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Fliess Operator Representations of Markov Jump Nonlinear Systems and their Parallel Interconnections

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***Abstract*— This paper provides a Fliess operator representation for a class of continuous-time Markov jump nonlinear systems. This representation allows one to describe the parallel sum and product interconnections of such systems. The paper concludes by showing that all parallel interconnections preserves rationality when the component operators have rational generating series.**

***Index Terms*— Formal power series, Chen-Fliess series, switched nonlinear systems, Poisson processes**

***AMS Subject Classifications*—05E99, 34A34, 47H30, 60G51, 93C10**

I. Introduction

Fliess operator theory is known to be a general framework for representing and analyzing nonlinear systems [7], [8], [12], [20]. For instance, any Volterra operator with analytic kernels can be written as a Fliess operator. It also allows one to generalize the convolution algebra for linear systems interconnections to the nonlinear setting [2], [9]–[11]. In [5], the authors extended the Fliess operator notion to include a class of switched systems, where the switching signal is a Poisson process which appears in the operator definition as Poisson integrals. Consider, for example, a jump nonlinear system with two modes defined by

$$\dot{z} = f_v(z) + g_v(z)\,u,\ z(0) = z_0,\ y = h(z), \tag{1}$$

where z is an n dimensional state vector, f, g and h are analytic functions, u is a suitable external input, and v is an arbitrary switching signal taking values in $\{0,1\}$. The state equation in (1) can be re-written as

$$\begin{aligned}\dot{z} = {} & (f_0(z) + g_0(z)\,u) \\ & + (f_1(z) - f_0(z))\,v + (g_1(z) - g_0(z))\,u\,v.\end{aligned} \tag{2}$$

Assuming that $v = dN$ constitutes the increments of a Poisson process, (2) becomes a jump stochastic differential equation. This equation can then be recursively integrated so that $z(t)$ can be represented as a weighted sum of iterated Stieltjes-Poisson integrals. While this formulation is sound theoretically, some caveats are needed for (2) to represent realistic systems. For example, (2) is normally in mode $v = 0$ unless a Poisson increment occurs, in which case the system transitions to mode $v = 1$. The system remains in $v = 1$ only for the duration of dN after which it returns to mode $v = 0$ until another Poisson jump occurs. The fact that one can only stay in mode $v = 1$ an infinitesimal amount of time is not a realistic assumption for modeling a fault-tolerant control system [21]. In addition, this operator requires the somewhat artificial assumption that $u = 1$ during jumps, which avoids the creation of additional integrals due to the integration by parts formula for Poisson processes.

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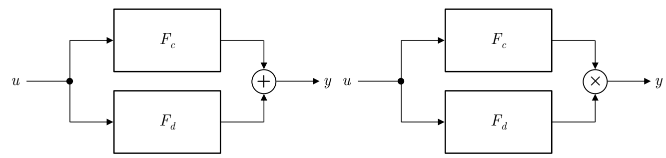

Fig. 1. Parallel sum (left) and parallel product (right) interconnections of two Fliess operators.

The first goal of this paper is to re-define the class of switched systems that can be written as Fliess operators driven by Poisson processes so that they can describe more realistic systems having a finite number of modes of operation. This includes recasting the switching signal by introducing memory and a level of correlation between the multiple modes. In particular, this signal should be capable of remembering its current state until an event managed by a Poisson process occurs. In addition, when there exists an ordering of the modes, this signal can only jump to the next consecutive mode (one step monotonically increasing) unless an event of the *nominal* Poisson process N_0 occurs. The latter causes the switching signal to return to mode 0 regardless of its current state. Adding memory implies introducing a new state, $\bar{z}$, for the switching signal. That is,

$$d\bar{z} = (1 - \bar{z})\,dN - \bar{z}\,dN_0. \tag{3}$$

So $\bar{z}$ takes the value 1 whenever a jump of the Poisson process N takes place and 0 when a jump of N_0 occurs. When there are no jumps in N and N_0, the state $\bar{z}$ maintains its current value. Thus, (3) is basically a simple birth-death Markov process. One can then set $v = \bar{z}$ in (2) and write

$$\frac{dz}{dt} = (f_0(z) + g_0(z)u)\,(1 - \bar{z}) + (f_1(z) + g_1(z)u)\,\bar{z}. \tag{4}$$

Representing (2) as in (4) removes the condition implying $u = 1$ during jumps in [5] since jumps are now in (3), and the system can stay in a particular mode for a non-negligible amount of time as established by (3).

The second goal of this paper is to characterize parallel interconnections of Fliess operators with Poisson jumps as shown in Figure 1. The parallel product interconnection requires the introduction of the quasi-shuffle product [13]. In addition, it is shown that such interconnections always preserve the rationality of generating series.

The paper is organized as follows. Section II gives preliminaries concerning Fliess operators with and without Poisson jumps as well as the basics on the property of rationality. The following section addresses the new class of Markov jump nonlinear systems and their representation as Fliess operators. In Section IV, a characterization of the generating series for parallel interconnections of Fliess operators with Poisson jumps is provided. This is followed by Section V in which it is shown that the parallel interconnections preserve the rationality of the component generating series. Conclusions are given in the final section of the paper.

