23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Stability of a hyperbolic system of the interconnected Schrödinger and wave equations

Jun-Min Wang[1], Fei Wang[1], and Xiangdong Liu[2]

***Abstract*— In this paper, we consider the stability of a hyperbolic system of the interconnected Schrödinger and wave equations, where the only distributed dissipative control is forced at the wave equation and there is no control fixed at the Schrödinger. The Schrödinger can be exponentially stabilized by the inter-change transmission between the Schrödinger and wave at the interconnection boundary. We show that the whole system is well-posed and exponentially stable in the energy Hilbert space. A numerical computation is presented for the distributions of the spectrum of the whole system, and it is found that the spectrum of the Schrödinger depends largely on the interconnected transmission parameter and the decay of the wave equation.**

I. INTRODUCTION

Extensive literature exists on control of the Schrödinger equation (see [3]-[8], [10], or [13]). In this paper, we consider the control problem of a hyperbolic system of the interconnected Schrödinger-wave equations (as shown in Fig. 1):

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ z_{tt}(x,t) = z_{xx}(x,t) + U(t), & 1 < x < 2, t > 0, \\ y(0,t) = z(2,t) = 0, & t \geq 0, \\ y(1,t) = kz_t(1,t), & t \geq 0, \\ z_x(1,t) = -iky_x(1,t), & t \geq 0, \end{cases} \tag{1}$$

where $k \neq 0 \in \mathbb{R}$ is a real constant, the Schrödinger and wave equations are connected at the boundary $x = 1$, and y and z denote the displacements of the Schrödinger and wave equations respectively. The distributed control $U(t)$ is forced only at the wave and there is no control fixed to the Schrödinger. The energy suppression of the Schrödinger is only through the interconnected boundary transmission between the Schrödinger and wave equations:

$$y(1,t) = kz_t(1,t), \; z_x(1,t) = -iky_x(1,t), \quad t \geq 0. \tag{2}$$

When we take the control $U(t)$ by the distributed velocity and displacements as

$$U(t) = -2bz_t(x,t) - b^2 z(x,t), \quad 1 < x < 2, \; t > 0, \tag{3}$$

where $b > 0$ is a positive feedback gain, the closed-loop system is given as

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1, \\ z_{tt}(x,t) = z_{xx}(x,t) & \\ \qquad -2bz_t(x,t) - b^2 z(x,t), & 1 < x < 2, \\ y(0,t) = z(2,t) = 0, & t \geq 0, \\ y(1,t) = kz_t(1,t), & t \geq 0, \\ z_x(1,t) = -iky_x(1,t), & t \geq 0. \end{cases} \tag{4}$$

The energy function of (4) is

$$E(t) = \frac{1}{2}\int_0^1 |y(x,t)|^2 dx + \frac{1}{2}\int_1^2 \left[|z_x(x,t)|^2 + b^2|z(x,t)|^2 + |z_t(x,t)|^2\right] dx. \tag{5}$$

The derivative of $E(t)$ with respect to the time t yields

$$\frac{dE(t)}{dt} = -2b\int_1^2 |z_t(x,t)|^2 \, dx \leq 0.$$

Hence, the energy $E(t)$ is non-increasing. An early attempt to consider the stability of the interconnected Schrödinger and wave equations is our recent paper [6]:

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1, \\ z_{tt}(x,t) = z_{xx}(x,t) - bz_t(x,t), & 1 < x < 2, \\ y(0,t) = z(2,t) = 0, & t \geq 0, \\ y(1,t) = kz_t(1,t), & t \geq 0, \\ z_x(1,t) = -iky_x(1,t), & t \geq 0, \end{cases} \tag{6}$$

where $b > 0$. The eigenvalues of system (6) in [6] is showed to be located in the left hand side of the complex plane, and hence the C_0-semigroup generated by the system operator achieves strong stability, but the result for exponential stability is still open in [6].

Compared to [6], for simplicity in this paper, we design the control $U(t)$ as (3) including both the distributed velocity $-2bz_t(x,t)$ and displacement $-b^2z(x,t)$, which can easily get the decay rate $-b$ of the wave subsystem if there is no interconnection with the Schrödinger. We show that the whole system (4) is exponentially stable in energy Hilbert space, which says that the Schrödinger without any control can be exponentially stable only through the boundary interconnected with the exponentially stabilized wave equation. Moreover, the simulations for the eigenvalues will demonstrate that the decay of the Schrödinger is depended largely on the interconnected parameter k and the decay rate $-b$ of the wave equation.

*This work was supported by the National Natural Science Foundation of China under Grant no. 61673061.
[1]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China. jmwang@bit.edu.cn
[2]School of Automation, Beijing Institute of Technology, Beijing 100081, P. R. China.

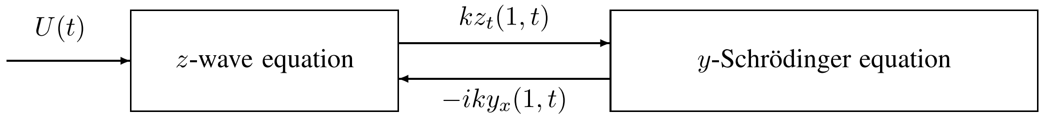

Fig. 1. Block diagram for a hyperbolic system of the interconnected Schrödinger-wave system

An interconnected Schrödinger and heat equations has been treated in [13]:

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ u_t(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ w(1,t) = u_x(1,t), & t \geq 0, \\ w(0,t) = ku(0,t), & t \geq 0, \\ u_x(1,t) = ikw_x(1,t), & t \geq 0, \end{cases} \tag{7}$$

where $k \neq 0$ is a constant. It is showed that the system (7) has two branches of eigenvalues along two parabolas, which are asymptotically symmetric relative to the line $\mathrm{Re}\lambda = -\mathrm{Im}\lambda$ in λ-plane. When $|k| \neq 1$, the system (7) is showed to have the Riesz basis property and exponential stability. Remarkably, the semigroup, generated by the system operator, is of Gevrey class $\delta > 2$, which says that the regularity of the Schrödinger has been greatly improved by the heat equation only through the boundary interconnections between the Schrödinger and heat equations. The stability and regularity of the interconnected heat-Schrödinger system in a two-dimensional torus region are obtained in [4].

