

MONOTONE MAPS FOR MANIPULATING MATRIX INEQUALITIES

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1. EXTENDED ABSTRACT

Let $f : (a, b) \rightarrow \mathbb{R}$. Given a self-adjoint matrix A with spectrum in (a, b) diagonalized by a unitary matrix U , that is,

$$A = U^* \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U$$

we define the expression $f(A)$ via the following formula.

$$f(A) = U^* \begin{pmatrix} f(\lambda_1) & 0 & \dots \\ 0 & f(\lambda_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U.$$

When f is given by a formula, such as a polynomial or rational function, the functional calculus reduces to substituting a matrix into the formula.

For example, let

$$f(x) = x^2,$$

and let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We get that

$$f(A) = A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Let A and B be self-adjoint matrices. We say $A \leq B$ if $B - A$ is positive semi-definite. We say $A < B$ if $B - A$ is positive definite.

Given $f : (a, b) \rightarrow \mathbb{R}$, we say f is **matrix monotone** if, for any natural number $n \in \mathbb{N}$, and any pair of n by n self-adjoint matrices A and B with spectrum in (a, b) ,

$$A \leq B \Rightarrow f(A) \leq f(B).$$

The class of matrix monotone functions is strict subset of the set of monotone functions. That is, there is a function f , which is monotone, which is not matrix monotone. The difference mirrors the difference between positive maps and completely positive maps.

1.1. A monotone map which is not matrix monotone. Let $f(x) = x^3$. The function f is monotone increasing on all of $\mathbb{R}$. Note,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \leq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is positive semidefinite. However,

$$f\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}, f\left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

and $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 4 & 1 \end{pmatrix}$ is not positive semidefinite since the determinant is -7 . Thus, x^3 is not matrix monotone on all of $\mathbb{R}$.

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1.2. A matrix monotone map. Let $f(x) = -x^{-1}$. We will show f is matrix monotone on $(0, \infty)$. Let $A \leq B$ with spectrum in the positive reals. Let $H = B - A$. By the fundamental theorem of calculus

$$f(B) - f(A) = \int_0^1 Df(A + \xi H)[H] d\xi$$

where $Df(X)[H] = \frac{d}{dt} f(X + tH)|_{t=0}$. So, it is enough to show that $Df(X)[H]$ is positive semidefinite. Now,

$$\begin{aligned} Df(X)[H] &= \lim_{t \rightarrow 0} \frac{f(X + tH) - f(X)}{t} \\ &= \lim_{t \rightarrow 0} \frac{-(X + tH)^{-1} + X^{-1}}{t} \\ &= \lim_{t \rightarrow 0} (X + tH)^{-1} H X^{-1} \\ &= X^{-1} H X^{-1}. \end{aligned}$$

Since, $H = B - A \geq 0$, $H^{1/2}$ exists, and, so,

$$Df(X)[H] = X^{-1} H X^{-1} = (H^{1/2} X^{-1})^* H^{1/2} X^{-1} \geq 0.$$

So, $-x^{-1}$ is matrix monotone.

1.3. Löwner's theorem. Let $\mathbb{H}$ denote the upper half plane in $\mathbb{C}$.

Theorem 1.1 (Löwner [7]). *Let $f : (a, b) \rightarrow \mathbb{R}$. The function f is matrix monotone if and only if f analytically continues to $\mathbb{H}$ as a function $F : \mathbb{H} \cup (a, b) \rightarrow \overline{\mathbb{H}}$ which is continuous on $\mathbb{H} \cup (a, b)$.*

For example $x^{1/3}$, $\log x$ and $-\frac{1}{x}$ are matrix monotone on $(1, 2)$ but x^3 and e^x are not. Similar results were obtained by Hansen [4] Agler, McCarthy and Young [1] and the author [10] on functions $f : (a, b)^d \rightarrow \mathbb{R}$ over various functional calculi of commuting operators.

We say a function $f : (a, b)^d \rightarrow \mathbb{R}$ is **locally matrix monotone** if $Df(X)[H] \geq 0$ whenever $X = (X_1, \dots, X_d)$ is a d -tuple of commuting self-adjoint matrices (note these can be jointly diagonalized) and $H = (H_1, \dots, H_d)$ is tuple of positive semi-definite matrices such that H points into the variety of tuples of commuting self-adjoints at X .