II. Preliminaries

A. Fliess operators

In the classical setting, Fliess operators are described by an infinite summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series. Specifically, let $X = \{x_0, x_1, \ldots, x_m\}$ be a set of letters called an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product. The length of a word $\eta \in X^*$ is the number of letters it contains, and it is written as $|\eta|$. Also, for $x \in X$, the number of times x appears in η is denoted by $|\eta|_x$. The set of all words of length n is written as X^n. A formal power series in X is any mapping of the form $X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. A series whose support is finite is called a polynomial, and the set of all polynomial is denoted by $\mathbb{R}^\ell\langle X\rangle$. Let $\mathcal{V}^m[a,b]$ be the set of m-dimensional measurable functions on $[a,b]$. For $u \in \mathcal{V}^m[a,b]$, define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : \ 1 \le i \le m\}$, where $\|u_i\|_{L_p} = \left(\int_a^b |u_i(s)|^p \, ds\right)^{1/p}$ is the usual L_p-norm. Let $L_p^m[a,b] := \{u \in \mathcal{V}^m[a,b], \|u\|_{L_p} < \infty\}$ and define iteratively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0+T] \to \mathcal{C}[t_0, t_0+T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t,t_0) = \int_{t_0}^t u_i(\tau) E_{\eta'}[u](\tau, t_0)\, d\tau, \tag{5}$$

where $x_i \in X$, $\eta' \in X^*$, and $u_0 = 1$. For convenience assume $t_0 = 0$ and let $E_\eta[u](t,0) = E_\eta[u](t)$. The input-output operator corresponding to $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is then

$$F_c[u](t) := \sum_{\eta \in X^*} (c,\eta)\, E_\eta[u](t),$$

which is called a *Fliess operator*. (Here $(c,\eta) := c(\eta)$.) The most general results regarding the convergence of classic Fliess operators were presented in [12]. There it was shown that if the generating series c is *locally convergent*, that is,

$$|(c,\eta)| \le KM^{|\eta|}|\eta|!, \ \forall \eta \in X^*, \tag{6}$$

for some $K, M \in \mathbb{R}^+$, then $F_c[u]$ converges uniformly and absolutely on $[0,T]$ for $u \in B_p^m(R)[0,T] := \{u \in L_p^m[0,T] : \|u\|_{L_p} \le R\}$ provided that T and R are sufficiently small. The set of all locally convergent series is denoted as $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. More recently in [4], it was shown that the notion of a Fliess operator can be generalized to admit a class of L_2-Itô stochastic processes. Specifically, such operators were defined as infinite summations of iterated Lebesgue and Stratonovich integrals, and conditions for their absolute mean square convergence were given.

B. Fliess operators with Poisson jumps

Fliess operators driven by Poisson processes were introduced in [5]. In this setting, it is assumed that there is an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\ge 0}$ is a filtration of Ω, $\mathcal{F}_0$ contains all the P-null subsets of $\mathcal{F}$, and $\mathbf{F}$ is right continuous. A Poisson process with intensity λ is defined as $N(t) := \sum_{i\ge 1} \mathbb{1}_{\{t \ge \tau_i\}}$, for $t \ge 0$, taking values in $\mathbb{N}$, where $\{\tau_i\}_{i\ge 0}$ is an increasing sequence of $\mathcal{F}_t$-stopping times with exponentially distributed inter-arrival times having parameter λ, and $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function. In particular, N is a *predictable* process. A process is called predictable on the probability space $([0,T]\times\Omega, \mathcal{P}, \mu\otimes P)$ under the measure $\mu \otimes P$ with μ being the Lebesgue measure if it is $\mathcal{P}$-measurable. $\mathcal{P}$ is known as the *predictable σ-algebra*, i.e., the σ-algebra generated by sets of the form $(s,t] \times F \subset [0,T] \times \Omega$ with $F \in \mathcal{F}_s$ for $0 \le s < t \le T$ and $\{0\} \times F'$ with $F' \in \mathcal{F}_0$. The integral of a predictable process H with respect to a Poisson process is defined as

$$\begin{aligned} J_N(H)_t &= \int_0^t H(s-)\, dN(s) \\ &= \sum_{k=1}^{N(t)} H(t-)[N(\tau_k \wedge t) - N(\tau_{k-1} \wedge t)], \end{aligned}$$

where $\tau \wedge t := \min\{\tau, t\}$, and $\{H(s-)\}_{s\ge 0}$ denotes the left-continuous version of a process $\{H(s)\}_{s\ge 0}$. Any left-continuous process is predicable. For a predictable process u, define $\|u\|_{L_p} = \max\{\|u_i\|_{L_p} : \ 1 \le i \le m\}$, where $\|u_i\|_{L_p} := \left(\mathbf{E}[\int_a^b |u_i(s)|^p \, ds]\right)^{1/p}$. The set of m-dimensional predictable processes having finite L_p-norm over $[a,b]$ is denoted as $L_p^m([a,b]\times\Omega, \mathcal{P}, \mu\otimes P)$, which in abbreviated form reads as $L_p^m[a,b]$. A ball in this space is denoted by $B_p^m(R)[0,T] := \{u \in L_p^m[a,b] : \|u\|_{L_p} \le R\}$. The L_p-norm for a random variable Z is taken to be $\|Z\|_p := (\mathbf{E}[Z^p])^{1/p}$. A more detailed description of the stochastic setting can be found in [16] and the references therein. In addition to the alphabet X introduced in Section II-A, consider $Y = \{y_1, y_2, \ldots, y_k\}$ and $XY = X \cup Y$. For each $\eta = q_i\eta' \in XY^*$ with $q_i \in XY$, define a Poisson-Lebesgue iterated integral E_η by first setting $E_\emptyset = 1$, then for $q_i \in X$ the iterated integral follows from (5) and for $q_i \in Y$ one has that

$$E_{y_i\eta'}[u](t) := \int_0^t E_{\eta'}[u](s-)\, dN_i(s), \tag{7}$$

where $\eta' \in XY^*$ and $u \in B_p^m(R)[0,T]$. The iterated integrals defined in (5) and (7) can be extended linearly to a polynomial $r \in \mathbb{R}\langle XY\rangle$ as

$$E_r[u](t) = \sum_{\eta \in \mathrm{supp}(r)} (r,\eta) E_\eta[u](t),$$

where $\mathrm{supp}(r) := \{\eta \in XY^* : (r,\eta) \neq 0\}$. A Fliess operator with Poisson jumps is defined as follows.

