dot{x}_3 &= u. \end{aligned} \tag{2}$$

It is easy to check that system (2) is an inherently nonlinear system for any feedback control. Under a linear feedback control, the monotone condition on the powers in [11] does not hold. To solve the stabilization problem of system (2), a natural idea is to design a feedback control law such that the closed-loop system has a HSDD according to a general dilation $r = [r_1, r_2, \cdots, r_n]$ and replace the monotone condition on the powers by the monotonicity of homogeneous degrees.

Power integral systems with the form (2) are special p-normal control systems [12], [13]. Many nonlinear control systems can be equivalently transformed into the p-normal form. For many p-normal control systems, stabilization problem has been widely investigated [14]-[17].

In this paper, for a class of power integral chain systems under a kind of nonlinear feedback, a necessary and sufficient condition for the asymptotic stability is obtained. Homogeneity with strictly decreasing degrees plays an important role in the construction of the Lyapunov/Chetaev function.

II. PRELIMINARIES

This section presents some fundamental theorem and some useful inequalities which will play important roles in obtaining the main results of this paper.

Theorem 1: (Lyapunov Stability Theorem) [1], [3] Consider a nonlinear system

$$\dot{x} = f(x), \quad x \in \Re^n, \tag{3}$$

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where $f(x)$ is Lipschitz continuous with respect to x, $f(0) = 0$. If there exists a locally positive definite function $V(x)$ such that

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) \tag{4}$$

is locally negative definite, then system (3) is asymptotically stable.

Theorem 2: (Chetaev Instability Theorem [18]) If there exists a continuously differentiable function $V(x)$ such that (i) the origin is a boundary point of the set $G = \{x \in \Re^n \mid V(x) > 0\}$; (ii) there exists a neighborhood U of the $x = 0$ such that $\dot{V}(x) > 0 \;\; \forall\, x \in U \cap G$, then $x = 0$ is an unstable equilibrium point of the system.

Lemma 1: (Jensen's inequality) [19] For $p \geq 1$ and $x_i \in \Re$, $i = 1, \cdots, n$, the following holds

$$|x_1+x_2+\cdots+x_n|^p \leq n^{p-1}(|x_1|^p+|x_2|^p+\cdots+|x_n|^p). \tag{5}$$

Lemma 2: [14] For $p \geq 1$ which is a ratio of positive odd integers, the following holds

$$x(x+a)^p \geq 2^{1-p}x^{p+1} + xa^p, \; \forall \; x, \; a \in \Re.$$

Lemma 3: [6] Let c and d be positive constants. Given any number $\gamma > 0$, the following inequality holds

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}, \forall \; x, \; y \in \Re.$$

Definition 1: [7] A continuous vector field $v : \mathrm{R}^n \to \mathrm{R}^n$ with $v = [v_1, \cdots, v_n]^{\mathrm{T}}$ is said to satisfy homogeneity with monotone degrees (HMD), if we can find positive real numbers $(r_1, \cdots, r_n)$ and real numbers $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ such that

$$v_i(\epsilon^{r_1}x_1, \cdots, \epsilon^{r_n}x_n) = \epsilon^{r_i+\tau_i}v_i(x) \tag{6}$$

for all $x \in \Re^n$, $\epsilon > 0$ and $i = 1, 2, \cdots, n$. The constants r_i's and τ_i's are called homogeneous weights and degrees, respectively.

III. Main Results

Consider the nonlinear control system composed of the chain of power integrals as follows:

$$\begin{array}{rcl} \dot{x}_i &=& x_{i+1}^{p_i}, \quad i = 1, 2, \cdots, n-1, \\ \dot{x}_n &=& u^{s_n}, \end{array} \tag{7}$$

where p_i's and s_n are ratios of positive odd integers. Here, we do not impose a monotonicity assumption on the powers of (7). In order to generalize the main result of [11], we define a special HMD:

Definition 2: A continuous vector field $v : \mathrm{R}^n \to \mathrm{R}^n$ with $v = [v_1, \cdots, v_n]^{\mathrm{T}}$ is said to satisfy homogeneity with strictly decreasing degrees (HSDD), if it has HMD defined in Definition 1 and the homogeneous degrees satisfy

$$\tau_1 > \tau_2 > \cdots > \tau_n.$$

We design the controller $u = f_n(x_1, \cdots, x_n)$, where f_n is defined recursively by

$$f_1(x_1) = k_1x_1, \tag{8}$$

$$f_{i+1}(x_1, \cdots, x_{i+1}) = f_i^{s_i/s_{i+1}} + k_{i+1}x_{i+1} \tag{9}$$

for each $i = 1, 2, \cdots, n-1$.

We are interested in the stability analysis of the closed-loop system:

$$\begin{array}{rcl} \dot{x}_i &=& x_{i+1}^{p_i}, \quad i = 1, 2, \cdots, n-1, \\ \dot{x}_n &=& -(f_n(x_1, x_2, \cdots, x_n))^{s_n}. \end{array} \tag{10}$$

Assumption 1: Nonlinear system (10) admits homogeneity with strictly decreasing degrees $\tau_1 > \tau_2 > \cdots > \tau_n$ with respect to the positive dilation weights $(r_1, r_2, \cdots, r_n)$, that is, Eq. (6) holds for all $x \in \Re^n$, $\epsilon > 0$ and $i = 1, 2, \cdots, n$, which is equivalent to

$$r_{i+1}p_i = \tau_i + r_i, \quad i = 1, 2, \cdots, n-1, \tag{11}$$

$$r_1s_1 = r_2s_2 = \cdots = r_ns_n = \tau_n + r_n. \tag{12}$$

Lemma 4: Let r_1, r_2, $\cdots$, r_s and n_1, n_2, $\cdots$, $n_s (s \geq 2)$ be any given positive constants. For any $\varepsilon > 0$, there exists a positive number A such that

$$|x_1|^{n_1}\cdots|x_s|^{n_s} \leq \varepsilon|x_1|^{\frac{n_1r_1+\cdots+n_sr_s}{r_1}} + A\sum_{i=2}^{s}|x_i|^{\frac{n_1r_1+\cdots+n_sr_s}{r_i}} \tag{13}$$

for all x_1, x_2, $\cdots$, $x_s \in \Re$.

Proof: (Mathematical Induction) As $s = 2$, by Lemma 3, we have that, for any $\varepsilon > 0$, there exists $\tilde{A} > 0$ such that

$$\begin{aligned} |x_1|^{n_1}|x_2|^{n_2} &= (|x_1|^{\frac{1}{r_1}})^{r_1n_1}(|x_2|^{\frac{1}{r_2}})^{r_2n_2} \\ &\leq \varepsilon|x_1|^{\frac{r_1n_1+r_2n_2}{r_1}} + \tilde{A}|x_2|^{\frac{r_1n_1+r_2n_2}{r_2}}. \end{aligned} \tag{14}$$

Suppose that Lemma 4 holds for the case of s, i.e. assume that (13) holds. In the rest of this proof, we consider the case of $s+1$. By (13), we have that

$$\begin{aligned} &|x_1|^{n_1}\cdots|x_s|^{n_s}|x_{s+1}|^{n_{s+1}} \\ &\leq (\varepsilon|x_1|^{\frac{n_1r_1+\cdots+n_sr_s}{r_1}} + A\sum_{i=2}^{s}|x_i|^{\frac{n_1r_1+\cdots+n_sr_s}{r_i}})|x_{s+1}|^{n_{s+1}}. \end{aligned} \tag{15}$$

By Lemma 3, for each $i \geq 1$, there exists $B_i > 0$ such that

$$\begin{aligned} &|x_i|^{\frac{n_1r_1+\cdots+n_sr_s}{r_i}}|x_{s+1}|^{n_{s+1}} \\ &\leq \; |x_i|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_i}} + B_i|x_{s+1}|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_{s+1}}}. \end{aligned} \tag{16}$$

