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Dynamics of heterogeneous mass chains via repeated compositions of Möbius transformations

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***Abstract*— This note studies the disturbance amplification in an interconnected heterogeneous mass chain in which one end is connected to a movable point. Each interconnection is represented by a general mechanical impedance function. The problem arises in the design of multi-storey buildings subjected to earthquake disturbances, or bidirectional control of vehicle platoons. Recurrence relations with respect to the number of masses are derived for the scalar transfer functions from the movable point displacement to a given intermass displacement. In particular, each relation takes the form of a Möbius transformation.**

I. Introduction

This note derives recurrence relations for the transfer function from a movable point displacement x_0 to a given intermass displacement $x_{i-1} - x_i$ in the heterogeneous mass chain of Fig. 1. More specifically, when a new mass is added to the chain, the transfer function in this new system is described by a function of that in the previous system. In particular, it is shown that each relation is in the form of a Möbius transformation. That is, adding new masses (and hence new interconnections) corresponds to repeatedly applying Möbius transformations.

In the case of homogeneous mass chain (identical masses and interconnections), [3] derived conditions on the interconnection for the stability and the uniform boundedness of the H_∞-norm of these transfer functions regardless of the number of masses. This means that masses can be freely added or taken out from the chain, while the stability and the H_∞-norm bound are still guaranteed. Conditions on the interconnection for tighter analytic bounds on the H_∞-norm of the first intermass displacement are given by [2]. These results are established by exploiting Möbius transformation formulations. The similar formulations provided in this note may be useful to investigate the effect of heterogeneity in the mass chain on these results.

Notation

$\mathbb{C}$ and $\mathbb{Z}^+$ denote the set of complex numbers and positive integers, respectively. The composition of two functions is denoted by $f \circ g(x) = f(g(x))$.

II. Problem Formulation

Consider a chain of n masses m_j connected by a mechanical impedance (the ratio of velocity to force) $Z_j(s)$, $j = 1, 2, \ldots, n$, as shown in Fig. 1. We assume that $m_j \neq 0$ and $Z_j(s) \not\equiv 0$, $\forall j$. Each interconnection provides an equal and opposite force on each mass and is assumed here to have negligible mass. The system is excited by a movable point $x_0(t)$ and the displacement of the ith mass from the left is denoted by $x_i(t)$, $i = 1, 2, \ldots, n$. Assume that the initial displacements of the movable point and the mass are all zero.

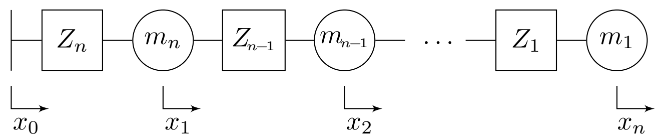

Fig. 1. Chain of n masses connected by mechanical impedances.

Note that the index of the displacements of masses start from the left while those of masses and impedances start from the right. This notation becomes natural when recurrence relations are derived. To avoid confusion, we use index i from the left in the mass chain and j from the right. Hence, $j = n - i + 1$.

The equations of motion in the Laplace transformed domain are given by

$$
\begin{aligned}
m_{n-i+1} s^2 \hat{x}_i &= s Z_{n-i+1}^{-1} \left(\hat{x}_{i-1} - \hat{x}_i\right) + s Z_{n-i}^{-1} \left(\hat{x}_{i+1} - \hat{x}_i\right) \\
&\qquad \text{for } i = 1, 2, \ldots, n-1, \\
m_1 s^2 \hat{x}_n &= s Z_1^{-1} \left(\hat{x}_{n-1} - \hat{x}_n\right)
\end{aligned}
$$

where $\hat{}$ denotes the Laplace transform. Let $h_j(s) := s Z_j(s) m_j$, $\alpha_j(s) := Z_{j+1}(s)/Z_j(s)$ and $\alpha_0 = 0$. Then we obtain

$$
\begin{aligned}
\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{pmatrix} &= \begin{pmatrix} L_{11} & -\alpha_{n-1} & & 0 \\ -1 & L_{22} & \ddots & \\ & \ddots & \ddots & -\alpha_1 \\ 0 & & -1 & L_{nn} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \hat{x}_0 \qquad (1) \\
&=: L^{-1} e_1 x_0
\end{aligned}
$$

where $L_{ii} = h_{n-i+1} + \alpha_{n-i} + 1$.