Definition 1: [5] A causal m-input, ℓ-output *Fliess operator* F_c, $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$, driven by $u \in B_p^m(R)[0,T]$ and k Poisson processes is formally defined as

$$F_c[u](t) = \sum_{\eta \in XY^*} (c,\eta)\, E_\eta[u](t), \tag{8}$$

where each E_η is given by (5) and (7).

Convergence conditions for these operators are given in the next theorem.

Theorem 1: [5] Suppose $c \in \mathbb{R}^\ell_{LC}\langle\langle XY\rangle\rangle$. Then there exist $R, T > 0$ such that for each predictable process $u \in B_p^m(R)[0,T]$ and Poisson processes $\{N_0, N_1, \ldots, N_k\}$, the series (8) converges uniformly and absolutely in the mean on $[0,T]$.

Naturally, when there is no switching signal, this setup reduces to the classical Fliess operator setting.

C. Rational formal power series

Some standard concepts regarding rational formal power series are provided next. Let Q be an arbitrary but finite alphabet. A series $c \in \mathbb{R}\langle\langle Q\rangle\rangle$ is called *invertible* if there exists a series $c^{-1} \in \mathbb{R}\langle\langle Q\rangle\rangle$ such that $cc^{-1} = c^{-1}c = 1$. In the event that c is not proper, it is always possible to write

$$c = (c,\emptyset)(1 - c'),$$

where $(c,\emptyset)$ is nonzero, and $c' \in \mathbb{R}\langle\langle Q\rangle\rangle$ is proper. It then follows that

$$c^{-1} = \frac{1}{(c,\emptyset)}(1-c')^{-1} = \frac{1}{(c,\emptyset)}(c')^*,$$

where $(c')^* := \sum_{i=0}^{\infty}(c')^i$. In fact, c is invertible if and *only if* c is not proper. Now let S be a subalgebra of the $\mathbb{R}$-algebra $\mathbb{R}\langle\langle Q\rangle\rangle$ under the catenation product. S is said to be *rationally closed* when every invertible $c \in S$ has $c^{-1} \in S$ (or equivalently, every proper $c' \in S$ has $(c')^* \in S$). The *rational closure* of any subset $E \subset \mathbb{R}\langle\langle Q\rangle\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle Q\rangle\rangle$ containing E.

Definition 2: [1] A series $c \in \mathbb{R}\langle\langle Q\rangle\rangle$ is *rational* if it belongs to the rational closure of $\mathbb{R}\langle Q\rangle$. The subset of all rational series in $\mathbb{R}\langle\langle Q\rangle\rangle$ is denoted by $\mathbb{R}_{rat}\langle\langle Q\rangle\rangle$.

From Definition 2 it is clear that every series in $\mathbb{R}_{rat}\langle\langle Q\rangle\rangle$ is generated from polynomials in Q in which a finite number of rational operations have been applied (scalar multiplication, addition, catenation, and inversion). In the case where Q contains an infinite number of letters, a polynomial is still written in terms of a finite subset of Q and, therefore, after applying a finite number of rational operations to a set of polynomials the resulting series must be rational in the sense of Definition 2. There exist other notions of rationality for infinite alphabets [15], but this paper is only concerned with the notion of rationality that is based on polynomials and the underlying finite alphabet generating their support.

It turns out that an entirely different characterization of a rational series is possible using a monoid structure on the set of $n \times n$ matrices over $\mathbb{R}$, denoted by $\mathbb{R}^{n\times n}$, where the product is conventional matrix multiplication and the unit is the identity matrix I.

Definition 3: [1] A *linear representation* of a series $c \in \mathbb{R}\langle\langle Q\rangle\rangle$ is any triple (μ,γ,λ), where $\mu : X^* \to \mathbb{R}^{n\times n}$ is a monoid morphism, and $\gamma, \lambda^T \in \mathbb{R}^{n\times 1}$ are such that

$$(c,\eta) = \lambda\mu(\eta)\gamma, \quad \forall \eta \in Q^*.$$

The integer n is the dimension of the representation.

Definition 4: [1] A series $c \in \mathbb{R}\langle\langle Q\rangle\rangle$ is called *recognizable* if it has a linear representation.

Theorem 2: [19] A formal power series is rational if and only if it is recognizable.

A consequence of a series being recognizable is that it is also local convergent. That is, $|(c,\eta)| \le |\lambda||\mu(\eta)||\gamma| \le KM^n$, where $\eta = x_{i_1}\cdots x_{i_n}$, $K = |\lambda||\gamma|$ and $M = \max\{|\mu(x_0)|, |\mu(x_1)|, \ldots, |\mu(x_m)|\}$ [12]. Therefore, a straightforward consequence of Theorem 2 is that $\mathbb{R}_{rat}\langle\langle X\rangle\rangle \subset \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$.

A third characterization of rationality is also possible. Define for any $q_i \in Q$ and $\eta = q_j\eta' \in Q^*$ the left-shift operator

$$q_i^{-1}(\eta) = \delta_{ij}\eta', \tag{9}$$

where δ_{ij} is the Kronecker delta. Higher order shifts are defined inductively via $(q_i\xi)^{-1}(\cdot) = \xi^{-1}q_i^{-1}(\cdot)$, where $\xi \in Q^*$. The left-shift operator is assumed to act linearly on $\mathbb{R}\langle\langle Q\rangle\rangle$.

Definition 5: [1] A subset $V \subset \mathbb{R}\langle\langle Q\rangle\rangle$ is called *stable* when $\xi^{-1}(c) \in V$ for all $c \in V$ and $\xi \in Q^*$.

The next theorem characterizes rational series in terms of stable vector spaces.

Theorem 3: [1] A series $c \in \mathbb{R}\langle\langle Q\rangle\rangle$ is rational if and only if there exists a stable finite dimensional $\mathbb{R}$-vector subspace of $\mathbb{R}\langle\langle Q\rangle\rangle$ containing c.

