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Distributed Nash Equilibrium Seeking for Non-Cooperative Games Subject to Coupled Inequality Constraints: A Time-Scale Separation Approach

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***Abstract*—This paper investigates the Nash equilibrium seeking problem for non-cooperative games subject to a coupled inequality constraint. Both the local cost functions and the constrained function are mutually coupled by the decision variables of players. A distributed seeking algorithm is proposed via local information interaction. First, a distributed observer with the projection property is introduced for each player to estimate the decision variables of all the other players. By using these estimations, a seeking algorithm with the projection property is then synthesized. The stability analysis is based on a time-scale separation approach. In particular, we first show that the fast dynamics composed by the distributed observer with an appropriate parameter guarantees that the estimation errors converge to an arbitrarily small neighborhood of the origin in finite time and maintain within it afterwards. Based on this result, we further show that the slow dynamics composed by the seeking algorithm achieves the convergence of the strategy profile to a neighborhood of the generalized Nash equilibrium of interest. an illustrative example is provided to verify the theoretical results.**

I. Introduction

In the past few decades, considerable attention has been paid to the research on game theory due to its various applications in biology, economy, production and other fields [1]. With the development of the game theory, the Nash equilibrium seeking in non-cooperative games is of significant interest from both theoretical and application perspectives [2].

In the general Nash equilibrium seeking problem, each player attempts to minimize its cost function by responding to its and other players' actions. This requires a fully connected network such that a full observation over all the players' actions in the network can be performed [3]. To relax this stringent network requirement, distributed algorithms are proposed borrowing from the cooperative control idea for multi-agent systems [4], [5]. In such distributed algorithms, players minimize their cost functions via local information interaction with only neighboring players.

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A number of distributed Nash equilibrium seeking algorithms have been proposed for both discrete-time and continuous-time dynamics. In particular, A gossip-based methodology was employed for seeking a Nash equilibrium of non-cooperative games [3]. The generalized convex game was solved in [6], [7] by using discrete-time distributed algorithms. A discrete-time stochastic algorithm was proposed in [8] such that players took actions in both simultaneous and asynchronous manners. Two-network zero-sum games with switching communications were studied in [9], where seeking algorithms with heterogeneous and homogenous stepsizes were developed, respectively. A systematic seeking methodology with local agent utility functions was presented in [10] for state-based potential games. Moreover, by introducing consensus-based distributed observers, continuous-time seeking algorithms were developed such that the Nash equilibrium was locally reached [1], [11]. Nevertheless, for the distributed seeking algorithms proposed in [1]-[3],[6]-[11], every player just attempts to optimize its local cost function without imposing any constraint on the players' actions represented by decision variables. A distributed Nash equilibrium seeking algorithm was proposed for aggregative games with decision variables being subject to linear coupled equality constraints [12]. By considering linear coupled inequality constraints, the Lemke's method was adopted for the generalized Nash equilibrium seeking in convex games with quadratic cost functions [13]. It is noted that the seeking algorithms in [12], [13] are inapplicable to the cases with coupled nonlinear inequality constraints.

In this paper, a distributed Nash equilibrium seeking algorithm is proposed for non-cooperative games. A nonlinear inequality constraint is imposed on the decision variables of players. The objective is to seek the generalized Nash equilibrium such that the individual cost function coupled with other decision variables is minimized, while the generalized Nash equilibrium of interest satisfies the imposed constraint. Compared with the previous works, the main contributions herein lie in two aspects. First, compared with [1]-[3], [6]-[11], we propose a distributed Nash equilibrium algorithm for the non-cooperative game problem subject to an inequality constraint, where a distributed observer with the projection property is introduced in the algorithm design such that all the decision variables are obtained by each player. Second, in contrast to [12], [13] with linear constraints, the considered inequality constraint is in a more general nonlinear form, and in turn, the seeking algorithm in this paper is also applicable to the cases in [12], [13].

The remaining sections are arranged as follows. Section II provides some useful mathematical preliminaries. Section III describes the problem to be solved. Section IV introduces our main results including seeking algorithm development and stability analysis. Section V performs simulations to verify the theoretical results. Section VI draws final conclusions.