Let us consider the determinant of the matrix L, d_n. It is obvious that $d_1(s) = h_1(s) + 1$ for $n = 1$. Suppose also $d_{-1} = 1$ and $d_0 = 1$. Using the Laplace expansion, we find that

$$
d_n(s) = \left(h_n(s) + \alpha_{n-1}(s) + 1\right) d_{n-1}(s) - \alpha_{n-1}(s) d_{n-2}(s) \qquad (2)
$$

for $n \in \mathbb{Z}^+$. Since $L^{-1} = \operatorname{adj} L / \det L$, (1) can be written

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as

$$\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\\ \vdots\\ \hat{x}_n\end{pmatrix} = \frac{1}{d_n}\begin{pmatrix}d_{n-1} & * & \dots & *\\ \vdots & \vdots & & \vdots\\ d_0 & * & \dots & *\end{pmatrix}\begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix}\hat{x}_0$$
$$= \frac{1}{d_n}\begin{pmatrix}d_{n-1}\\ \vdots\\ d_0\end{pmatrix}\hat{x}_0.$$

Hence, the transfer functions from the disturbance x_0 to the ith intermass displacement $x_{i-1} - x_i$ in the heterogeneous mass chain of Fig. 1 are given by

$$F_n^{(i)}(s) := \frac{d_{n-i+1}(s) - d_{n-i}(s)}{d_n(s)} \tag{3}$$

for $i = 1, \dots, n$.

III. Intermass Displacements

A. Recurrence relations in mass chains

It is shown in this section that the transfer functions $F_n^{(i)}$ in (3) are represented as compositions of Möbius transformations. For this purpose, let us first define the following recursion with respect to i:

$$\begin{aligned} d_n^{(i)} &= (h_n + \alpha_{n-1} + 1)d_{n-1}^{(i-1)} - \alpha_{n-1}d_{n-2}^{(i-2)}\\ d_n^{(-1)} &= d_n^{(0)} = 1. \end{aligned} \tag{4}$$

Note that $d_n^{(n)}$ is equivalent to d_n defined by (2), and will be denoted as d_n in the sequel.

Theorem 1: Let $F_n^{(i)}$ be defined by (3). For any $i = 1, 2, \dots$, the sequence $(F_n^{(i)})_{n=i}^{\infty}$ satisfies the following recurrence relation:

$$F_n^{(i)}(s) = \frac{d_{n-1}^{(i-2)}(s)\alpha_{n-i}(s)F_{n-1}^{(i)}(s) + h_{n-i+1}(s)}{\prod_{k=1}^{i}\alpha_{n-k}(s)F_{n-1}^{(i)}(s) + d_n^{(i)}(s)} \tag{5}$$

where $F_{i-1}^{(i)}(s) = 0$, $\alpha_0 = 0$ $\alpha_j(s) = Z_{j+1}(s)/Z_j(s)$ $h_j(s) = sZ_j(s)m_j$ for $j = 1, 2, \dots$, and $d_n^{(i)}$ is as defined in (4).

Sketch of proof: Define

$$\begin{aligned} &P(n,i)\\ &= (d_{n-i+1} - d_{n-i})\left[\prod_{k=1}^{i}\alpha_{n-k}(d_{n-i} - d_{n-i-1}) + d_{n-1}d_n^{(i)}\right]\\ &\quad - \alpha_{n-i}d_n d_{n-1}^{(i-2)}(d_{n-i} - d_{n-i-1}) - h_{n-i+1}d_{n-1}d_n. \end{aligned}$$

From (3) and (5), we see that the theorem is equivalent to $P(n,i) = 0$ for all $i \in \mathbb{Z}^+$ and $i \le n \in \mathbb{Z}^+$. The proof will follow by induction after establishing the following facts:

1) $P(n,1) = 0$ for all $n \ge 1$.
2) $P(n,2) = 0$ for all $n \ge 2$.
3) $P(n,i) = 0$ for any $i \ge 3, n \ge i$
   if $P(n,i-1) = P(n-1,i-1) = P(n-1,i-2) = 0$,

by repeatedly using (2) and (4) . ■

Remark: It can be easily seen that the coefficient $\prod_{k=1}^{i}\alpha_{n-k}$ in the denominator equals Z_n/Z_{n-i}. Hence, (5) can also be written as

$$F_n^{(i)}(s) = \frac{d_{n-1}^{(i-2)}Z_{n-i+1}F_{n-1}^{(i)} + sZ_{n-i}Z_{n-i+1}m}{Z_nF_{n-1}^{(i)} + d_n^{(i)}Z_{n-i}}.$$

The recurrence relation (5) describes a sequence of transfer functions in the complex variable s. It can also be interpreted as compositions of Möbius transformations for a fixed $s \in \mathbb{C}$, or equivalently fixed $h_j \in \mathbb{C}$ and $\alpha_{j-1} \in \mathbb{C}$, $j = 1, \dots, n$; writing

$$f_n^{(i)}(z) = \frac{d_{n-1}^{(i-2)}\alpha_{n-i}z + h_{n-i+1}}{\prod_{k=1}^{i}\alpha_{n-k}z + d_n^{(i)}},$$

we see that the sequence $F_n^{(i)}$ for $n = i-1, i, i+1, \dots$ is the same as 0, $f_i^{(i)}(0)$, $f_{i+1}^{(i)} \circ f_i^{(i)}(0), \dots$, for given h_j and α_{j-1}, $j = 1, \dots, n$.

B. Homogeneous mass chains

For homogeneous mass chains, since $Z_j(s) = Z(s), m_j = m, \forall j$, (5) is simplified to

$$F_n^{(i)}(s) = \frac{d_{i-2}(s)F_{n-1}^{(i)}(s) + h(s)}{F_{n-1}^{(i)}(s) + d_i(s)}$$

for $n = i, i+1, \dots$, where $F_{i-1}^{(i)}(s) = 0$, $h(s) = sZ(s)m$ and d_i is as defined in (2). The following results are previously established:

Theorem 2 ([3]): For $0 \not\equiv Z(s)$ positive real, all poles in the transfer function $F_n^{(i)}(s)$ have negative real parts for any $n \in \mathbb{Z}^+$ if $sZ(s)m$ does not take values in the interval $(-4, 0)$ for any s with $\mathrm{Re}(s) = 0$.

Theorem 3 ([3]): Suppose $Z(s) = (k/s + Y_1(s))^{-1}$ where k is a positive constant and $Y_1(s)$ is a positive-real admittance satisfying $Y_1(0) > 0$. Suppose $h(j\omega) = mj\omega Z(j\omega)$ does not intersect the interval $[-4, 0)$ for any $\omega \ge 0$. Then

$$\sup_{n\ge i}\left\|F_n^{(i)}(s)\right\|_\infty$$

is finite for any $i = 1, 2, \dots$.

Theorem 4 ([2]): If

$$h(s) = \frac{a^2s^2}{(1-a)s^2 + 2s + 1}, \quad a > 0,$$

then

$$\sup_{n\in\mathbb{Z}^+}\|F_n^{(1)}(s)\|_\infty \le a.$$

C. Heterogeneity and uniform boundedness

The present formulation (5) may be useful to investigate the effect of heterogeneity on the uniform boundedness results in the homogeneous mass chains. For the first intermass displacement ($i = 1$), the recursive relation (5) becomes

$$F_n^{(1)} = \frac{\alpha_{n-1}F_{n-1}^{(1)} + h_n}{\alpha_{n-1}F_{n-1}^{(1)} + h_n + 1}.$$

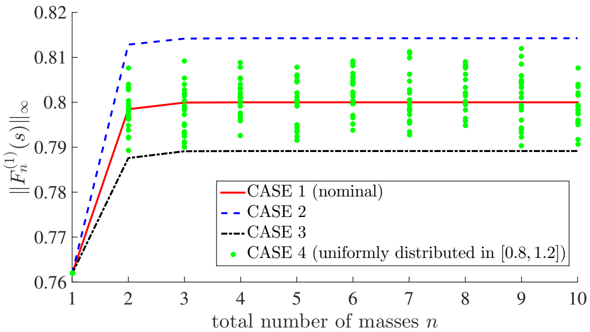

Fig. 2. H_∞-norm of the first intermass displacement transfer function in a chain of n masses, $F_n^{(1)}$, for different mass distributions.