III. Markov Jump Nonlinear Systems Represented by Fliess Operators

Consider a switching signal, as given in (3), which includes more "modes":

$$\begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_k \end{pmatrix} = \begin{pmatrix} (\bar{z}_0 - \bar{z}_1)dN_1 - \bar{z}_1\, dN_0 \\ \vdots \\ (\bar{z}_{k-1} - \bar{z}_k)\, dN_k - \bar{z}_k\, dN_0 \end{pmatrix}, \tag{10}$$

where $\bar{z}_0 = 1$ for all time. This switching signal has been defined so that mode i cannot be entered unless the system is already in mode $i-1$. Observe that the probability of two independent Poisson processes jumping at the same time instant is zero. Moreover, the event inter-arrival times are exponentially distributed for Poisson processes, which means that the system will remain in a particular mode a non-negligible amount of time. Figure 2 illustrates (10) for $k=4$ and $v = \sum_{i=1}^{k}\bar{z}_i$ for some Poisson processes with arbitrary intensities.

It is now straightforward to extend (4) to the case with multiple modes as in the next definition.

Definition 6: A Markov jump nonlinear system with $k+1$ modes driven by $u \in B_p^m(R)[0,T]$ and switched signal $v = \sum_{i=1}^{k}\bar{z}_i$ with $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_k)^\top$ as in (10) is given by

$$d\tilde{z} = \sum_{i=0}^{k}\left(f_i(\tilde{z}) - \sum_{j=1}^{m} g_{ij}(\tilde{z})\,u_j\right)(\bar{z}_i - \bar{z}_{i+1})\,dt, \;\; \tilde{z}(0) = \tilde{z}_0, \tag{11}$$
$$y = h(z),$$

where $\tilde{z}$ is an n dimensional state vector defined on some neighborhood $W \subseteq \mathbb{R}^n$ containing $\tilde{z}_0$, $\bar{z}_{k+1} = 0$ for all times, and f_i, g_{ij} and h are real analytic functions on W for $i = 1,2,\ldots,m$ and $j = 0,1,\ldots,k$.

The next theorem states that (11) can be represented as a Fliess operator with Poisson jumps.

Theorem 4: A Markov jump nonlinear system with $k+1$ modes as in (11) driven by a predictable $u \in B_p^m(R)[0,T]$ and a Poisson switching signal of k-types as defined in (10) can always be written uniquely in the form of (8) for some $c \in \mathbb{R}^{\ell}_{LC}\langle\langle XY\rangle\rangle$.

Proof: Only a sketch of the proof is presented here since it follows the same basic steps used in proving [5, Theorem 6]. Defining the state vector $z = (\tilde{z}^\top, \bar{z}^\top)^\top$, one has that (11) can be written as

$$dz(t) = \sum_{i=0}^{k}\left(A_i(z) + \sum_{j=1}^{m} B_{ij}(z)u_j\right)dt + \sum_{i=0}^{k} D_i(z)\,dN_i(t), \tag{12}$$

where $A_i(z) = (f_i(\tilde{z})^\top(\bar{z}_i - \bar{z}_{i+1}), 0, \ldots, 0)^\top$, $B_{ij}(z) = (g_{ij}(\tilde{z})^\top(\bar{z}_i - \bar{z}_{i+1}), 0, \ldots, 0)^\top$, $D_0(z) = (0, -\bar{z}_1, \ldots, -\bar{z}_k)^\top$, and $D_i(z) = (0, \ldots, (\bar{z}_{i-1} - \bar{z}_i), \ldots, 0)^\top$ with the only non zero component in the $i+1$ position and $\bar{z}_0 = 1$. Thus, without loss of generality, assume $k = 0$ and drop the indexes so that integrating (12) gives

$$z(t) = \int_0^t A(z)\,ds + \int_0^t B(z)\,u(s)\,ds + \int_0^t D(z(s-))\,dN(s). \tag{13}$$

From the properties of the Poison integral, $z(t) = z(t-) + D(z(t-))$ when $\Delta N(t) \neq 0$ and 0 otherwise. Given a differentiable function, F, the change of variables formula for Poisson integrals produces

$$F(z(t)) - F(z(0)) = \int_0^t \left(A(z(s))\,\frac{\partial}{\partial z}F(z(s))\right) ds + \int_0^t \left(B(z(s))\,\frac{\partial}{\partial z}F(z(s))\right) u(s)\,ds$$

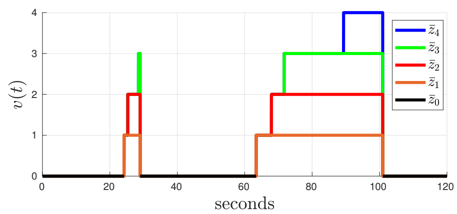

Fig. 2. Realization of a switching signal $v = \sum_{i=1}^{k} \bar{z}_i$ for $k = 4$.

$$+\int_0^t (F(z(s)) - F(z(s-)))\, dN(s). \qquad (14)$$

Identify the operators $L_A F(z) := \frac{\partial F(z)}{\partial z} A(z)$, $L_B F(z) := \frac{\partial F(z)}{\partial z} B(z)$ and $\Delta_D F(z) := F(z + D(z)) - F(z)$. Now, let $F(z)$ in (14) be replaced by either $A(z)$, $B(z)$ or $D(z)$, and then substitute for $A(z)$, $B(z)$ and $D(z)$ into (13). This yields

$$\begin{aligned} z(t) =& z(0) + A(z(0)) \int_0^t ds + B(z(0)) \int_0^t u(s)\, ds \\ &+ D(z(0)) \int_0^t dN(s) + R_1(z(t)), \end{aligned}$$

where $R_1(z(t))$ contains all the iterated integrals of order 2 whose integrands do not depend on $z(0)$. In light of (5) and (7), define $X = \{x_0, x_1\}$, $Y = \{y_1\}$. An operator L_η on the output function h is defined for any $\eta = q_{i_k} \cdots q_{i_1} \in XY^*$ as