Theorem 1.2 (Agler, McCarthy, Young [1], Pascoe [10]). *Let $f : (a, b)^2 \rightarrow \mathbb{R}$. The function f is locally matrix monotone if and only if f analytically continues to $\mathbb{H}^2$ as function $F : \mathbb{H}^2 \cup (a, b)^2 \rightarrow \overline{\mathbb{H}}$ which is continuous on $\mathbb{H}^2 \cup (a, b)^2$.*

The original work of Agler, McCarthy, Young required that f be C^1 . In more than two variables similar results hold, but the analytic continuation is in a more special class of functions from $\mathbb{H}^d$ to $\overline{\mathbb{H}}$.

2. MANIPULATING SYSTEMS OF MATRIX INEQUALITIES

Our goal now is to generalize Löwner's theorem to systems of matrix inequalities. That is, we want to have a multi-variable analogue of Löwner's theorem which gives an easy criterion to check if a function in several variables is matrix monotone on general tuples of matrices that might not commute. First, we will give two examples which have been studied thoroughly.

2.1. The Schur complement. Given a block 2 by 2 matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

we define the Schur complement to be

$$A/A_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21}.$$

Classically, it was shown that A is positive definite if and only if A is self-adjoint, A_{22} is positive definite, and A/A_{22} is positive definite.

Theorem 2.1 (Anderson [2]). *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

be positive definite block 2 by 2 matrices. If $A \leq B$, then $A/A_{22} \leq B/B_{22}$.

This says that the Schur complement is matrix monotone on positive definite block 2 by 2 matrices. This fact has been rediscovered, reimagined and applied many times, including work of Liu-Wang[9], Bhatia [3].

2.1.1. *Matrix means.* Pusz and Woronowicz [14] showed that given a pair of positive definite matrices A and B of the same size there exists a maximum positive definite matrix C such that

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} \text{ is positive-semidefinite,}$$

and, moreover, C is given by the concrete formula

$$C = A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

They called $A \# B$ the **matrix geometric mean**. Notably, $A \# B$ is equal to the ordinary geometric mean $\sqrt{AB}$ when A and B are 1 by 1 matrices.

Theorem 2.2 (Pusz and Woronowicz[14]). *Let (A_1, B_1) and (A_2, B_2) be pairs of positive definite n by n matrices. If $A_1 \leq A_2$ and $B_1 \leq B_2$, then $A_1 \# B_1 \leq A_2 \# B_2$.*

The above theorem was generalized by Kubo-Ando[8], Hansen[5], and many others. The analysis in Hansen makes heavy use of the Nevanlinna representation, and classifies many related means. Many of the current developments concern means of more than 2 matrices.

2.2. **The noncommutative Löwner theorem.** We define the **matrix universe**

$$\mathcal{M}^d = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{C})^d.$$

We endow $\mathcal{M}^d$ with the disjoint union topology. Let $\mathcal{D} \subseteq \mathcal{M}^d$. We say $f : \mathcal{D} \rightarrow \mathcal{M}^g$ is a **noncommutative function** if it is a pointwise limit of g -tuples of noncommutative free polynomials in d variables. Any tuple of noncommutative polynomials or rational functions will be a noncommutative function. The Schur complement and the matrix geometric mean give examples of noncommutative functions.

An Archimedian linear pencil is a map $L(X_1, \dots, X_n) = \sum_{i=1}^d L_i \otimes X_i$ such that there is a point $(X_1, \dots, X_n) \in \mathcal{M}^d$ such that L takes a positive definite value. Here the L_i are some fixed matrices. We say $A \leq_L B$ if $L(B - A)$ is positive semi-definite. Define the interval $\mathcal{I}_L$ by the formula

$$\mathcal{I}_L = \{X \in \mathcal{M}^d | 0 \leq_L X \leq_L 1\}.$$

A noncommutative function $f : \mathcal{I}_L \rightarrow \mathcal{M}^1$ is **matrix monotone** if

$$A \leq_L B \Rightarrow f(A) \leq f(B).$$

We define Π to be matrices in $\mathcal{M}^1$ with positive imaginary part. (That is, $\text{Im } X = (X - X^*)/2i$, positive semi-definite.) Similarly, Π_L to be the tuples of matrices Y in $\mathcal{M}^d$ such that $L(Y)$ has positive imaginary part. These are the noncommutative analogues of upper half planes.

