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Distributed Saddle-Point Seeking via Continuous-time Multi-Agent Systems

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***Abstract*— We presents a continuous-time multi-agent system for distributed saddle-point seeking subject to bounded constraints. In the system, two groups of agents are employed for computing the two state vectors in a saddle-point, respectively. Each agent seeks for consensus with the agents in the same group, and simultaneously optimize its local objective functions by competing with the agents in the opposite group. A projection operator is introduced into the dynamics of each agent for dealing with bounded constraints. Two types of local interactions are considered. First, we consider proportional consensus protocols only. In this case, the gradient term of each agent is equipped with a disminishing gain. Second, we consider proportional-integral consensus protocols but without diminishing gain. In both cases, it is shown that the proposed systems can converge to the saddle-point set of a convex-concave function provided the communication topology is undirected and connected.**

I. Extended Abstract

Recent years have seen a flurry of research on collective dynamics analysis and distributed control in networked systems due to their wide applications in science and engineering, see [1], [2], [3], [4], [5], [6], [7] and references therein. A typical problem of interest is to achieve a networked-level objective, such as consensus, by a group of cooperative agents via local interactions. Among these, consensus-based distributed optimization has received considerable attentions, as it exists widely in various applications including machine learning [8], signal processing [9], etc. In this problem, the networked-level objective is to optimize a global function, whose information is distributedly known by the agents. The optimal solution is attained via local interaction among agents. Compared with the centralized methods, the distributed ones have the advantage in scalability, robustness and privacy protection.

A. Problem Formulation

We are interested in seeking a saddle-point of $F(x,y)$ on $X\times Y$ by using a distributed method based on multi-agent networks. Suppose that the information of $F(x,y)$ is distributed over two groups of disjoint agents $\mathscr{V}_1=\{1,2,\cdots,n_1\}$ and $\mathscr{V}_2=\{1,2,\cdots,n_2\}$, i.e., $F(x,y)$ can be expressed as

$$F(x,y)=\sum_{i\in\mathscr{V}_1} f_1^i(x,y)=\sum_{i\in\mathscr{V}_2} f_2^i(x,y), \tag{1}$$

where f_ℓ^i ($i\in\mathscr{V}_\ell$, $\ell\in\{1,2\}$) is a convex-concave function assigned to agent $i\in\mathscr{V}_\ell$ only. Denote the states of agents $i\in\mathscr{V}_1$ and $i\in\mathscr{V}_2$ by $x_i\in\mathbb{R}^{d_1}$ and $y_i\in\mathbb{R}^{d_2}$, respectively. All agents in the two groups can share state information with others. Our objective is to achieve

$$\lim_{t\to\infty} x_i(t)=x^*,\quad \forall i\in\mathscr{V}_1, \tag{2a}$$

$$\lim_{t\to\infty} y_i(t)=y^*,\quad \forall i\in\mathscr{V}_2, \tag{2b}$$

with $(x^*,y^*)\in X^*\times Y^*$ by designing proper dynamics of all agents.

B. Multi-Agent Models

First, we introduce the multi-agent system with diminishing gains. The dynamics of all agents are given as

$$\begin{cases} \dot{x}_i\in & \sum_{j\in\mathscr{V}_1} a_{ij}(x_j-x_i)-\alpha(t)\partial_x f_1^i(x_i,\tilde{y}_i) \\ & +\mathscr{P}_X(x_i)-x_i,\quad i\in\mathscr{V}_1, \\ \dot{y}_i\in & \sum_{j\in\mathscr{V}_2} a_{ij}(y_j-y_i)+\alpha(t)\partial_y f_2^i(\tilde{x}_i,y_i) \\ & +\mathscr{P}_Y(y_i)-y_i,\quad i\in\mathscr{V}_2, \end{cases} \tag{3}$$

where $[a_{ij}]$ is the adjacent matrix of the underlying communication graph $\mathscr{G}$ among agents, $\alpha(t)>0$ is a continuous gain satisfying $\lim_{t\to\infty}\alpha(t)=0$, $\int_{t_0}^{\infty}\alpha(t)\mathrm{d}t=+\infty$, $\tilde{y}_i$ $(i\in\mathscr{V}_1)$ and $\tilde{x}_i$ $(i\in\mathscr{V}_2)$ are defined as

$$\tilde{y}_i=\frac{\sum_{j\in\mathscr{V}_2} a_{ij}\mathscr{P}_Y(y_j)}{\sum_{j\in\mathscr{V}_2} a_{ij}},\quad \tilde{x}_i=\frac{\sum_{j\in\mathscr{V}_1} a_{ij}\mathscr{P}_X(x_j)}{\sum_{j\in\mathscr{V}_1} a_{ij}}. \tag{4}$$

Second, we introduce the multi-agent system with constant gains. The dynamics of all agents are given as

$$\begin{cases} \dot{x}_i\in & \sum_{j\in\mathscr{V}_1} a_{ij}(x_j-x_i)+\int_0^t\sum_{j\in\mathscr{V}_1} a_{ij}(x_j(s)-x_i(s))\mathrm{d}s \\ & -\alpha\partial_x f_1^i(x_i,\tilde{y}_i)+\mathscr{P}_X(x_i)-x_i,\quad i\in\mathscr{V}_1, \\ \dot{y}_i\in & \sum_{j\in\mathscr{V}_2} a_{ij}(y_j-y_i)+\int_0^t\sum_{j\in\mathscr{V}_2} a_{ij}(y_j(s)-y_i(s))\mathrm{d}s \\ & +\alpha\partial_y f_2^i(\tilde{x}_i,y_i)+\mathscr{P}_Y(y_i)-y_i,\quad i\in\mathscr{V}_2, \end{cases} \tag{5}$$

where $\alpha>0$ is a constant gain.

Assume that each f_ℓ^i ($i\in\mathscr{V}_\ell$, $\ell\in\{1,2\}$) is coercive, i.e., $\lim_{\|(x,y)\|\to\infty}|f_\ell^i(x,y)|=\infty$. The communication graph $\mathscr{G}$ is undirected and connected. In addition, there is a constant $L>0$ such that $\|\gamma\|\le L$, where $\gamma\in\partial_x f_1^i(x,y)\bigcup\partial_y f_2^i(x,y)$, $\forall i\in\mathscr{V}_1$, $\forall j\in\mathscr{V}_2$, $\forall x\in X,\forall y\in Y$. We can theoretically show that both of the above two multi-agent systems can achieve (2).

*This presentation is a summarization of our works on distributed saddle-pointing seeking, which are cooperated with these researchers: Jun Wang, Tingwen Huang, Qingshan Liu, Huaqing Li, Wenfeng Hu.

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