Applying (16) to (15) yields that there exists $\hat{A}$ such that

$$\begin{aligned} &|x_1|^{n_1}\cdots|x_s|^{n_s}|x_{s+1}|^{n_{s+1}} \\ &\leq \varepsilon|x_1|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_1}} + \hat{A}\sum_{i=2}^{s+1}|x_i|^{\frac{n_1r_1+\cdots+n_{s+1}r_{s+1}}{r_i}}. \end{aligned} \tag{17}$$

■

Proposition 1: Suppose nonlinear system (10) satisfies Assumption 1, $s_1 \geq s_2 \geq \cdots > s_n \geq 1$ and $k_i \neq 0$. Then, by a diffeomorphism transformation, (10) is equivalent to the equations as follows:

$$\begin{aligned} \dot{e}_1 &= \frac{k_1}{k_2^{p_1}}(e_2 - e_1^{\frac{s_1}{s_2}})^{p_1} =: g_1(e_1, e_2), \\ \dot{e}_i &= \frac{k_i}{k_{i+1}^{p_i}}(e_{i+1} - e_i^{\frac{s_i}{s_{i+1}}})^{p_i} + \frac{s_{i-1}}{s_i}e_{i-1}^{\frac{s_{i-1}}{s_i}-1}g_{i-1}(e_1, \cdots, e_i) \\ &=: g_i(e_1, \cdots, e_{i+1}), \qquad i = 2, 3, \cdots, n-1, \\ \dot{e}_n &= -k_ne_n^{s_n} + \frac{s_{n-1}}{s_n}e_{n-1}^{\frac{s_{n-1}}{s_n}-1}g_{n-1}(e_1, \cdots, e_n). \end{aligned} \tag{18}$$

Proof: Construct a nonlinear transformation

$$e_i = f_i(x_1, \cdots, x_i), \quad i = 1, 2, \cdots, n, \tag{19}$$

where each f_i is defined by (8) and (9). It is easy to check that the inverse mapping of (19) is

$$x_1 = k_1^{-1} e_1, \; x_i = k_i^{-1}(e_i - e_{i-1}^{s_{i-1}/s_i}), \quad i = 2, \cdots, n. \tag{20}$$

Since $s_1 \geq s_2 \geq \cdots > s_n \geq 1$, both the transformation (19) and its inverse mapping (20) are smooth, which implies that the transformation described by (19) is a diffeomorphism. A straightforward computation shows that system (10) is equivalently transformed into (18). ■

Lemma 5: Under Assumption 1, each function defined by the right hand side of (18) satisfies

$$|g_i(e_1, \cdots, e_{i+1})| \leq C_i(\sum_{k=1}^{i}\sum_{j=k}^{i}|e_j|^{\frac{\tau_k + r_i}{r_j}} + |e_{i+1}|^{\frac{\tau_i + r_i}{r_{i+1}}}), \tag{21}$$

where each C_i is a constant dependent on g_i.

Proof: (Mathematical Induction) For the case of $i = 1$, from (5) it follows that

$$\begin{aligned} |g_1(e_1, e_2)| &\leq C_1(|e_1|^{\frac{s_1 p_1}{s_2}} + |e_2|^{p_1}) \\ &= C_1(|e_1|^{\frac{\tau_1 + r_1}{r_1}} + |e_2|^{\frac{\tau_1 + r_1}{r_2}}), \end{aligned} \tag{22}$$

where C_1 is dependent on g_1. Suppose that the lemma holds for the case of i, i.e. (21) holds. In the following, let us estimate $|g_{i+1}|$. From (18), Lemma 1 and Assumption 1, it follows that there exists a constant $A > 0$ such that