For example, we had that the Schur complement was monotone with respect to the ordering induced by

$$L(X_{11}, X_{12}, X_{21}, X_{22}) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

For the matrix geometric mean, we had that it was monotone with respect to

$$L(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For classical one variable matrix monotonicity, we can take $L(X) = X$.

Theorem 2.3 (The noncommutative Löwner theorem. P-Tully-Doyle[12], Pascoe [13]). *Let L be an Archimedian linear pencil. Let $f : \mathcal{I}_L \rightarrow \mathcal{M}^1$ be a self-adjoint valued noncommutative function on $\mathcal{I}_L$. The function f is matrix monotone if and only if it extends to a continuous noncommutative function on $\Pi_L \cup \mathcal{I}_L$ which is analytic on Π_L and takes values in $\overline{\Pi}$.*

The results in [13] relax some additional smoothness hypotheses that on f assumed in [12]. Moreover, the results in [13] allow for block matrix outputs and more. In both cases, the results can be stated on general matrix convex sets. The current author and Tully-Doyle [12] also developed analogues of the Nevanlinna representation, for which the analogue of the “support” was described by Williams [15].

Recall the Schur complement,

$$X/X_{22} = X_{11} - X_{12}X_{22}^{-1}X_{21}.$$

We can see that the natural extension of its formula has positive imaginary part via the following formula:

$$\operatorname{Im} X/X_{22} = \left((X_{22}^*)^{-1} X_{12}^* \right)^* \left[\operatorname{Im} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \right] \left((X_{22}^*)^{-1} X_{12}^* \right)$$

which witnesses the matrix monotonicity by our theorem. Moreover, for any noncommutative rational function, one can elicit similar “algebraic certificates” using the NC rational Positivstellensatz [11], which relies on algorithms of Helton-Klep-Nelson [6].

REFERENCES

- [1] J. Agler, J.E. McCarthy, and N.J. Young. Operator monotone functions and Löwner functions of several variables. *Ann. of Math.*, 176:1783–1826, 2012.
- [2] William N. Anderson. Shorted operators. *SIAM Journal on Applied Mathematics*, 20(3):520–525, 1971.
- [3] Rajendra Bhatia and Rajeeva L. Karandikar. Monotonicity of the matrix geometric mean. *Mathematische Annalen*, 353(4):1453–1467, 2012.
- [4] F. Hansen. Operator monotone functions of several variables. *Math. Inequal. Appl.*, 6:1–17, 2003.
- [5] Frank Hansen. Means and concave products of positive semi-definite matrices. *Mathematische Annalen*, 264:119–128, 03 1983.
- [6] J.W. Helton, I. Klep, and C. Nelson. Noncommutative polynomials nonnegative on a variety intersect a convex set. *J. Funct. Anal.*, 266(12):6684–6752, 2014.
- [7] K. Löwner. Über monotone Matrixfunktionen. *Math. Z.*, 38:177–216, 1934.
- [8] Fumio Kubo and Tsuyoshi Ando. Means of positive linear operators. *Mathematische Annalen*, 246:205–224, 1979.
- [9] Jianzhou Liu and Jian Wang. Some inequalities for schur complements. *Linear Algebra and its Applications*, 293(1):233 – 241, 1999.
- [10] J E. Pascoe. *Boundary behaviour of Pick functions in several variables*. PhD thesis, 2015.
- [11] J E. Pascoe. Positivstellensätze for noncommutative rational expressions. *Proc. Amer. Math. Soc.*, 2017. to appear.
- [12] J E. Pascoe and Ryan Tully-Doyle. Free Pick functions: representations, asymptotic behavior and matrix monotonicity in several noncommuting variables. *J. Funct. Anal.*, 273:283–328, 2017.
- [13] J.E. Pascoe. The noncommutative Löwner theorem for matrix monotone functions over operator systems. *Linear Algebra and its Applications*, 541:54 – 59, 2018.
- [14] W. Pusz and S.L. Woronowicz. Functional calculus for sesquilinear forms and the purification map. *Reports on Mathematical Physics*, 8(2):159 – 170, 1975.
- [15] John D. Williams. Analytic function theory for operator-valued free probability. to appear in Crelle’s journal, 2013.