$$L_\eta h = L_{q_{i_1}} \cdots L_{q_{i_k}} h, \qquad (15)$$

where

$$L_{q_i} h(z) = \begin{cases} \frac{\partial h}{\partial z}(z) A(z) & : \quad q_i = x_0 \\ \frac{\partial h}{\partial z}(z) B(z) & : \quad q_i = x_1 \\ \Delta_D h(z) & : \quad q_i = y_1 \end{cases}$$

with $\Delta_{D_i} h(z) : h(z + D_i(z)) - h(z)$, and $L_\emptyset h = h$. Identifying $L_{x_0} = L_A$, $L_{x_1} = L_B$ and $L_{y_1} = \Delta_D$, repeating this procedure iteratively, and using (15) yields

$$z(t) = F_{c_z}[u](t) = \sum_{\eta \in XY^*} L_\eta \varphi(z(0))\, E_\eta[u](t), \qquad (16)$$

where φ denotes the identity map. Thus, (A, B, D, φ, z_0) realizes the operator F_{c_z} driven by u and a Poisson process N, where $(c_z, \eta) = L_\eta \varphi(z(0))$, $\forall \eta \in XY^*$. It was proved in [5, Lemma 24] that the coefficients of c_z satisfy (6), and therefore, as a result of Theorem 1, series (16) is convergent in the mean for all $t \in [0, T]$. In addition, a slight extension of Fliess' Fundamental Lemma (see [5, Lemma 25]) gives

$$h(z) = \sum_{\eta \in XY^*} L_\eta (h \circ \varphi)(z(0)) E_\eta[u](t),$$

when z is written as in (16). Thus, for a real analytic function h, one has that

$$h(z(t)) = F_c[u](t) = \sum_{\eta \in XY^*} L_\eta h(z(0))\, E_\eta[u](t),$$

where $c \in \mathbb{R}\langle\langle XY \rangle\rangle$ and $(c, \eta) = L_\eta h(z(0))$. Finally, from [5, Theorem 17], the generating series c is unique, which completes the proof. ■

IV. Parallel Interconnections and the Quasi-Shuffle Algebra

The parallel interconnection of two Fliess operators is formed componentwise by the addition or multiplication of their outputs as shown in Figure 1. In the classical setting, given F_c and F_d with $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X \rangle\rangle$, these parallel interconnections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup d}$ [7], where the operation $\sqcup\!\sqcup$ is the *shuffle product* [17]. The shuffle of two words in X^* is defined inductively as

$$(x_i \eta) \sqcup\!\sqcup (x_j \xi) = x_i(\eta \sqcup\!\sqcup (x_j \xi)) + x_j((x_i \eta) \sqcup\!\sqcup \xi) \qquad (17)$$

with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ given any $\eta, \xi \in X^*$ and $x_i, x_j \in X$ [7], [18]. The definition is extended linearly to any two series $c, d \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ by letting

$$c \sqcup\!\sqcup d = \sum_{\eta, \xi \in X^*} (c, \eta)(d, \xi)\, \eta \sqcup\!\sqcup \xi, \qquad (18)$$

where again the product on $\mathbb{R}^\ell$ is defined componentwise. For a fixed $\nu \in X^*$, the coefficient $(\eta \sqcup\!\sqcup \xi, \nu) = 0$ if $|\eta| + |\xi| \neq |\nu|$. Hence, the infinite sum in (18) is always well defined since the family of polynomials $\{\eta \sqcup\!\sqcup \xi\}_{\eta, \xi \in X^*}$ is locally finite [1, Section 3 of Chapter I]. It is well-known that addition and the shuffle product preserve local convergence [20]. The unit for this product is $\mathbf{1} := 1\emptyset$.

A. Parallel sum interconnection

In the case of Fiess operators with Poisson jumps, the parallel sum interconnection is characterized by the following theorem.

Theorem 5: Let $c, d \in \mathbb{R}^\ell_{LC}\langle\langle XY \rangle\rangle$, then $F_c + F_d = F_{c+d}$ with $c + d \in \mathbb{R}^\ell_{LC}\langle\langle XY \rangle\rangle$.

Proof: Observe

$$\begin{aligned} F_c[u](t) + F_d[u](t) &= \sum_{\eta \in XY^*} [(c, \eta) + (d, \eta)] E_\eta[u](t) \\ &= F_{c+d}[u](t). \end{aligned}$$

It is trivial to show that $c + d$ must be local convergent. ■

B. Parallel product interconnection

The parallel product interconnection is not as straightforward as the parallel sum interconnection as it involves the so-called *quasi-shuffle* product. The first objective is to give a brief description of the quasi-shuffle product. Then, this product is used to describe the generating series of the parallel product interconnection.

The main reason why (17) appears for the parallel product interconnection in the standard case is that it codifies the product of Stieltjes iterated integrals using the standard integration by parts formula. However, it cannot account for products of iterated integrals that include Poisson integrals. For instance, observe that from the integration by parts formula for Poisson integrals,

$$\begin{aligned} &\int_0^t u_1(s-)\, dN_1(s) \int_0^t u_2(s-)\, dN_2(s) = \\ &\quad \int_0^t u_1(s-) \int_0^s u_2(\tau-)\, dN_1(\tau) dN_2(s) \\ &\quad + \int_0^t u_2(s-) \int_0^s u_1(\tau-)\, dN_2(\tau) dN_1(s) \\ &\quad + \delta_{12} \int_0^t u_1(s-) u_2(s-)\, \langle dN_1, dN_2 \rangle_s, \end{aligned} \qquad (19)$$

where u_1 and u_2 are suitable predictable Poisson integrable processes, $\langle \cdot, \cdot \rangle_s$ denotes the quadratic variation process, and δ_{12} is the usual Kronecker delta [16]. Using $Y = \{y_0, y_1, y_2\}$, it then follows that (19) can be written as

$$\begin{aligned} E_{y_1}[u](t) E_{y_2}[u](t) &= E_{y_1 y_2}[u](t) + E_{y_2 y_1}[u](t) \\ &\quad + \delta_{12} \int_0^t u_1(s-) u_2(s-)\, \langle dN_1, dN_2 \rangle_s. \end{aligned} \qquad (20)$$