II. Preliminaries

Notations. Throughout this paper, $\mathbb{R}^n$ denotes real vectors of dimension n, superscript T denotes the transpose of a vector or a matrix, I_n denotes an $n \times n$ unit vector, $\mathbf{1_n}$ denotes an n-dimensional vector with all its entries being 1, $\|\cdot\|$ denotes the Euclidean norm of a real vector, $\mathrm{col}(x_1, x_2, \cdots, x_n)$ denotes a column stack vector composed by vectors or scalars x_1 to x_n, and $\mathrm{diag}\{y_1, y_2, \cdots, y_n\}$ denotes a diagonal matrix with diagonal entries being scalars y_1 to y_n. Let ∇f be the gradient of a function f and $\mathcal{J}_F$ be the Jacobian matrix of a map F. In addition, given a constant $r \geq 0$, we define sets $\Omega_r = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ and $\bar{\Omega}_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$.

Function properties. A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex over set Ω if $f(x) - f(y) \geq \nabla f(y)^T(x - y)$, $\forall x, y \in \Omega$. f is locally θ-Lipschitz if there exists a definition domain Ω such that $\|f(x) - f(y)\| \leq \theta\|x - y\|$, $\forall x, y \in \Omega$. In addition, a map $F : \mathbb{R}^n \to \mathbb{R}^n$ is (strictly) monotone over set Ω if $(x - y)^T(F(x) - F(y)) \geq 0$ $(\omega\|x - y\|^2)$, $\forall x, y \in \Omega$, where ω is a positive constant. According to [12], a differentiable map F is (strictly) monotone over Ω if and only if the Jacobian matrix $\mathcal{J}F(x)$ is positive (semi-)definite for each $x \in \Omega$.

Graph theory. The information exchange among players is established by a topology graph $\mathcal{G} \triangleq \{\mathcal{N}, \mathcal{E}\}$ consisting of a node set $\mathcal{N} \triangleq \{1, 2, \cdots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$. $(j, i) \in \mathcal{E}$ indicates that player j's information is available to player i. For an undirected graph, $(j, i) \in \mathcal{E} \Leftrightarrow (i, j) \in \mathcal{E}$, and it is called connected if each player has a path to every other player. In addition, the (symmetric) adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ associated with an undirected graph $\mathcal{G}$ is defined such that $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise; and its (symmetric) Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

III. Problem Statement

Consider an n-player game subject to the inequality constraint. The set of players is denoted by $\mathcal{N}$. Each player is assigned with a decision variable $x_i \in \mathbb{R}$ and a differentiable cost function $f_i(x)$: $\Omega \to \mathbb{R}$, where $x = \mathrm{col}(x_1, x_2, \cdots, x_n) \in \Omega$ is the strategy profile of this game, and $\Omega = \bar{\Omega}_r \times \bar{\Omega}_r \times \cdots \times \bar{\Omega}_r$ is the definition domain of x with r being a positive constant. Suppose that player i has no access to player j's decision variable provided that player j is not a neighbor of player i. Moreover, the strategy profile is subject to an inequality constraint, which is defined by the following constrained set:

$$\chi = \{x \in \mathbb{R}^n | g(x) \leq 0\}, \tag{1}$$

where the constrained function $g : \Omega \to \mathbb{R}$ is differentiable. Thus, the feasible strategy set of this game is $\mathcal{K} = \Omega \cap \chi$. In what follows, we assume that $\mathcal{K} \neq \emptyset$.

Consider the aforementioned game subject to the coupled inequality constraint. The objective herein is to develop a Nash equilibrium seeking strategy such that all the players obtain the generalized Nash equilibrium, which is defined as follows.

Definition 1. *A strategy profile x^* is called a generalized Nash equilibrium of the game if*

$$f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*), \quad \forall x \in \mathcal{K}, \tag{2}$$

where $x_{-i} = \mathrm{col}(x_1, \cdots, x_{i-1}, x_{i+1} \cdots, x_n)$, $i \in \mathcal{N}$.

In fact, the generalized Nash equilibrium is an optimal strategy profile involved in set $\mathcal{K}$ in the sense that no player can further reduce its associated cost function by unilaterally changing its own decision variable. In addition, to clarify the subsequent analysis, we define a gradient vector

$$F(x) = \mathrm{col}(\nabla_{x_1} f_1(x), \nabla_{x_2} f_2(x), \cdots, \nabla_{x_n} f_n(x)). \tag{3}$$

Before moving on, some necessary assumptions about the studied game are imposed as follows.