$$\begin{aligned} |g_{i+1}| &= \left| \frac{k_{i+1}}{k_{i+2}^{p_{i+1}}}(e_{i+2} - e_{i+1}^{\frac{s_{i+1}}{s_{i+2}}})^{p_{i+1}} + \frac{s_i}{s_{i+1}} e_i^{\frac{s_i}{s_{i+1}} - 1} g_i \right| \\ &\leq A(|e_{i+1}|^{\frac{s_{i+1} p_{i+1}}{s_{i+2}}} + |e_{i+2}|^{p_{i+1}} + |e_i|^{\frac{s_i}{s_{i+1}} - 1}|g_i|) \\ &= A(|e_{i+1}|^{\frac{\tau_{i+1} + r_{i+1}}{r_{i+1}}} + |e_{i+2}|^{\frac{\tau_{i+1} + r_{i+1}}{r_{i+2}}}) + |e_i|^{\frac{r_{i+1}}{r_i} - 1}|g_i|). \end{aligned} \tag{23}$$

By the induction assumption, applying (21) to (23), we have that

$$\begin{aligned} |g_{i+1}| \leq\; & A(|e_{i+1}|^{\frac{\tau_{i+1} + r_{i+1}}{r_{i+1}}} + |e_{i+2}|^{\frac{\tau_{i+1} + r_{i+1}}{r_{i+2}}}) \\ & + AC_i \sum_{k=1}^{i}\sum_{j=k}^{i}(|e_j|^{\frac{\tau_k + r_i}{r_j}} + |e_{i+1}|^{\frac{\tau_i + r_i}{r_{i+1}}})|e_i|^{\frac{r_{i+1}}{r_i} - 1}. \end{aligned} \tag{24}$$

Using Lemma 4 to the last term of (24), we obtain that there exists $B > 0$ such that

$$\begin{aligned} & (|e_j|^{\frac{\tau_k + r_i}{r_j}} + |e_{i+1}|^{\frac{\tau_i + r_i}{r_{i+1}}})|e_i|^{\frac{r_{i+1}}{r_i} - 1} \\ \leq\; & B(|e_j|^{\frac{\tau_k + r_{i+1}}{r_j}} + |e_i|^{\frac{\tau_k + r_{i+1}}{r_i}} + |e_{i+1}|^{\frac{\tau_i + r_{i+1}}{r_{i+1}}} + |e_i|^{\frac{\tau_i + r_{i+1}}{r_i}}). \end{aligned} \tag{25}$$

From (24) and (25), it follows the conclusion of the case of $i+1$. So, by Mathematical Induction, the proof is complete. ■

Theorem 3: Suppose nonlinear system (10) satisfies Assumption 1, $s_1 \geq s_2 \geq \cdots > s_n \geq 1$ and $k_i \neq 0$. Then system (10) is asymptotically stable if and only if $k_i > 0$.

Proof: Construct the following Lyapunov/Chetaev function

$$V(e) = \sum_{i=1}^{n} \frac{l_i}{\alpha_i} e_i^{\alpha_i}, \tag{26}$$

where

$$l_i = -k_{i+1}^{p_i} k_i^{-1}, \; i = 1, 2, \cdots, n-1, \; l_n = -k_n^{-1} \tag{27}$$

and every $\alpha_i > 1$ is a ratio of an even integer and an odd integer satisfying

$$\frac{r_i}{r_{i+1}}\alpha_i \leq \alpha_{i+1} < \frac{r_i}{r_{i+1}}\alpha_i + \frac{\tau_i - \tau_{i+1}}{r_{i+1}}, \quad i = 1, 2, \cdots, n-1. \tag{28}$$

Since $\tau_i > \tau_{i+1}$ for all i, given α_1 one can successively determine α_2, α_3, $\cdots$, α_n by (28). The derivative of (26) along (18) can be easily computed as

$$\begin{aligned} \dot{V} = & \sum_{i=1}^{n-1} e_i^{\alpha_i - 1}(e_i^{\frac{s_i}{s_{i+1}}} - e_{i+1})^{p_i} + e_n^{\alpha_n - 1 + s_n} \\ & + \sum_{i=2}^{n} \frac{l_i s_{i-1}}{s_i} e_i^{\alpha_i - 1} e_{i-1}^{\frac{s_{i-1}}{s_i} - 1} g_{i-1}(e_1, \cdots, e_i). \end{aligned} \tag{29}$$