In some instances the solution of the Euler-Bernoulli beam can be obtained from the Schrödinger equation [8], but significant differences arise due to boundary conditions [11]. The interconnected heat and Euler-Bernoulli equations has been considered in [12] and [17] with different boundary connections respectively, where the exponential stability and Gevrey regularity are established due to the analytic regularity of the heat equation.

The rest of this paper is organized as follows. In the next section, we present the well-posedness of the system. Section III is devoted to the spectral analysis and to get the asymptotic eigenvalues of the system. The Riesz basis property and exponential stability are established in Section IV. Some numerical computations of the eigenvalues are presented in Section V. Some conclusion is given in last Section VI.

II. Well-posedness of the system

In this section, we show the well-posedness of system (4). For simplicity in arguments, we introduce the following transformation:

$$\begin{cases} w(x,t) = y(1-x,t), & 0 < x < 1, t > 0, \\ u(x,t) = z(x+1,t), & 0 < x < 1, t > 0, \end{cases} \tag{8}$$

then system (4) becomes

$$\begin{cases} w_t(x,t) + iw_{xx}(x,t) = 0, & 0 < x < 1, \\ u_{tt}(x,t) = u_{xx}(x,t) & \\ \qquad -2bu_t(x,t) - b^2u(x,t), & 0 < x < 1, \\ w(1,t) = u(1,t) = 0, & t \geq 0, \\ w(0,t) = ku_t(0,t), & t \geq 0, \\ u_x(0,t) = ikw_x(0,t), & t \geq 0. \end{cases} \tag{9}$$

We consider system (9) in the energy Hilbert space

$$\mathcal{H} = L^2(0,1) \times H^1_E(0,1) \times L^2(0,1),$$

where $H^1_E(0,1) = \{g \in H^1(0,1) | g(1) = 0\}$. The norm in $\mathcal{H}$ is induced by the inner product:

$$\langle X_1, X_2 \rangle = \int_0^1 \Big[f_1(x)\overline{f_2(x)} + g_1'(x)\overline{g_2'(x)} + b^2 g_1(x)\overline{g_2(x)} + h_1(x)\overline{h_2(x)} \Big] dx, \tag{10}$$

where $X_s = (f_s, g_s, h_s) \in \mathcal{H}$, $s = 1, 2$. Now we can define two linear operators $\mathcal{A}$ and $\mathcal{B}$, respectively, by

$$\begin{cases} \mathcal{A}(f,g,h) = (-if'',\ h,\ g''),\ \forall (f,g,h) \in D(\mathcal{A}), \\ D(\mathcal{A}) = \begin{cases} (f,g,h) \in \left(H^2 \times H^2 \times H^1_E\right) \cap \mathcal{H} \ \Big| \\ f(1) = 0, f(0) = kh(0), \\ g'(0) = ikf'(0), \end{cases} \end{cases} \tag{11}$$

and

$$\mathcal{B}(f,g,h) = (0, 0, -2bh - b^2 g), \forall (f,g,h) \in D(\mathcal{B}) = \mathcal{H}, \tag{12}$$

where it is found that $\mathcal{B}$ is bounded in $\mathcal{H}$. Then system (9) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{dX(t)}{dt} = (\mathcal{A} + \mathcal{B})X(t), \quad X(0) = X_0 \tag{13}$$

where $X(t) = (w(\cdot,t), u(\cdot,t), u_t(\cdot,t))$ and X_0 denotes the initial datum.

Lemma 2.1: In $\mathcal{H}$, $\mathcal{A}$ is a skew-adjoint operator with compact resolvents and $\mathcal{B}$ is a bounded operator. Hence, $\mathcal{A} + \mathcal{B}$ generates a C_0-group $e^{(\mathcal{A}+\mathcal{B})t}$ and the spectrum $\sigma(\mathcal{A} + \mathcal{B})$ consists of isolated eigenvalues only.

Proof: It is easy to verify that $\mathcal{A}$ is a skew-adjoint operator with compact resolvents in $\mathcal{H}$, and hence $\mathcal{A}$ has only isolated eigenvalues located on the imaginary axis, and generates a unitary group in $\mathcal{H}$. Since $\mathcal{B}$ is bounded in $\mathcal{H}$, the standard perturbation result ([9]) implies that $\mathcal{A} + \mathcal{B}$ has compact resolvents and generates a C_0-group $e^{(\mathcal{A}+\mathcal{B})t}$ on $\mathcal{H}$. Moreover, the spectrum $\sigma(\mathcal{A}+\mathcal{B})$ consists of isolated eigenvalues only. The proof is complete. ∎

Lemma 2.2: $\mathcal{A}+\mathcal{B}$ is dissipative in $\mathcal{H}$ and hence $e^{(\mathcal{A}+\mathcal{B})t}$, generated by $\mathcal{A}+\mathcal{B}$, is a contractions C_0-group in $\mathcal{H}$.

Proof: Let $X = (f, g, h) \in D(\mathcal{A}+\mathcal{B})$. A direct computation gives

$$\begin{aligned} & \mathrm{Re}\langle(\mathcal{A}+\mathcal{B})X, X\rangle = \mathrm{Re}\langle\mathcal{B}X, X\rangle \\ = \ & \mathrm{Re}\left\langle(0,0,-2bh-b^2g),\ (f,g,h)\right\rangle \\ = \ & -2b\int_0^1 |h|^2dx \le 0. \end{aligned} \tag{14}$$

Hence $\mathcal{A}+\mathcal{B}$ is dissipative. The proof is complete. ∎

III. Spectral analysis

In this section, we consider the eigenvalue problem of system (9). Let $(\mathcal{A}+\mathcal{B})X = \lambda X$ and $\lambda \in \sigma(\mathcal{A}+\mathcal{B})$, where $0 \neq X = (f, g, h) \in D(\mathcal{A}+\mathcal{B})$. Then we have that $h(x) = \lambda g(x)$ and that $f(x)$ and $g(x)$ satisfy the eigenvalue problem

$$\begin{cases} f''(x) - i\lambda f(x) = 0, \\ g''(x) = (\lambda^2+2b\lambda+b^2)g(x) = (\lambda+b)^2g(x), \\ f(1) = g(1) = 0, \\ f(0) = \lambda k g(0),\ g'(0) = ikf'(0). \end{cases} \tag{15}$$