The last term in the summand on the right-hand side cannot be described using only the alphabet Y. In fact, the general setting in which products of iterated Poisson integrals are considered requires an infinite alphabet. The extra letters, in addition to those in Y, account for all possible finite products of inputs. Letting $\bar{Y} = Y \cup \{y_{1,2}\}$, it follows that the product $E_{y_1}[u](t)E_{y_2}[u](t)$ is encoded symbolically in terms of a shuffle-like product on $\bar{Y}^*$

$$y_1 \circledast y_2 = y_1y_2 + y_2y_1 + y_{1,2} \in \mathbb{R}\langle \bar{Y}\rangle,$$

where $y_{1,2}$ is associated with the input product u_1u_2 in the integral with respect to the quadratic variation process $\langle N_1, N_2\rangle$ in (19).

In a more general context, define the commutative bracket operation on letters in an arbitrary alphabet $Q = \{q_0, q_1, \ldots, q_m\}$ to be $[q_iq_j] = q_{i,j}$, where $q_{i,j}$ is a letter not necessarily in Q. This operation is assumed to be associative, i.e., $[[q_iq_j]q_l] = [q_i[q_jq_l]]$ for $q_i, q_j, q_l \in Q$. Iterated brackets may therefore be denoted by $q_{i_1,\ldots,i_n} := [[[q_{i_1}q_{i_2}]\cdots]q_{i_n}]$. The augmented alphabet $\bar{Q}$ contains Q as well as all finitely iterated brackets $Q_{i_1,\ldots,i_n}$. The monoid of words with letters from $\bar{Q}$ is denoted $\bar{Q}^*$. The quasi-shuffle product $\circledast$ on $\bar{Q}^*$ can then be defined recursively for words $\eta = \eta_1\cdots\eta_n$ and $\xi = \xi_1\cdots\xi_m$, where $\eta_i, \xi_j \in \bar{Q}$, as

$$\begin{aligned}\eta \circledast \xi = {} & \eta_1(\eta_1^{-1}(\eta) \circledast \xi) + \xi_1(\eta \circledast \xi_1^{-1}(\xi)) \\ & + [\eta_1\xi_1]\big(\eta_1^{-1}(\eta) \circledast \xi_1^{-1}(\xi)\big).\end{aligned} \tag{21}$$

with $\emptyset \circledast \eta = \eta \circledast \emptyset = \eta$ for $\eta \in \bar{Q}^*$ and $\eta_1^{-1}(\cdot)$ is the left-shift operator defined in (9). This implies that

$$E_\eta[u](t) \cdot E_\xi[u](t) = E_{\eta\circledast\xi}[u](t), \tag{22}$$

with $\eta \circledast \xi \in \mathbb{R}\langle \bar{Q}\rangle$. Observe that since $|\eta|, |\xi| < \infty$, then $\mathrm{supp}(\eta \circledast \xi)$ is generated by a finite subset of $\bar{Q}$. The quasi-shuffle product $\circledast$ is linearly extended to series $c, d \in \mathbb{R}\langle\langle \bar{Q}\rangle\rangle$ so that

$$\begin{aligned}c \circledast d &= \sum_{\eta,\xi\in\bar{Q}^*} (c,\eta)(d,\xi)\eta \circledast \xi \\ &= \sum_{\nu\in\bar{Q}^*} \underbrace{\sum_{\eta,\xi\in\bar{Q}^*} (c,\eta)(d,\xi)(\eta \circledast \xi, \nu)}_{(c\circledast d,\nu)}\,\nu.\end{aligned} \tag{23}$$

Note that the coefficient $(\eta \circledast \xi, \nu) \neq 0$ only when $\nu \in Q^*$ is such that $|\eta| + |\xi| - \min(|\eta|, |\xi|) \leq |\nu| \leq |\eta| + |\xi|$. Therefore, $(c \circledast d, \nu)$ is finite since the set $I_\circledast(\nu) := \{(\eta, \xi) \in \bar{Q}^* \times \bar{Q}^* : (\eta \circledast \xi, \nu) \neq 0\}$ is finite. Hence, the property of local finiteness holds for $c \circledast d$, and thus the resulting series is also summable. It can be shown that the quasi-shuffle product is commutative, associative and distributes over addition [6], [13]. Therefore, the vector space $\mathbb{R}\langle\langle \bar{Q}\rangle\rangle$ endowed with the quasi-shuffle product forms a commutative $\mathbb{R}$-algebra, the so-called *quasi-shuffle algebra* with multiplicative identity element $\mathbf{1} := 1\emptyset$. In particular, if $Q = XY$, then the product $\circledast$ acquires some extra properties in terms of the bracket operation. These are:

i. $[x_ix_j] = 0$, $[x_iy_j] = 0$, $[y_iy_j] = \delta_{ij}y_{i,i}$
ii. $[[[y_{i_1}y_{i_2}]\cdots]y_{i_n}]] = \delta_{i_1i_2}\cdots\delta_{i_{n-1}i_n}y_{i_1,\ldots,i_1}$.
iii If the Poisson iterated integrals come from the series solution of (11), then $y_{i,\ldots,i} = y_i$.

Equation (20) can now be written as

$$E_{y_1}[u](t)E_{y_2}[u](t) = E_{y_1y_2}[u](t) + E_{y_2y_1}[u](t) + \delta_{12}E_{y_{1,1}}[u](t)$$

since $\langle N_i, N_j\rangle_t = \delta_{ij}N_i(t)$.