Assumption 1 (Connectivity)**.** *The underlying graph $\mathcal{G}$ characterizing the communication network among players is undirected and connected.*

Assumption 2 (Lipschitz)**.** *The constrained function $g(x)$, the gradients $\nabla_{x_i} f_i(x)$, $i \in \mathcal{N}$, and $\nabla_{x_i} g(x)$ are θ-Lipschitz over set Ω.*

Assumption 3 (Monotonicity)**.** *The gradient vector $F(x)$ is strictly monotone, and the constrained function $g(x)$ is monotone over set Ω.*

Remark 1. *According to [12], Assumptions 2-3 guarantee that the studied game admits a unique generalized Nash equilibrium introduced in Definition 1.*

Next, motivated by [14], we have the following result.

Lemma 1. *Suppose that Assumptions 1-3 hold. The strategy profile $x^* \in \mathcal{K}$ is a generalized Nash equilibrium if and only if there exists a Lagrangian multiplier $\lambda_i^* \geq 0$ for each player $i \in \mathcal{N}$ such that the following KKT condition holds:*

$$\begin{aligned} &\nabla_{x_i} f_i(x^*) + \lambda_i^* \nabla_{x_i} g(x^*) = 0, \\ &g(x^*) \leq 0, \quad \lambda_i^* g(x^*) = 0. \end{aligned} \tag{4}$$

In addition, the set of multipliers $\{\lambda_i^*\}$ satisfying the KKT condition (4) *is closed, convex and bounded.*

In terms of Lemma 1, the Nash equilibrium seeking problem can be transformed into searching a strategy profile

$x^* \in \Omega$ as well as a multiplier $\lambda^* = \mathrm{col}(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*)$ such that the KKT condition (4) is guaranteed.

IV. Main results

In this section, a distributed seeking algorithm is exploited such that the strategy profile converges to the generalized Nash equilibrium introduced in Definition 1. We first propose a feasible algorithm with the help of an improved distributed observer, and then analyze the system stability using a time-scale separation approach.

A. Seeking algorithm synthesis

To begin with, we assign a Lagrangian function for each player $i \in \mathcal{N}$ in the following form:

$$\mathcal{L}_i(x_i, \lambda_i) = f_i(x_i, x_{-i}) + \lambda_i g(x_i, x_{-i}), \quad \lambda_i \geq 0. \tag{5}$$

Inspired by Saddle Point Theorem [14], we have the following result.

Lemma 2. *Suppose that Assumptions 1-3 hold. The strategy profile $x^* \in \Omega$ is a generalized Nash equilibrium satisfying the KKT condition* (4) *with a multiplier λ^* if and only if each (x_i^*, λ_i^*) is a saddle point of the Lagrangian function $\mathcal{L}_i(x_i, \lambda_i)$ given by* (5)*.*

To search a saddle point of the Lagrangian function (5), the primal-dual gradient dynamics, which are popular in the solution to the distributed optimization problem [15], [16], are implemented to develop the seeking algorithm. More specifically,

$$\dot{x}_i = -\nabla_{x_i}\mathcal{L}_i(x_i, \lambda_i) = -\nabla_{x_i} f_i(x) - \lambda_i \nabla_{x_i} g(x), \tag{6a}$$

$$\dot{\lambda}_i = [\nabla_{\lambda_i}\mathcal{L}_i(x_i, \lambda_i)]^+_{\lambda_i} = [g(x)]^+_{\lambda_i}, \tag{6b}$$

where operator $[z]^+_\lambda$ is defined such that $[z]^+_\lambda = z$ if $z > 0$ or $\lambda > 0$, and otherwise, $[z]^+_\lambda = 0$. This operation suffices to guarantee that $\lambda_i(t) \geq 0$, $\forall t \geq 0$ with the initial $\lambda_i(0) \geq 0$ [16]. Next, a lemma is presented to discuss the equilibrium of algorithm (6).