We can get

$$\begin{cases} f(x) = c_1\sinh(\sqrt{i\lambda})(1-x), \\ g(x) = c_2\sinh(\lambda+b)(1-x), \end{cases} \tag{16}$$

where c_1 and c_2 are constants. By the boundary conditions $f(0) = \lambda k g(0)$ and $g'(0) = ikf'(0)$, we have

$$\begin{cases} c_1\sinh(\sqrt{i\lambda}) = c_2k\lambda\sinh(\lambda+b), \\ c_2(\lambda+b)\cosh(\lambda+b) = c_1ik\sqrt{i\lambda}\cosh(\sqrt{i\lambda}). \end{cases} \tag{17}$$

We claim in (17) that $c_1 \neq 0$. Actually, if $c_1 = 0$, then g satisfies

$$\begin{cases} g''(x) = (\lambda^2+2b\lambda+b^2)g(x) = (\lambda+b)^2g(x), \\ g(1) = g(0) = g'(0) = 0, \end{cases}$$

which yields $g \equiv 0$. Contradiction! Hence, $c_1 \neq 0$. Similarly, we have $c_2 \neq 0$. It follows from (17) that

$$\begin{aligned} c_1c_2\Big[&(\lambda+b)\cosh(\lambda+b)\sinh(\sqrt{i\lambda}) \\ &-ik^2\lambda\sqrt{i\lambda}\sinh(\lambda+b)\cosh(\sqrt{i\lambda})\Big] = 0. \end{aligned}$$

By $c_1c_2 \neq 0$, we get equation (15) has the nontrivial solution if and only if

$$\begin{aligned} &(\lambda+b)\cosh(\lambda+b)\sinh(\sqrt{i\lambda}) \\ &\quad -ik^2\lambda\sqrt{i\lambda}\sinh(\lambda+b)\cosh(\sqrt{i\lambda}) = 0. \end{aligned}$$

have zeros. Hence, we get the following lemma immediately.

Lemma 3.1: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (11) and (12) respectively, and let

$$\begin{aligned} \Delta(\lambda) = \ & (\lambda+b)\cosh(\lambda+b)\sinh(\sqrt{i\lambda}) \\ & -ik^2\lambda\sqrt{i\lambda}\sinh(\lambda+b)\cosh(\sqrt{i\lambda}). \end{aligned} \tag{18}$$

Then

$$\sigma(\mathcal{A}+\mathcal{B}) = \sigma_p(\mathcal{A}+\mathcal{B}) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\}. \tag{19}$$

Lemma 3.2: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (11) and (12) respectively. For each $\lambda \in \sigma_p(\mathcal{A}+\mathcal{B})$, we have $\mathrm{Re}\lambda < 0$.

Proof: By Lemma 2.2, since $\mathcal{A}+\mathcal{B}$ is dissipative, for each $\lambda \in \sigma(\mathcal{A}+\mathcal{B})$, we have $\mathrm{Re}\lambda \le 0$. So we only need to show there is no eigenvalue on the imaginary axis. Let $\lambda = \pm i\mu^2 \in \sigma_p(\mathcal{A}+\mathcal{B})$ with $\mu \in \mathbb{R}^+$ and $X = (f, g, h) \in D(\mathcal{A}+\mathcal{B})$ be its associated eigenfunction of $\mathcal{A}+\mathcal{B}$. Then by (14), we have

$$\mathrm{Re}\langle(\mathcal{A}+\mathcal{B})X, X\rangle = -2b\int_0^1 |h|^2dx = 0.$$

Hence $h(x) = 0$. By the second equations of (15), we have $g(x) = 0$. Then by the first equation of (15) and its boundary conditions we have:

$$f''(x) = i\lambda f(x), \quad f(0) = f'(0) = f(1) = 0,$$

which yields $f(x) = 0$. Hence, $X = (f, g, h) = 0$. Therefore, there is no eigenvalue on the imaginary axis. ∎

Theorem 3.3: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (11) and (12) respectively, and let $\Delta(\lambda)$ given by (18). The eigenvalues of $\mathcal{A}+\mathcal{B}$ have the following asymptotic expressions: for $n \in \mathbb{N}$ and as $n \to \infty$,

$$\lambda_{1n}^{\pm} = -b \pm n\pi i + \mathcal{O}(n^{-1/2}), \tag{20}$$

and

$$\lambda_{2n} = -\beta_{1n} + \beta_{2n}i + \left(n - \frac{1}{2}\right)^2\pi^2 i + \mathcal{O}(n^{-1}), \tag{21}$$

where $\beta_{1n} > 0$ and β_{2n} are two real constants, and β_{1n} satisfies the relationship

$$2^{-1}k^2\beta_{1n} = \frac{\gamma_{1n}^2 - 1 + \gamma_{2n}^2}{(\gamma_{1n}-1)^2 + \gamma_{2n}^2} = \frac{e^{4b-4\beta_{1n}} - 1}{(\gamma_{1n}-1)^2 + \gamma_{2n}^2}, \tag{22}$$

with

$$\begin{cases} \gamma_{1n} = e^{2b-2\beta_{1n}}\cos\left[2\beta_{2n} + 2\left(n-\frac{1}{2}\right)^2\pi^2\right], \\ \gamma_{2n} = e^{2b-2\beta_{1n}}\sin\left[2\beta_{2n} + 2\left(n-\frac{1}{2}\right)^2\pi^2\right]. \end{cases} \tag{23}$$

Moreover, as $n \to \infty$, we have

$$\beta_{1n} \not\to 0. \tag{24}$$

Therefore, as $n \to \infty$,

$$\mathrm{Re}\lambda_{1n}^{\pm} \to -b < 0,\ \mathrm{Re}\lambda_{2n}^{\pm} = -\beta_{1n}(<0) \not\to 0, \tag{25}$$

which says that the imaginary axis is not an asymptote of the eigenvalues of $\mathcal{A}+\mathcal{B}$.