From (29), Lemma 2 and Assumption 1, it follows that

$$\begin{aligned} \dot{V}(x) \geq & \sum_{i=1}^{n-1} 2^{1-p_i} e_i^{\alpha_i - 1 + \frac{s_i p_i}{s_{i+1}}} - \sum_{i=1}^{n-1} |e_i^{\alpha_i - 1} e_{i+1}^{p_i}| + e_n^{\alpha_n - 1 + s_n} \\ & - \sum_{i=2}^{n} \frac{l_i s_{i-1}}{s_i} |e_i^{\alpha_i - 1} e_{i-1}^{\frac{s_{i-1}}{s_i} - 1} g_{i-1}(e_1, \cdots, e_i)| \\ = & \sum_{i=1}^{n-1} 2^{1-p_i}|e_i|^{\mu_i} + |e_n|^{\mu_n} - \sum_{i=1}^{n-1} |e_i^{\alpha_i - 1} e_{i+1}^{p_i}| \\ & - \sum_{i=2}^{n} \frac{l_i s_{i-1}}{s_i} |e_i^{\alpha_i - 1} e_{i-1}^{\frac{r_i}{r_{i-1}} - 1} g_{i-1}(e_1, \cdots, e_i)|, \end{aligned} \tag{30}$$

where

$$\mu_i = \alpha_i - 1 + \frac{s_i p_i}{s_{i+1}} = \frac{\tau_i + \alpha_i r_i}{r_i}, \quad i = 1, 2, \cdots, n-1,$$

and

$$\mu_n = \alpha_n - 1 + s_n = \frac{\tau_n + \alpha_n r_n}{r_n}.$$

By (30) and Lemma 4, for any $\varepsilon > 0$, there is $\bar{A}$ such that

$$\begin{aligned} |e_i^{\alpha_i - 1} e_{i+1}^{p_i}| &\leq \varepsilon |e_i|^{\frac{\alpha_i r_i - r_i + r_{i+1} p_i}{r_i}} + \bar{A}|e_{i+1}|^{\frac{\alpha_i r_i - r_i + r_{i+1} p_i}{r_{i+1}}} \\ &= \varepsilon |e_i|^{\frac{\tau_i + \alpha_i r_i}{r_i}} + \bar{A}|e_{i+1}|^{\frac{\tau_i + \alpha_i r_i}{r_{i+1}}} \\ &= \varepsilon |e_i|^{\mu_i} + \bar{A}|e_{i+1}|^{\hat{\mu}_{i+1}}, \end{aligned} \tag{31}$$

where

$$\hat{\mu}_{i+1} = \frac{\tau_i + \alpha_i r_i}{r_{i+1}} > \frac{\tau_{i+1} + \alpha_{i+1} r_{i+1}}{r_{i+1}} = \mu_{i+1}$$

due to $\tau_i > \tau_{i+1}$ and (28). Moreover, by Lemma 5, we have

$$\begin{aligned} & |e_i|^{\alpha_i - 1}|e_{i-1}|^{\frac{r_i}{r_{i-1}} - 1}|g_{i-1}| \\ \leq\; & C_{i-1}\sum_{k=1}^{i-1}\sum_{j=k}^{i-1}|e_i|^{\alpha_i - 1}|e_{i-1}|^{\frac{r_i}{r_{i-1}} - 1}(|e_j|^{\frac{\tau_k + r_{i-1}}{r_j}} + |e_i|^{\frac{\tau_{i-1} + r_{i-1}}{r_i}}). \end{aligned} \tag{32}$$

