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On Spectral Properties of Directed Scale-Free Networks

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***Abstract*— In this paper, we study the spectral properties of directed scale-free networks. First, we study the algebraic connectivity of a directed scale-free network, which is the eigenvalue of its Laplacian matrix whose real part is the second smallest. This is known as an important measure for the diffusion speed of many diffusion processes over networks (e.g. consensus, information spreading, epidemics). We also investigate the eigenvalue of the Laplacian matrix with maximum real part. We propose an algorithm, extending that of Barabasi and Albert, to generate directed scale-free networks, and show by simulations the relations between the spectral properties and network size, exponents of in/out-degree distributions, and minimum in/out degrees.**

I. Introduction

Recently the scale of many real networks have grown larger and their topologies become more complex. In response, many network models have been proposed to analyze the topological property of real networks [1]–[4]. In particular, analyzing the second smallest eigenvalue of the network's *Laplacian* matrix has attracted much attention [3], [5]–[8]. This special eigenvalue is referred to as the *algebraic connectivity* of networks, and is known as an important measure for the diffusion speed of many diffusion processes over networks (e.g. consensus, information/innovation spreading, epidemics). Moreover, the maximum eigenvalue of the associated Laplacian matrix is known to affect the robustness against inter-agent/node communication delay for consensus reaching over undirected networks [5]. So far, the spectral properties of undirected [5], directed small-world [6] (only the algebraic connectivity), and undirected scale-free [3] networks have been studied.

In many real, scale-free networks such as social networking service (SNS) and World Wide Web (WWW), however, the edges may not be bidirectional. For example, in Twitter, we can follow some (popular) people, but they do not necessarily follow us; in WWW, a webpage can have links to some (well-known) pages, which may not have links back to that webpage. These have motivated us to study the spectral properties of *directed* scale-free networks.

In this paper, we first propose a new algorithm that provably generates directed scale-free networks. This algorithm is a natural extension of the Barabasi and Albert (BA) model [2]: starting from an initial directed network, one node is added at a time with m_{in} *in-edges* from, and m_{out} *out-edges* to, the existing nodes by preferential attachment.

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Using this algorithm, we investigate the impacts of *structural* properties of directed scale-free networks (size, exponents of in/out-degree distributions, and minimum in/out degrees) on the spectral properties. We note that in [9], an algorithm was reported which also generates directed scale-free networks. However, there was no study in [9] on network's spectral properties.

The outline of the rest of this paper is as follows. In Section II, we introduce preliminaries on algebraic graph theory and directed scale-free networks. In Section III, we present an algorithm which generates directed scale-free networks and we provide simulation results on the relations between the structural properties and the spectral properties of directed scale-free networks. Finally, our conclusions are stated in Section IV.

II. Preliminaries

For an arbitrary directed graph, we can represent its topology by the Laplacian matrix. Of particular importance is the eigenvalue of the Laplacian matrix whose real part is the second smallest; we refer to the real part of this eigenvalue as the *algebraic connectivity*.[1] It is well-known [8] that the algebraic connectivity determines the diffusion rate of many diffusion processes over networks. Moreover, the maximum eigenvalue of Laplacian matrix measures the robustness against inter-agent/node communication delay for consensus reaching over undirected networks [5]. In this paper we shall also study the eigenvalue of the Laplacian matrix whose real part is the largest.

Next, we introduce the *scale-free* property, which is found to be a common feature in many real networks [2]. This property means roughly that many nodes are connected with only a handful of other nodes, while some (hub) nodes with a large number of nodes. Let k_{in} (resp. k_{out}) be the in-degree (resp. out-degree) of a node, namely the number of in-edges (resp. out-edges) of that node. Also let $P(k_{in})$, $P(k_{out})$ be the in-degree distribution and the out-degree distribution, respectively; these are the ratios of the number of nodes with in-degree k_{in} or out-degree k_{out} with respect to the total number of nodes in the network. The scale-free property of directed networks refers to that $P(k_{in})$, $P(k_{out})$ follow the power laws [10]:

$$P(k_{in}) \sim {k_{in}}^{-\gamma_{in}}, \qquad P(k_{out}) \sim {k_{out}}^{-\gamma_{out}},$$

where $\sim$ means "proportional to" and γ_{in}, γ_{out} are called the exponents of the in-degree distribution and the out-

[1] Algebraic connectivity is originally defined for undirected graphs and refers to the second smallest eigenvalue of the corresponding Laplacian matrix [11].

degree destribution, respectively. As an example, the in/out-degree distributions of WWW follow power laws with $\gamma_{in} \simeq 2.1, \gamma_{out} \simeq 2.7$ [10].