Proof: By $\Delta(\lambda) = 0$, we have

$$\begin{aligned} & ik^2\lambda\sqrt{i\lambda}(e^{(\lambda+b)} - e^{-(\lambda+b)})(e^{\sqrt{i\lambda}} + e^{-\sqrt{i\lambda}}) \\ & -(\lambda+b)(e^{(\lambda+b)} + e^{-(\lambda+b)})(e^{\sqrt{i\lambda}} - e^{-\sqrt{i\lambda}}) = 0. \end{aligned} \tag{26}$$

Let $\lambda=\rho^2\neq 0$. Since $\mathrm{Re}\lambda<0$, we have

$$\arg\rho\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right),$$

and hence (26) can be rewritten as

$$\begin{aligned}&0=k^2(e^{(\lambda+b)}-e^{-(\lambda+b)})(e^{\sqrt{i}\rho}+e^{-\sqrt{i}\rho})\\&+\sqrt{i}\left(\rho^{-1}+b\rho^{-3}\right)(e^{(\lambda+b)}+e^{-(\lambda+b)})(e^{\sqrt{i}\rho}-e^{-\sqrt{i}\rho}).\end{aligned}\tag{27}$$

There are two cases:

i) When

$$\arg\rho\in\mathcal{S}_1=\left(\frac{11\pi}{16},\ \frac{3\pi}{4}\right],$$

we have

$$\arg\left(\sqrt{i}\rho\right)\in\left(\frac{15\pi}{16},\pi\right],$$

and

$$\arg(\lambda)=\arg(\rho^2)\in\left(\frac{11\pi}{8},\ \frac{3\pi}{2}\right],$$

with the estimates

$$\left|e^{\sqrt{i}\rho}\right|\to 0,\quad \left|e^{-\sqrt{i}\rho}\right|\to+\infty.$$

So, equation (27) becomes

$$e^{2(\lambda+b)}-1+\mathcal{O}(\rho^{-1})=0.\tag{28}$$

Since $e^{2(\lambda+b)}-1=0$ has the solutions

$$\tilde{\lambda}_{1n}^{-}=-b-n\pi i,\ n\in\mathbb{N}.$$

Applying the Rouché's theorem, we get the solutions of equation (28): as $n\to\infty$,

$$\lambda_{1n}^{-}=-b-n\pi i+\mathcal{O}(n^{-1/2}),\quad n\in\mathbb{N}.\tag{29}$$

ii) When

$$\arg\rho\in\mathcal{S}_2=\left[\frac{\pi}{4},\ \frac{5\pi}{16}\right),$$

we have

$$\arg\left(\sqrt{i}\rho\right)\in\left[\frac{\pi}{2},\ \frac{9\pi}{16}\right)$$

and

$$\arg\lambda=\arg(\rho^2)\in\left[\frac{\pi}{2},\ \frac{5\pi}{8}\right).$$

Moreover, equation (26) can be rewritten as

$$\begin{aligned}&0=k^2(e^{(\lambda+b)}-e^{-(\lambda+b)})(e^{\sqrt{i}\rho}+e^{-\sqrt{i}\rho})\\&+\sqrt{i}\rho^{-1}(e^{(\lambda+b)}+e^{-(\lambda+b)})(e^{\sqrt{i}\rho}-e^{-\sqrt{i}\rho})+\mathcal{O}(\rho^{-3}).\end{aligned}\tag{30}$$

It is noting that the equation

$$(e^{(\lambda+b)}-e^{-(\lambda+b)})(e^{\sqrt{i}\rho}+e^{-\sqrt{i}\rho})=0$$

goes to

$$e^{(\lambda+b)}-e^{-(\lambda+b)}=0\ \text{ or }\ e^{\sqrt{i}\rho}+e^{-\sqrt{i}\rho}=0.\tag{31}$$

The equation (31) yields two branch solutions:

$$\begin{cases}\tilde{\lambda}_{1n}^{+}=-b+n\pi i, & n\in\mathbb{N},\\ \tilde{\rho}_{2n}=\left(n-\frac{1}{2}\right)\sqrt{i}\pi, & n\in\mathbb{N},\\ \tilde{\lambda}_{2n}=\tilde{\rho}_{2n}^2=\left(n-\frac{1}{2}\right)^2\pi^2 i, & n\in\mathbb{N}.\end{cases}\tag{32}$$

By applying the Rouché's theorem again, we get the solutions of equation (30): for $n\in\mathbb{N}$ and $n\to\infty$,

$$\lambda_{1n}^{+}=-b+n\pi i+\mathcal{O}(n^{-1/2}),\tag{33}$$

and

$$\rho_{2n}=\tilde{\rho}_{2n}+\alpha_n=\left(n-\frac{1}{2}\right)\sqrt{i}\pi+\alpha_n,\ \alpha_n=\mathcal{O}(n^{-1}).\tag{34}$$

For the second branch ρ_{2n}, we need to get the more accurate estimate for α_n. Since

$$\lambda_{2n}=\rho_{2n}^2=\left(n-\frac{1}{2}\right)^2\pi^2 i+2\alpha_n\left(n-\frac{1}{2}\right)\sqrt{i}\pi+\mathcal{O}(n^{-2})\tag{35}$$

and

$$e^{\lambda_{2n}}=\beta_n e^{\alpha_n(2n-1)\sqrt{i}\pi}+\mathcal{O}(n^{-2}),\beta_n=e^{(n-1/2)^2\pi^2 i}.\tag{36}$$

Substituting these into equation (30), we have

$$\begin{aligned}&0=\mathcal{O}(\rho_{2n}^{-3})+k^2\left[e^{\sqrt{i}(\tilde{\rho}_{2n}+\alpha_n)}+e^{-\sqrt{i}(\tilde{\rho}_{2n}+\alpha_n)}\right]\\&+\sqrt{i}\rho_{2n}^{-1}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}\left(e^{\sqrt{i}(\tilde{\rho}_{2n}+\alpha_n)}-e^{-\sqrt{i}(\tilde{\rho}_{2n}+\alpha_n)}\right).\end{aligned}$$