Now, let us estimate the terms of the right-hand side of (32). As $k=i-1$, we have that $j=i-1$ and

$$\begin{aligned} & |e_i|^{\alpha_i-1}|e_{i-1}|^{\frac{r_i}{r_{i-1}}-1}|e_j|^{\frac{\tau_k+r_{i-1}}{r_j}} \\ = & |e_i|^{\alpha_i-1}|e_{i-1}|^{\frac{\tau_{i-1}+r_i}{r_{i-1}}} \\ \leq & \varepsilon|e_{i-1}|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_{i-1}}}+\hat{A}|e_i|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_i}}, \end{aligned} \quad (33)$$

where

$$\frac{\tau_{i-1}+\alpha_i r_i}{r_{i-1}} \geq \frac{\tau_{i-1}+\alpha_{i-1}r_{i-1}}{r_{i-1}}=\mu_{i-1}$$

and

$$\frac{\tau_{i-1}+\alpha_i r_i}{r_i} > \frac{\tau_i+\alpha_i r_i}{r_i}=\mu_i.$$

As $k<i-1$, we have that

$$\begin{aligned} & |e_i|^{\alpha_i-1}|e_{i-1}|^{\frac{r_i}{r_{i-1}}-1}|e_j|^{\frac{\tau_k+r_{i-1}}{r_j}} \\ \leq & \varepsilon|e_j|^{\frac{\tau_k+\alpha_i r_i}{r_j}}+A(|e_i|^{\frac{\tau_k+\alpha_i r_i}{r_i}}+|e_{i-1}|^{\frac{\tau_k+\alpha_i r_i}{r_{i-1}}}). \end{aligned}$$

Considering $k<i-1$ and $k\leq j<i$, we have

$$\frac{\tau_k+\alpha_i r_i}{r_j} \geq \frac{\tau_j+\alpha_j r_j}{r_j}=\mu_j,$$

$$\frac{\tau_k+\alpha_i r_i}{r_i} > \frac{\tau_i+\alpha_i r_i}{r_i}=\mu_i$$

and

$$\frac{\tau_k+\alpha_i r_i}{r_{i-1}} > \frac{\tau_{i-1}+\alpha_{i-1}r_{i-1}}{r_{i-1}}=\mu_{i-1}.$$

Moreover,

$$\begin{aligned} & |e_i|^{\alpha_i-1}|e_{i-1}|^{\frac{r_i}{r_{i-1}}-1}|e_i|^{\frac{\tau_{i-1}+r_{i-1}}{r_i}} \\ = & |e_i|^{\frac{\alpha_i r_i-r_i+\tau_{i-1}+r_{i-1}}{r_i}}|e_{i-1}|^{\frac{r_i}{r_{i-1}}-1} \\ \leq & \varepsilon|e_{i-1}|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_{i-1}}}+A|e_i|^{\frac{\tau_{i-1}+\alpha_i r_i}{r_i}}, \end{aligned}$$

where

$$\frac{\tau_{i-1}+\alpha_i r_i}{r_{i-1}} \geq \frac{\tau_{i-1}+\alpha_{i-1}r_{i-1}}{r_{i-1}}=\mu_{i-1},$$

$$\frac{\tau_{i-1}+\alpha_i r_i}{r_i} > \frac{\tau_i+\alpha_i r_i}{r_i}=\mu_i.$$

By the above discussion below (30), letting $\varepsilon>0$ be sufficiently small, we conclude that there exist constants K_i $(i=1,2,\cdots,n)$ such that

$$\dot{V}(x) \geq \sum_{i=1}^{n} K_i|e_i|^{\mu_i}+h(e_1,e_2,\cdots,e_n),$$

where $h(e_1,e_2,\cdots,e_n)$ is composed of higher order terms. Therefore, if ε is sufficiently small, there exists a domain $D\subset\Re^n$ such that $\dot{V}(e)$ is positive definite on D, that is,

$$\dot{V}(e)>0, \quad \forall\, e\in D\backslash\{0\}. \quad (34)$$

From (27), it is easily seen that

$$l_i<0\ (i=1,2,\cdots,n) \Leftrightarrow k_i>0\ (i=1,2,\cdots,n). \quad (35)$$

If $k_i>0$ $(i=1,2,\cdots,n)$, it is clear that $V(e)$ is negative definite due to (35) and (26). This, together with (34), implies that the zero solution of (10) is asymptotically stable by Lyapunov Stability Theorem. Therefore the positivity of k_i's is sufficient for the asymptotic stability of (10) .