Barabasi and Albert introduced an algorithm to generate undirected scale-free networks [2]. This algorithm has two essential ingredients: "growth" and "preferential attachment". First, the network grows by adding one new node at each iteration. Second, the probability that the new node is connected to an existing node is proportional to the latter's degree. It was shown [2] that the degree distribution of undirected scale-free networks generated by BA algorithm follows a power law.

In this paper we study directed scale-free networks and their spectral properties. For this, we shall design an algorithm to generate directed scale-free networks, by extending the BA algorithm but maintaining the two main ingredients – "growth" and "preferential attachment".

III. Spectral Properties of Directed Scale-Free Networks

A. Algorithm for Generating Directed Scale-Free Networks

First, we present an algorithm to generate directed scale-free (DSF) networks of N nodes, by extending the BA algorithm.

Algorithm DSF:

1) Initially let $\mathcal{D}_0$ be a directed graph with $m_0 (> 1)$ nodes that contains a spanning tree.
2) At each iteration t (≥ 1), add a new node with $m_{in} \in [1, m_0]$ in-edges from and $m_{out} \in [1, m_0]$ out-edges to the existing nodes. The probability $\Pi_{i,in}$ (resp. $\Pi_{i,out}$) that an existing node i with in-degree $k_{i,in}$ (resp. out-degree $k_{i,out}$) establishes an in-edge from (resp. out-edge to) the existing node is

$$\Pi_{i,in} = \frac{k_{i,in}}{\sum_j k_{j,in}}, \qquad (1)$$

$$\text{resp.} \quad \Pi_{i,out} = \frac{k_{i,out}}{\sum_j k_{j,out}}. \qquad (2)$$

The above summations are over all the existing nodes. No self-loop edges or multiple edges are allowed.

3) If $t = N - m_0 - 1$, stop. Otherwise advance t to $t+1$ and go to Step 2).

In Step 2) of the DSF Algorithm, the network grows with one new node at each iteration, and the probabilities $\Pi_{i,in}$, $\Pi_{i,out}$ in (1), (2) mean *preferential attachment*: the higher in-degree (resp. out-degree) an existing node has, the more likely it establishes an in-edge from (resp. out-edge to) the newly added node.

The growth and preferential attachment features lead to that the network generated by the DSF Algorithm has scale-free property, as asserted by the following theorem.

Theorem 1: The network generated by the DSF Algorithm has scale-free property, i.e.

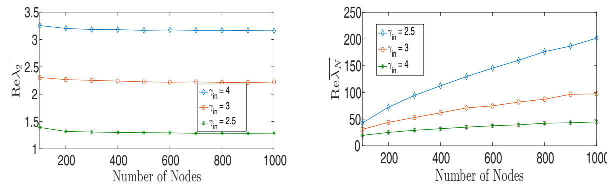
$$P(k_{in}) \sim k_{in}{}^{-\gamma_{in}}, \qquad P(k_{out}) \sim k_{out}{}^{-\gamma_{out}},$$

where $\gamma_{in} = 2 + \frac{m_{in}}{m_{out}}$, $\gamma_{out} = 2 + \frac{m_{out}}{m_{in}}$.

TABLE I

Parameter Settings for Fig. 1.

γ_{in}	γ_{out}	m_{in}, m_{out}
2.5	4	$m_{in} = 2, m_{out} = 4$
3	3	$m_{in} = m_{out} = 3$
4	2.5	$m_{in} = 4, m_{out} = 2$



(a) Re$\overline{\lambda_2}$ (b) Re$\overline{\lambda_N}$

Fig. 1. The impacts of network size and exponents of in/out-degree distributions on (a) Re$\overline{\lambda_2}$ and (b) Re$\overline{\lambda_N}$ (averaged over 100 simulation runs).

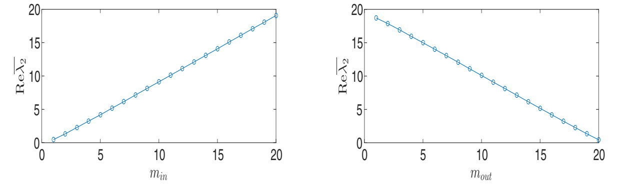
By Theorem 1, the directed networks generated by the DSF Algorithm have power-law in/out-degree distributions, with exponents $\gamma_{in}, \gamma_{out}$ determined solely by the number m_{in} of in-edges and the number m_{out} of out-edges of the newly added nodes. Hence the exponents may easily be varied by changing the values of m_{in}, m_{out}.