By using the equality $e^{\sqrt{i}\tilde{\rho}_{2n}}=-e^{-\sqrt{i}\tilde{\rho}_{2n}}$, the above equation can be rewritten as

$$\begin{aligned}&0=\mathcal{O}(\rho_{2n}^{-3})+k^2\left[e^{\sqrt{i}\alpha_n}-e^{-\sqrt{i}\alpha_n}\right]\\&+\sqrt{i}\tilde{\rho}_{2n}^{-1}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}\left(e^{\sqrt{i}\alpha_n}+e^{-\sqrt{i}\alpha_n}\right).\end{aligned}$$

By Taylor series expression, we have

$$2\sqrt{i}k^2\alpha_n+2\sqrt{i}\tilde{\rho}_{2n}^{-1}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}+\mathcal{O}(\tilde{\rho}_{2n}^{-2})=0,$$

which yields

$$\alpha_n=-k^{-2}\tilde{\rho}_{2n}^{-1}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}+\mathcal{O}(\tilde{\rho}_{2n}^{-2}).$$

Hence,

$$\rho_{2n}=\tilde{\rho}_{2n}-k^{-2}\tilde{\rho}_{2n}^{-1}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}+\mathcal{O}(\tilde{\rho}_{2n}^{-2})$$

and

$$\begin{aligned}\lambda_{2n}&=\rho_{2n}^2=\tilde{\rho}_{2n}^2-2k^{-2}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}+\mathcal{O}(\tilde{\rho}_{2n}^{-1})\\&=\left(n-\frac{1}{2}\right)^2\pi^2 i-2k^{-2}\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1}+\mathcal{O}(n^{-1}).\end{aligned}\tag{37}$$

From above expression of λ_{2n}, λ_{2n} has the expression (21). By Lemma 3.2, we have

$$\beta_{1n}>0.$$

It is further found that

$$\begin{aligned}
&e^{2(\lambda_{2n}+b)} = e^{2b-2\beta_{1n}} e^{2\left[\beta_{2n}+\left(n-\frac{1}{2}\right)^2\pi^2\right]i} + \mathcal{O}(n^{-1}) \\
= \quad & e^{2b-2\beta_{1n}} \cos\left[2\beta_{2n} + 2\left(n-\frac{1}{2}\right)^2 \pi^2\right] \\
& + i e^{2b-2\beta_{1n}} \sin\left[2\beta_{2n} + 2\left(n-\frac{1}{2}\right)^2 \pi^2\right] + \mathcal{O}(n^{-1}) \\
= \quad & \gamma_{1n} + i\gamma_{2n} + \mathcal{O}(n^{-1}),
\end{aligned}$$

where γ_{1n} and γ_{2n} are given as (23). So, we have

$$\begin{aligned}
&\frac{e^{2(\lambda_{2n}+b)}+1}{e^{2(\lambda_{2n}+b)}-1} = \frac{1+\gamma_{1n}+i\gamma_{2n}}{\gamma_{1n}-1+i\gamma_{2n}} + \mathcal{O}(n^{-1}) \\
= \quad & \frac{\gamma_{1n}^2 - 1 + \gamma_{2n}^2}{(\gamma_{1n}-1)^2+\gamma_{2n}^2} - i\frac{2\gamma_{2n}}{(\gamma_{1n}-1)^2+\gamma_{2n}^2} + \mathcal{O}(n^{-1}),
\end{aligned}$$

and γ_{1n} and γ_{2n} satisfy the relationships:

$$\begin{cases} \gamma_{1n}^2 + \gamma_{2n}^2 = e^{4b-4\beta_{1n}}, \\ (\gamma_{1n}-1)^2 + \gamma_{2n}^2 = \gamma_{1n}^2 + \gamma_{2n}^2 + 1 - 2\gamma_{1n}. \end{cases}$$

Comparing to expressions (37) and (21), β_{1n} satisfies the relationship (22). Finally, we claim that

$$\beta_{1n} \not\to 0, \quad \text{as } n \to \infty.$$

Actually, if $\beta_{1n} \to 0$, then the right side of (22) does not go to 0. This is contradiction! The proof is complete. ■

IV. Exponential stability

In this section, we establish the Riesz basis property and stability of system (9). Due to fact that $\mathcal{B}$ is bounded in $\mathcal{H}$. It is natural to use the bounded perturbation conclusion to consider the completeness of system (9). So, we cite the Keldysh theorem [2, pp.170, Theorem 4] here.

Lemma 4.1: Let K be a compact self-adjoint operator in a Hilbert space $\mathcal{H}$ with $\ker K = \{0\}$ and eigenvalues $\lambda_j(K), j = 1, 2, \ldots$.. Assume that

$$\sum_{j=1}^{\infty} |\lambda_j(K)|^r < \infty$$

for some $r \geq 1$, and let S be a compact operator such that $I + S$ is invertible. Then the system of generalized eigenfunctions of the operator

$$A := K(I+S)$$

is complete in $\mathcal{H}$.

Now we can get the completeness of system (9).

Theorem 4.2: Let $\mathcal{A}$ and $\mathcal{B}$ be defined by (11) and (12) respectively. Then system (9) is complete in the sense that the generalized eigenfunctions of $\mathcal{A}+\mathcal{B}$ are complete in Hilbert space $\mathcal{H}$.

Proof: Since $\mathcal{A}$ is a skew-adjoint operator with compact resolvents and $0, \infty \in \rho(A)$, $(i\mathcal{A})^{-1}$ is a compact self-adjoint operator with $\ker(i\mathcal{A})^{-1} = \{0\}$. We have

$$\left\{\lambda_k\left((i\mathcal{A})^{-1}\right)\right\}_{k=1}^{\infty} \in \ell^2.$$

Since

$$\begin{aligned}
(i(\mathcal{A}+\mathcal{B}))^{-1} &= (i\mathcal{A})^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1} \\
&= (i\mathcal{A})^{-1}(I - \mathcal{B}\mathcal{A}^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1}),
\end{aligned}$$

$\mathcal{B}\mathcal{A}^{-1}$ and $\mathcal{B}\mathcal{A}^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1}$ are compact and

$$I - \mathcal{B}\mathcal{A}^{-1}(I+\mathcal{B}\mathcal{A}^{-1})^{-1}$$

is invertible, so the proof follows from Lemma 4.1. ■

We cite another conclusion to get the Riesz basis property (see [14], [15] or [16]).