On the other hand, if there exists a $k_i<0$, by (35) there exists an $l_j>0$. In this case, by (26) we know that the set $G:=\{e\in\Re^n \mid V(e)>0\}$ is not empty and $e=0$ is a boundary point of G. Therefore, from (34) and Chetaev Instability Theorem, it follows that the zero solution of (10) is unstable. This implies that the positivity of k_i's is also necessary for the asymptotic stability of (10) ■

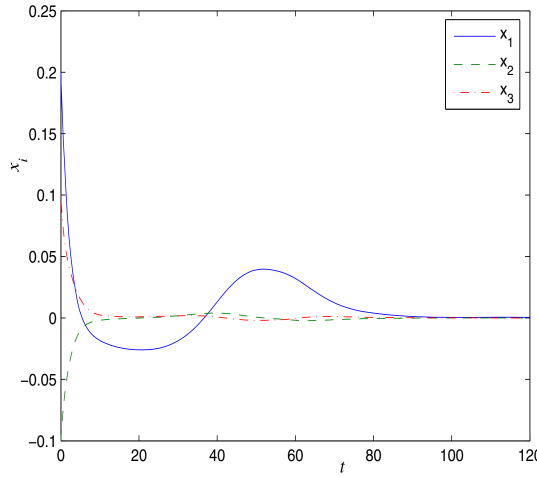

Fig. 1. Time response curves as $k_1=k_2=k_3=1$.

IV. SIMULATIONS

Consider the nonlinear system as follows:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3^{9/7}, \\ \dot{x}_3 &= -((k_1x_1^{81/45}+k_2x_2)^{83/81}+k_3x_3). \end{aligned} \quad (36)$$

It is easy to check that the system above is in the form (10) with the monotone homogeneous degrees

$$(\tau_1,\tau_2,\tau_3)=(\frac{4}{5},\frac{4}{7},0)$$

relative to dilation Δ_r with

$$(r_1,r_2,r_3)=(1,\frac{9}{5},\frac{83}{45}).$$

Fig.1 shows the time response curves as $k_1=k_2=k_3=1$ and Fig.2 shows the time response curves as $k_1=3$, $k_2=1$ and $k_3=2$. The simulations show that the nonlinear system with $k_1>0$, $k_2>0$ and $k_2>0$ is asymptotically stable.

V. CONCLUSION

For a class of power integral chain systems controlled by a nonlinear feedback, the asymptotic stability is analyzed by using homogeneity with strictly decreasing degrees (HSDD), technique of homogeneous domination, Lyapunov Stability Theorem and Chetaev Instability Theorem. In the future work, we will investigate the general nonlinear systems admitting HSDD.

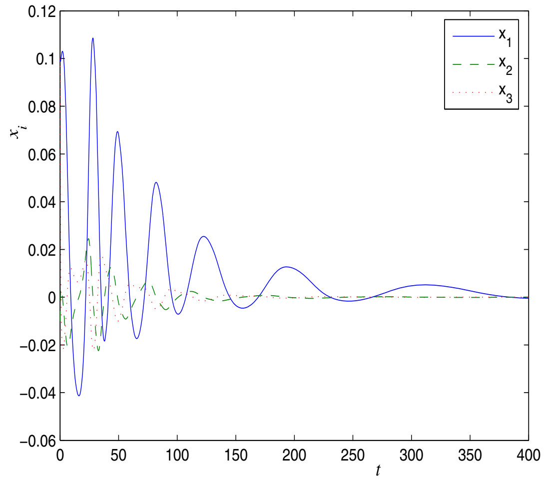

Fig. 2. Time response curves as $k_1=3$, $k_2=1$ and $k_3=2$.

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