The next theorem shows that the quasi-shuffle product preserves local convergence.

Theorem 6: If $c, d \in \mathbb{R}^\ell_{LC}\langle\langle Q\rangle\rangle$, then $c \circledast d \in \mathbb{R}^\ell_{LC}\langle\langle \bar{Q}\rangle\rangle$.

Proof: The proof is done by computing a bound for the coefficient $(c \circledast d, \nu)$, where $\nu \in \bar{Q}^*$. This requires one to first show that for any $\nu \in \bar{Q}^n$

$$\sum_{|\eta|=l,|\xi|=n-l} (\eta \circledast \xi, \nu) \leq \binom{n}{l}. \tag{24}$$

It is straightforward from (21) that the left-shift operator acts on the quasi-shuffle product as

$$\begin{aligned}q^{-1}(\eta \circledast \xi) = {} & q^{-1}(\eta) \circledast \xi + \eta \circledast q^{-1}(\xi) \\ & + \delta_{q,[q_iq_j]}(q_i^{-1}(\eta) \circledast q_j^{-1}(\xi)),\end{aligned} \tag{25}$$

where $q \in Q$, $\eta = q_i\eta', \xi = q_j\xi' \in \bar{Q}^*$ and $\delta_{q,[q_iq_j]} = 1$ if $q = [q_iq_j]$ and 0 otherwise. Next, use induction over $|\eta| + |\xi| = n$ to prove (24). The case for $|\eta| + |\xi| = 0, 1$ follows directly. Assume now that (24) holds up to $|\eta| + |\xi| = n - 1$ and let $\nu = q\bar{\nu} \in Q^n$ with $q \in Q$. It follows from (25) and Pascal's rule that

$$\begin{aligned}\sum_{|\eta|=l,|\xi|=n-l} (\eta \circledast \xi, \nu) &= \sum_{|\eta|=l,|\xi|=n-l} (q^{-1}(\eta \circledast \xi), \bar{\nu}) \\ &= \sum_{|\eta|=l,|\xi|=n-l} (q^{-1}(\eta) \circledast \xi, \bar{\nu}) + \sum_{|\eta|=l,|\xi|=n-l} (\eta \circledast q^{-1}(\xi), \bar{\nu}) \\ &\quad + \sum_{\substack{|\eta_1\eta|=l,|\xi_1\xi|=n-l\\ \eta_1,\xi_1\in Q}} \delta_{q,[\eta_1\xi_1]}\underbrace{(\eta_1^{-1}(\eta) \circledast \xi_1^{-1}(\xi), \bar{\nu})}_{=0} \\ &\leq \sum_{|\eta|=l-1,|\xi|=n-l} (\eta \circledast \xi, \bar{\nu}) + \sum_{|\eta|=l,|\xi|=n-l-1} (\eta \circledast \xi, \bar{\nu}) \\ &\leq \binom{n-1}{l-1} + \binom{n-1}{l} = \binom{n}{l}.\end{aligned}$$

A key observation before computing a bound for $(c \circledast d, \nu)$ is that words of length n can be produced by the quasi-shuffling of words $\eta, \xi \in \bar{Q}^*$ where $|\eta| + |\xi| > n$. However, if $|\eta| + |\xi| > 2n$ then $\zeta \in \mathrm{supp}(\eta \circledast \xi)$ satisfies $|\zeta| > n$. Now let the local growth constants of c and d be $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. From (23) and defining $\bar{n} = 2n$, it follows that

$$\begin{aligned}|(c \circledast d, \nu)| &\leq \sum_{\eta,\xi\in\bar{Q}^*} |(c,\eta)||(d,\xi)||(\eta \circledast \xi, \nu)| \\ &= \sum_{l=0}^{\bar{n}} \sum_{\eta=l,\xi=\bar{n}-l} |(c,\eta)||(d,\xi)||(\eta \circledast \xi, \nu)| \\ &\leq \sum_{l=0}^{\bar{n}} \sum_{\eta=l,\xi=\bar{n}-l} K_cM_c^lK_dM_d^{\bar{n}-l}l!(\bar{n}-l)!\,|(\eta \circledast \xi, \nu)|.\end{aligned}$$

Define $K_1 = \max\{K_c, K_d\}$ and $M_1 = \max\{M_c, M_d\}$. Since $l!(n-l)! \leq n!$, it follows for any $\nu \in \bar{Q}^*$ that

$$\begin{aligned}|(c \circledast d, \nu)| &\leq K_1^2M_1^nn!\sum_{l=0}^{\bar{n}} \sum_{\eta=l,\xi=\bar{n}-l} |(\eta \circledast \xi, \nu)| \\ &\leq K_1^2M_1^nn!\sum_{l=0}^{\bar{n}} \binom{\bar{n}}{l} = K_1^2M_1^n2^{\bar{n}}n! = K_1^2(4M_1)^nn!,\end{aligned}$$

which concludes the proof since $|(c \circledast d, \eta)|$ satisfies (6) with $K = K_1^2$ and $M = 4M_1$ for all $\nu \in \bar{Q}^*$. ■

The following theorem describes the generating series for the parallel product connection of Fliess operators with Poisson jumps.

Theorem 7: If $c, d \in \mathbb{R}^\ell_{LC}\langle\langle XY\rangle\rangle$, then $F_cF_d = F_{c\circledast d}$ with $c \circledast d \in \mathbb{R}^\ell_{LC}\langle\langle XY\rangle\rangle$.