Lemma 4.3: Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{A}$ be the generator of a C_0-semigroup $T(t)$ on $\mathcal{H}$. Suppose that the following conditions hold:

(1) $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$ and $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k=1}^{\infty}$ consists of only isolated eigenvalues of finite algebraic multiplicity;

(2) For $m_a(\lambda_k) := \dim E(\lambda_k, \mathcal{A})\mathcal{H}$, where $E(\lambda_k, \mathcal{A})$ denotes the Riesz-projection associated with λ_k, it has

$$\sup_{k\geq 1} m_a(\lambda_k) < \infty;$$

(3) There is a constant α such that

$$\sup\{\mathrm{Re}\lambda | \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\mathrm{Re}\lambda | \lambda \in \sigma_2(\mathcal{A})\}$$

and

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

Then the following assertions hold:

(i) There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$, $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$, and $\{E(\lambda_k, \mathcal{A})\mathcal{H}_2\}_{k=1}^{\infty}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$, i.e., $\forall x_2 \in \mathcal{H}_2$,

$$x_2 = \sum_{k=1}^{\infty} E(\lambda_k, \mathcal{A})x_2.$$

(ii) There exist two positive constants C_1, C_2 independent of k and x_2 such that

$$\begin{aligned}
C_1 \sum_{k=1}^{\infty} \|E(\lambda_k, \mathcal{A})x_2\|^2 &\leq \left\|\sum_{k=1}^{\infty} E(\lambda_k, \mathcal{A})x_2\right\|^2 \\
&\leq C_2 \sum_{k=1}^{\infty} ||E(\lambda_k, \mathcal{A})x_2||^2.
\end{aligned}$$

Furthermore,

$$\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2}.$$

(iii) If $\sup_{k\geq 1} ||E(\lambda_k, \mathcal{A})|| < \infty$, then

$$D(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}.$$

(iv) $\mathcal{H}$ can decompose into the topological direct sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

if and only if

$$\sup_{n\geq 1} ||\sum_{k=1}^{n} E(\lambda_k, \mathcal{A})|| < \infty.$$

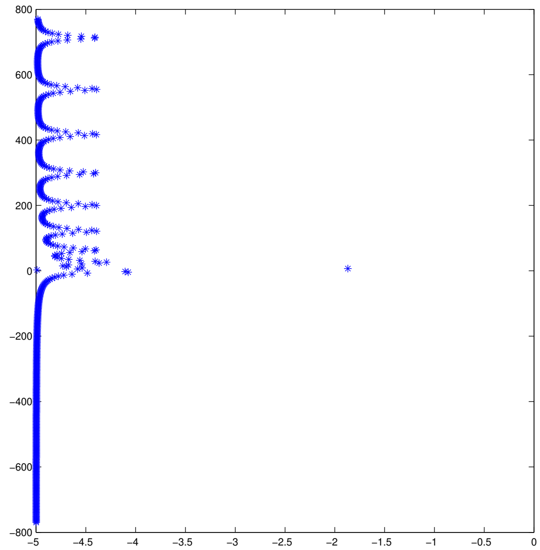

Fig. 2. Spectrum for $b = 5, k = 0.5$

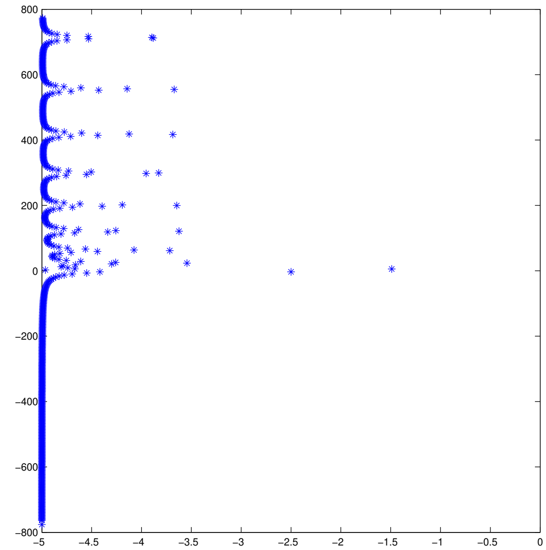

Fig. 4. Spectrum for $b = 5, k = 0.7$

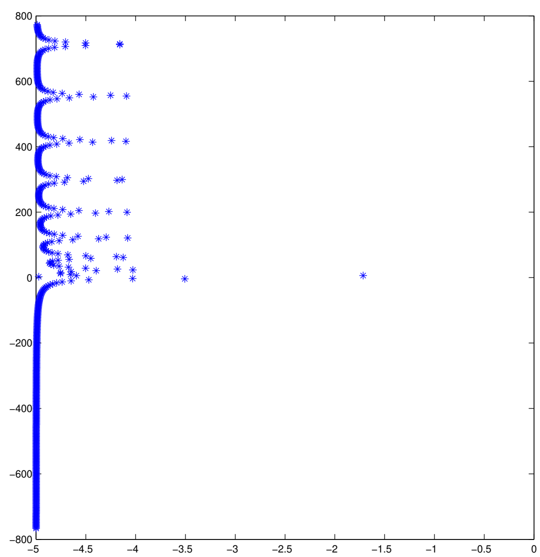

Fig. 3. Spectrum for $b = 5, k = 0.6$

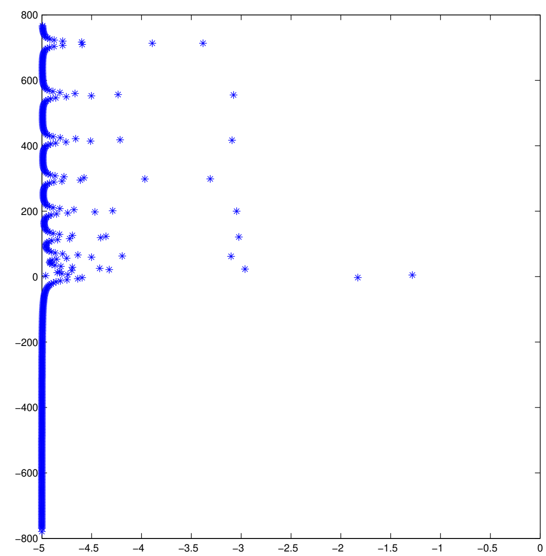

Fig. 5. Spectrum for $b = 5, k = 0.8$

Now we can get the Riesz basis property of system (9).