B. Topological Impacts on Spectral Properties

We show simulation results on the spectral properties of the directed scale-free networks generated by the DSF Algorithm in Section III-A. We illustrate the topological impacts on the algebraic connectivity of directed scale-free networks. We shall focus on three factors: size, exponents of in/out-degree distributions, and minimum in/out-degrees.

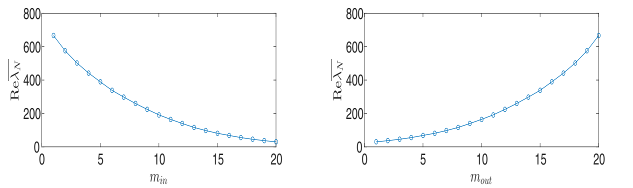
First (size), a ring graph of $m_0 = 4$ nodes is set as the initial network and let $m_{in} = m_{out} = 3$. Vary N from 100 to 1000 and compute the corresponding algebraic connectivity and maximum eigenvalue. This investigation is important because growth is one of the two main features of scale-free networks. In Fig. 1(a), we observe that algebraic connectivity Re$\overline{\lambda_2}$ stays roughly the same as N increases. This means diffusion rate does not drop as the network expands, which makes directed scale-free networks an ideal model for scalable (fast) diffusion. On the other hand, in Fig. 1, we observe that Re$\overline{\lambda_N}$ monotonously increases as networks grow larger.

Second (exponents of in/out-degree distributions), we consider the same initial network as above, but change m_{in}, m_{out} to obtain different $\gamma_{in}, \gamma_{out}$ (see Table I). $\gamma_{in}, \gamma_{out}$ reflect 'degrees' of preferential attachment, the second main feature of scale-free networks. Observe that algebraic connectivity increases (resp. decreases) as the exponent of in-degree (resp. out-degree) distribution increases, consistently for different network sizes. This impact of the exponent of in-degree distribution on the algebraic connectivity is the same as that of the exponent of degree distribution in the undirected case [3]. What is interesting in the current directed networks is that the impact of the exponent of out-degree distribution is in the reverse direction. Hence for fast



(a) in-degree (b) out-degree

Fig. 2. The impacts of minimum (a) in-degree and (b) out-degree on algebraic connectivity $\mathrm{Re}\overline{\lambda_2}$ (averaged over 100 simulation runs)



(a) in-degree (b) out-degree

Fig. 3. The impacts of minimum (a) in-degree and (b) out-degree on $\mathrm{Re}\overline{\lambda_N}$ (averaged over 100 simulation runs)

diffusion, it is desired to have high exponent of in-degree distribution and low exponent of out-degree distribution. On the contrary, we observe that $\mathrm{Re}\overline{\lambda_N}$ decreases (resp. increases) as the exponent of in-degree (resp. out-degree) distribution increases for any network sizes.

Third (minimum in/out-degree), we study the impact of minimum in/out-degree on the spectral properties; this is for comparison with [3] on the undirected scale-free case. For this study we set the complete graph with $m_0 = 21$ as the initial graph and increase m_{in} with the constraint $m_{in} + m_{out} = 21$. In Fig. 2 each plotted point is an average of 100 simulation runs. Observe that algebraic connectivity increases (resp. decreases) as the minimum in-degree (resp. minimum out-degree) increases. The impact of the minimum in-degree on algebraic connectivity is the same as that of the minimum degree in the undirected case [3], while that of the minimum out-degree is in the reverse direction. As shown in Fig. 3(b), we observe that $\mathrm{Re}\overline{\lambda_N}$ (averaged over 100 simulation runs) increases as the minimum out-degree increases, which corresponds to the impact of the minimum degree in the undirected case [3]. In contrast, the impact of the minimum in-degree is in the reverse direction (see Fig. 3(a)).

IV. Conclusions

We have proposed an algorithm, extending that of Barabasi and Albert [2], to generate directed scale-free networks. Using this algorithm, we have investigated by simulations the topological impacts on the spectral properties of directed scale-free networks. In future work, we aim to investigate the spectral properties of hierarchical networks [4], which have both scale-free and small-world property.

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