One may consider adopting the design such that $h_n(s) = h(s)$ (i.e., $Z_n(s) = h(s)/(sm_n)$). Then

$$F_n^{(1)} = \frac{(m_{n-1}/m_n)F_{n-1}^{(1)} + h}{(m_{n-1}/m_n)F_{n-1}^{(1)} + h + 1}. \tag{6}$$

Clearly, the distribution of masses affects the norm bound. To see this, let us consider the four different mass distributions in the mass chain of Fig. 1:

CASE 1: $m_j = 1$ for $j = 1, 2, \ldots, n$ (nominal case)

CASE 2: $\begin{cases} m_j = 0.8 \text{ for } j = n, n-2, \ldots \\ m_j = 1.2 \text{ for } j = n-1, n-3, \ldots \end{cases}$

CASE 3: $\begin{cases} m_j = 1.2 \text{ for } j = n, n-2, \ldots \\ m_j = 0.8 \text{ for } j = n-1, n-3, \ldots \end{cases}$

CASE 4: $0.8 \le m_j \le 1.2$ for $j = 1, 2, \ldots, n$.

The interconnection impedances $Z_j(s)$ are selected such that

$$sZ_j(s)m_j = h(s) = \frac{0.8^2 s^2}{0.2s^2 + 2s + 1}.$$

for $j = 1, 2, \ldots, n$. This corresponds to $a = 0.8$ in Theorem 4 and indeed it is observed in Fig. 2 that the H_∞-norm $\|F_n^{(1)}\|_\infty$ is bounded by 0.8 for CASE 1, the nominal homogeneous mass chain. We may also see that the H_∞-norm is larger than the nominal case for any n in CASE 2, i.e., the mass alternates between 0.8 and 1.2 from left to right in the mass chain of Fig. 1. In CASE 3, the mass distribution pattern is flipped and the H_∞-norm is smaller than the nominal case for any n. For CASE 4, 20 different mass distributions were generated for each n so that m_j, $j = 1, 2, \ldots, n$, are uniformly distributed random numbers in the interval $[0.8, 1.2]$. It may be observed that the H_∞-norm in CASE 4 for each distribution pattern is between that in CASE 2 and that in CASE 3.

From these numerical results, it looks promising that we will be able to establish uniform boundedness in heterogeneous mass chains. Deriving explicit conditions will be considered as a future work.

D. Possible solution path

One of the great features of Möbius transformations is their close connection to linear algebra [1]; let us associate with every Möbius transformation $g(z) = (az+b)/(cz+d)$ a corresponding matrix

$$[g] := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Compositions of Möbius transformations can then be conveniently represented by multiplication of corresponding matrices:

$$[g_2][g_1] = [g_2 \circ g_1].$$

Denote the corresponding matrix to $f_n^{(i)}$ in section III-A as

$$\left[f_n^{(i)}\right] = \begin{bmatrix} d_{n-1}^{(i-2)}\alpha_{n-i} & h_{n-i+1} \\ \prod_{k=1}^{i}\alpha_{n-k} & d_n^{(i)} \end{bmatrix}$$

and define

$$\begin{bmatrix} \xi_{n+1}^{(i)} \\ \zeta_{n+1}^{(i)} \end{bmatrix} = \left[f_{n+1}^{(i)}\right] \begin{bmatrix} \xi_n^{(i)} \\ \zeta_n^{(i)} \end{bmatrix}, \quad \begin{bmatrix} \xi_i^{(i)} \\ \zeta_i^{(i)} \end{bmatrix} := \begin{bmatrix} F_i^{(i)} \\ 1 \end{bmatrix} \tag{7}$$

for $n \ge i$. This formulation was a key in [2] to develop a scale free design method in homogeneous mass chains and the similar techniques may be applied to the heterogeneous case.

IV. Conclusions

The interconnection of a chain of n masses has been studied in which neighbouring masses are connected by two-terminal mechanical impedances. One end of the chain is connected to a movable point. Formulas for the transfer functions from the movable point displacement to a given intermass displacement have been derived in the form of composition sequences generated by Möbius transformations. Effects of heterogeneity on the results provided for the homogeneous mass chains in [2], [3] have been numerically investigated.

Acknowledgements

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References

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