Theorem 4.4: System (9) is a Riesz spectral system (in the sense that its generalized eigenfunctions form a Riesz basis in $\mathcal{H}$). Thus, the spectrum determined growth condition holds, that is, $s(\mathcal{A}+\mathcal{B}) = \omega(\mathcal{A}+\mathcal{B})$, with $s(\mathcal{A}+\mathcal{B}) = \sup\{Re\lambda|\lambda \in \sigma(\mathcal{A}+\mathcal{B})\}$ being the spectral bound of $\mathcal{A}+\mathcal{B}$ and $\omega(\mathcal{A}+\mathcal{B})$ being the growth bound of the semigroup $e^{(\mathcal{A}+\mathcal{B})t}$.

Proof: We take $\sigma_2(\mathcal{A}+\mathcal{B}) = \sigma(\mathcal{A}+\mathcal{B})$, $\sigma_1(\mathcal{A}+\mathcal{B}) = \{\infty\}$, then it is easy to see that conditions (2) and (3) in Lemma 4.3 are true. Finally, Theorem 4.2 implies that $\mathcal{H}_1 = \{0\}$. Therefore, the first assertion of Lemma 4.3 says that there is a sequence of generalized eigenvectors of $\mathcal{A} + \mathcal{B}$ that forms a Riesz basis in $\mathcal{H}$. Accordingly, the spectrum determined growth condition can be obtained by a direct consequence of the Riesz basis property of $\mathcal{A} + \mathcal{B}$. ■

As a consequence of Theorem 4.4, we have the exponential stability for system (9).

Theorem 4.5: Let $b > 0$ and $k \neq 0$. Then system (9) is exponentially stable.

Proof: Theorem 4.4 ensures the spectrum-determined

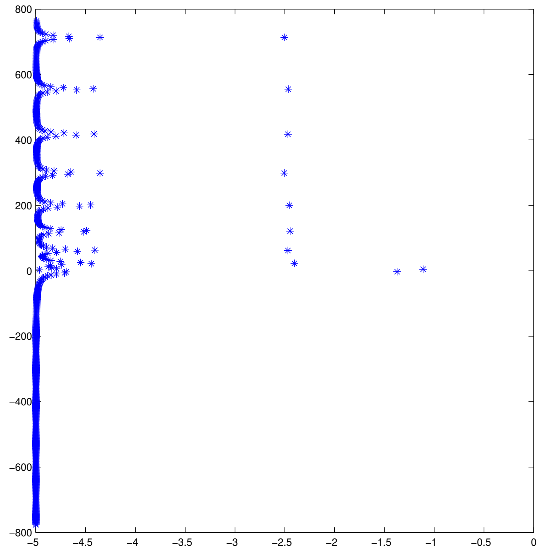

Fig. 6. Spectrum for $b = 5, k = 0.9$

growth condition $s(\mathcal{A}+\mathcal{B}) = \omega(\mathcal{A}+\mathcal{B})$, Lemma 3.2 says that $\mathrm{Re}\lambda < 0$ provided $\lambda \in \sigma(\mathcal{A}+\mathcal{B})$ and Theorem 3.3 shows that imaginary axis is not an asymptote of $\sigma(\mathcal{A}+\mathcal{B})$. Therefore $s(\mathcal{A}+\mathcal{B}) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{A}+\mathcal{B})\} < 0$ and system (9) is exponentially stable. ■

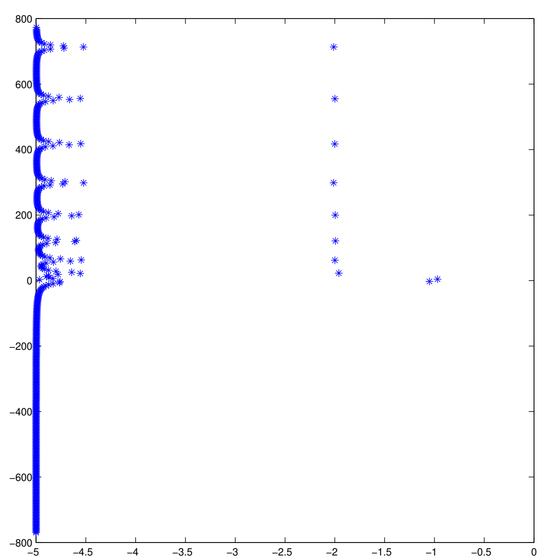

Fig. 7. Spectrum for $b = 5, k = 1.0$

V. SIMULATIONS

The Legendre spectral method [1] is adopted for system (9) to present a numerical calculations of the spectrum of