Proof: From (22), the parallel product connection of two Fliess operators is

$$\begin{aligned}F_c[u](t)\,F_d[u](t) &= \sum_{\eta\in XY^*} (c,\eta)E_\eta[u](t) \sum_{\xi\in XY^*} (d,\xi)E_\xi[u](t) \\ &= \sum_{\eta,\xi\in XY^*} (c,\eta)(d,\xi)E_{\eta\circledast\xi}[u](t)\end{aligned}$$

$$= F_{c \circledast d}[u](t).$$

In light of Theorem 6, this completes the proof. ∎

V. Rationality of Parallel Product Interconnections

In this section the question of whether the parallel product connection preserves rationality is addressed. In other words, is the quasi-shuffle product of two rational series again rational? The question was affirmatively answered in [14]. However, it was unclear the context in which rationality was defined (finite or infinite alphabets) and no proof was provided. Here a complete proof is given for rational series on finite alphabets. Let $\bar{Q}$ be an infinite alphabet that is the augmentation of an arbitrary finite alphabet Q via the bracket operation. From brevity assume $\ell = 1$. In light of Definition 2 and the remark thereafter, it is clear that a rational series over $\bar{Q}$ is always constructed using a finite number of scalar multiplications, additions, catenations and inversions applied to a finite set of polynomials over $\bar{Q}$. In which case, a rational series c in this setting is understood as coming from a finite sub-alphabet $Q_c \subset \bar{Q}$. Since Theorems 2 and 3 hold for any series over a finite alphabet, it then follows that these theorems still hold in the present context. Also note that for the parallel product connection the underlying alphabets for the generating series of F_c and F_d are always the same since the inputs are identical. But there is no additional complexity introduced if the alphabets are allowed to be distinct. So let $Q_c, Q_d \subset \bar{Q}$ be finite sub-alphabets of $\bar{Q}$ corresponding to the generating series c and d and with cardinalities N_c and N_d, respectively. Define $[Q_cQ_d] = \{[q_i^c q_j^d] : q_i^c \in Q_c, q_j^d \in Q_d, i = 1,\ldots,N_c, j = 1\ldots,N_d\}$. In this context, the main theorem of the section is given first.

Theorem 8: If $c,d \in \mathbb{R}_{rat}\langle\langle \bar{Q}\rangle\rangle$ with underlying finite alphabets $Q_c, Q_d \in \bar{Q}$, then $e = c \circledast d \in \mathbb{R}_{rat}\langle\langle \bar{Q}\rangle\rangle$ with underlying alphabet $Q_e = Q_c \cup Q_d \cup [Q_cQ_d] \in \bar{Q}$.

Proof: In light of (21), the series $e = c \circledast d$ is clearly defined over the finite alphabet Q_e. Therefore, both Theorems 2 and 3 apply for characterizing the rationality in $\mathbb{R}\langle\langle X_e\rangle\rangle$. Hence, the goal is to construct a stable finite dimensional vector space V_e which contains e. Since c and d are both rational, let V_c and V_d be stable finite dimensional vector subspaces of $\mathbb{R}\langle\langle Q_c\rangle\rangle$ and $\mathbb{R}\langle\langle Q_d\rangle\rangle$ containing c and d, respectively. Let $\{\bar{c}_i\}_{i=1}^{n_c}$ and $\{\bar{d}_j\}_{j=1}^{n_d}$ denote their corresponding bases. Define

$$V_e = \mathrm{span}_{\mathbb{R}}\{\bar{c}_i \circledast \bar{d}_j \,:\, i = 1,\ldots,n_c,\ j = 1,\ldots,n_d\}.$$

Clearly, $V_e \subset \mathbb{R}\langle\langle Q_e\rangle\rangle$ is finite dimensional. If one writes

$$c = \sum_{i=1}^{n_c} \alpha_i \bar{c}_i,\ \ d = \sum_{j=1}^{n_d} \beta_j \bar{d}_j,$$

where $\alpha_i, \beta_j \in \mathbb{R}$, it then follows directly that

$$e = c \circledast d = \sum_{i,j=1}^{n_c,n_d} \alpha_i\beta_j\ \bar{c}_i \circledast \bar{d}_j \in V_e.$$

So it only remains to be shown that V_e is stable. Writing $c = (c,\emptyset) + \sum_{i=0}^{N_c} q_i^c\ (q_i^c)^{-1}(c)$ and $d = (d,\emptyset) + \sum_{i=0}^{N_d} q_i^d\ (q_i^d)^{-1}(d)$ and using the bilinearity of the quasi-shuffle, it follows from (25) that

$$\begin{aligned} q^{-1}(e) &= q^{-1}(c) \circledast d + c \circledast q^{-1}(d) \\ &\quad + \sum_{i,j=0}^{N_c,N_d} \delta_{q,[q_i^c q_j^d]}((q_i^c)^{-1}(c) \circledast (q_j^d)^{-1}(d)). \end{aligned}$$

But since $V_c, V_d \subset \mathbb{R}\langle\langle Q_e\rangle\rangle$ are stable vector spaces by assumption, it is immediate that $(q_i^c)^{-1}(c) \in V_c$ and $(q_j^d)^{-1}(d) \in V_d$, and therefore $q^{-1}(e) \in V_e$ as well. It then follows that V_e is a stable vector space, and hence e is rational. ∎

The next corollary follows directly.

Corollary 1: If $c,d \in \mathbb{R}_{rat}\langle\langle \bar{Q}\rangle\rangle$ are rational series with underlying finite alphabets $Q_c, Q_d \in \bar{Q}$, then $F_cF_d = F_{c\circledast d}$ with $e = c \circledast d$ rational, where $Q_e = Q_c \cup Q_d \cup [Q_cQ_d]$.

Note that when $Q = XY$ then $\bar{Q} = Q_c = Q_d = Q_e = XY$, and thus, the corollary obviously holds.

VI. Conclusions

A Fliess operator representation was provided for a Markov jump nonlinear systems used to model fault-tolerant control systems. In particular, it was shown that known properties of classical Fliess operators regarding convergence and rationality are maintained by this representation. In addition, the generating series of the parallel interconnections (sum and product) were described for Fliess operators with Poisson jumps.

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