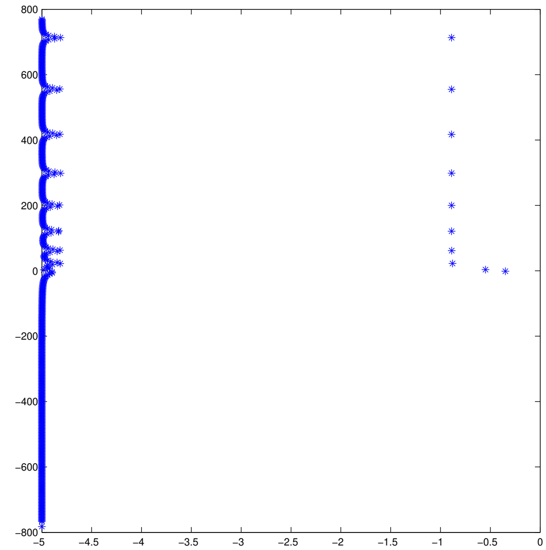

Fig. 8. Spectrum for $b = 5, k = 1.5$

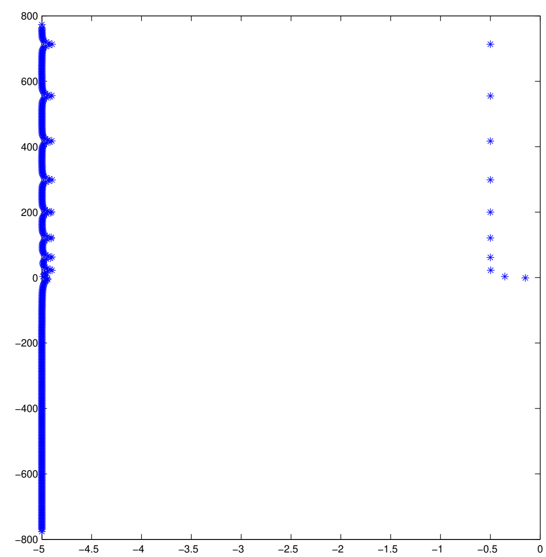

Fig. 9. Spectrum for $b = 5, k = 2.0$

the feedback gains $b = 5$ and the interconnected parameter $k = 0.5$, 0.6, 0.7, 0.8, 0.9, 1.0, 1.5 and 2.0 respectively.

From Fig. 2 - 9, it is found that all eigenvalues of system (9) are located on the left half side of the complex plane. As k from 0.5 to 2.0, a branch of spectrum goes to the imaginary axis which separates all eigenvalues into two parts: the left side is for the wave and the other right side is for the Schrödinger. From the figures, it is indicated that as $|k|$ large, the decay of the Schrödinger changes weak.

It is also nothing that if there are no interconnections between the wave and the Schrödinger, the decay rate of the wave will be -5 if we take $b = 5$. From the figures 2 - 9, it is found that the real part of the left lower part of the eigenvalues is very closer to -5, which says that this part of the wave is just a little effect on the Schrödinger.

VI. CONCLUSIONS

In this paper, the stability of a hyperbolic system of the interconnected Schrödinger and wave equations is treated. It is showed that only with the distributed dissipative velocity and displacement are forced at the wave equation, the whole system of the interconnected Schrödinger and wave equations will be exponentially stable. The numerical computation of the spectrum illustrates the correction of the stability result. Moreover, from the simulation of the spectrum, it is found that the spectrum of the Schrödinger depends largely on the interconnected transmission parameter and the decay of the wave equation.

REFERENCES

[1] G. Chen, S.G. Krantz, D.L. Russell, C.E. Wayne, H.H. West, M.P. Coleman, Analysis, designs, and behavior of dissipative joints for coupled beams, SIAM Journal on Applied Mathematics. 1989; **49(6)**: 1665-1693.

[2] I. Gohberg, Classes of Linear Operator, vol.I, Brikhäuset Verlag, Basel-Boston-Berlin, 1990.

[3] B.Z. Guo, Z.C. Shao, Regularity of a Schrödinger equation with Dirichlet control and colocated observation, Systems and Control Letters. 2005; **54**: 1135-1142.

[4] Y.P. Guo, J.M. Wang, D.X. Zhao, Stability of an interconnected Schrödinger-heat system in a torus region, Mathematical Methods in the Applied Sciences. 2016: **39(13)**: 3735-3749.

[5] M. Krstic, B.Z. Guo, A. Smyshlyaev, Boundary controllers and observers for the linearized Schrödinger equation, SIAM J. Control Optim. 2011; **49(4)**: 1479-1497.

[6] L. Lu, J.M. Wang, Transmission problem of Schrödinger and wave equation with viscous damping. Applied Mathematics Letters. 2016; **54**: 7-14.

[7] E. Machtyngier, Exact controllability for the Schrödinger equation, SIAM J. Control Optim. 1994; **32**: 24-34.

[8] E. Machtyngier, E. Zuazua, Stabilization of the Schrödinger equation, Portugaliae Mathematica. 1994; **51(2)**: 243-256.

[9] A. Pazy, Semigroups of Linear Operator and Applications to Partial Differential Equations. Springer-Verlag, New York (1967).

[10] K.-D. Phung, Observability and control of Schrödinger equation, SIAM J. Control Optim. 2001; **40**: 211-230.

[11] A. Smyshlyaev, B. Z. Guo, M. Krstic, Arbitrary decay rate for Euler-Bernoulli beam by backstepping boundary feedback, IEEE Trans. Autom. Control. 2009; **54(5)**: 1134-1140.

[12] J.M. Wang, M. Krstic, Stability of an interconnected system of Euler-Bernoulli beam and heat equation with boundary coupling, Control, Optimisation and Calculus of Variations. 2015; **21**: 1029-1052.

[13] J.M. Wang, B.B. Ren, M. Krstic, Stabilization and Gevrey regularity of a Schrödinger equation in boundary feedback with a heat equation. IEEE Transactions on automatic control. 2012; **57(1)**:179-185.

[14] J.M. Wang, G.Q. Xu, S.P. Yung, Exponential stability for variable codfficients Rayleigh beams under boundary feedback control: a Rieaz basis approach, System Control Letters. 2004; **51(1)**:33-50.

[15] J.M. Wang, S.P. Yung, Staility of a nonuniform Rayleugh beam with indefinite damping. Systems and Control Letters. 2006; **55**: 863-870.

[16] G.Q. Xu, S.P. Yung, The expansion of a semigroup and a Riesz basis criterion. Journal of Differential Equations. 2005; **210(1)**: 1-24.

[17] Q. Zhang, J.M. Wang, B.Z. Guo, Stabilization of the Euler-Bernoulli equation via boundary connection with heat equation, Mathematics of Control, Signals, and Systems. 2014; **26(1)**: 77-118.