Lemma 3. *For an equilibrium $(\bar{x}, \bar{\lambda})$ of algorithm* (6)*, $\bar{x}$ is the generalized Nash equilibrium introduced in Definition 1.*

Proof. Based on (6b), $[g(\bar{x})]^+_{\bar{\lambda}_i} = 0$ implies that $\bar{\lambda}_i g(\bar{x}) = 0$ and $g(\bar{x}) \leq 0$, $i \in \mathcal{N}$. Moreover, from (6a), we have that

$$\nabla_{x_i} f_i(\bar{x}) + \bar{\lambda}_i \nabla_{x_i} g_i(\bar{x}) = 0, \; i \in \mathcal{N}. \tag{7}$$

Based on the above argument, equilibrium $(\bar{x}, \bar{\lambda})$ can be shown to satisfy the KKT condition (4). According to Lemma 1, $\bar{x}$ is exactly the generalized Nash equilibrium. □

It is worth noticing that algorithm (6) is available under a fundamental assumption that each player has access to all the others' decision variables. Nonetheless, this assumption is impractical in practical applications by considering insufficient resource sharing and privacy protection. To address this issue, we introduce a distributed observer for each player using the projection algorithm:

$$\dot{\hat{x}}_{ij} = \begin{cases} \rho_{ij}, & \text{if } \hat{x}_{ij} \in \Omega_r, \\ & \text{or } \hat{x}_{ij} = r \text{ with } \rho_{ij} \leq 0, \\ & \text{or } \hat{x}_{ij} = -r \text{ with } \rho_{ij} \geq 0, \\ 0 & \text{if } \hat{x}_{ij} = r \text{ with } \rho_{ij} > 0, \\ & \text{or } \hat{x}_{ij} = -r \text{ with } \rho_{ij} < 0, \end{cases} \tag{8a}$$

with

$$\rho_{ij} = \sigma\left(-\sum_{k=1}^n a_{ik}(\hat{x}_{ij} - \hat{x}_{kj}) - a_{ij}(\hat{x}_{ij} - x_j)\right), \tag{8b}$$

where $\hat{x}_{ij}$ denotes the player i's estimation on the j-th decision variable, σ is a sufficiently large positive constant to be determined, and a_{ij} denotes the (i,j)-th entry of the adjacency matrix A associated with the underlying graph $\mathcal{G}$. It is trivial to show that each estimation $\hat{x}_{ij}(t) \in \bar{\Omega}_r$, $\forall t \geq 0$ given $\hat{x}_{ij}(0) \in \bar{\Omega}_r$ for $i, j \in \mathcal{N}$.

Next, define $\hat{x}_i = \mathrm{col}(\hat{x}_{i1}, \hat{x}_{i2}, \cdots, \hat{x}_{in})$. By implementing the estimation $\hat{x}_i$ outputted from the distributed observer (8) instead of the unavailable strategy profile x, the seeking algorithm (6) is revised into

$$\dot{x}_i = \begin{cases} u_i, & \text{if } x_i \in \Omega_r, \\ & \text{or } x_i = r \text{ with } u_i \leq 0, \\ & \text{or } x_i = -r \text{ with } u_i \geq 0, \\ 0 & \text{if } x_i = r \text{ with } u_i > 0, \\ & \text{or } x_i = -r \text{ with } u_i < 0. \end{cases} \tag{9a}$$

with

$$u_i = -\nabla_{\hat{x}_i} f_i(\hat{x}_i) - \lambda_i \nabla_{\hat{x}_i} g(\hat{x}_i), \tag{9b}$$

and

$$\dot{\lambda}_i = \begin{cases} g(\hat{x}_i) & \text{if } 0 < \lambda_i < \lambda_{\max}, \\ & \text{or } \lambda_i = 0 \text{ with } g(\hat{x}_i) \geq 0, \\ & \text{or } \lambda_i = \lambda_{\max} \text{ with } g(\hat{x}_i) \leq 0, \\ 0 & \text{if } \lambda_i = 0 \text{ with } g(\hat{x}_i) < 0, \\ & \text{or } \lambda_i = \lambda_{\max} \text{ with } g(\hat{x}_i) > 0, \end{cases} \tag{9c}$$

where $\lambda_{\max}$ is a sufficiently large control parameter to be determined. Likewise, the seeking algorithm (9) using the projection algorithm guarantees that each decision variable $x_i(t) \in \bar{\Omega}_r$ and multiplier $\lambda_i(t) \in [0, \lambda_{\max}]$, $\forall t \geq 0$ given $x_i(0) \in \bar{\Omega}_r$ and $\lambda_i(0) \in [0, \lambda_{\max}]$ for $i \in \mathcal{N}$. Note from (8) and (9) that, in our seeking algorithm, player i updates its private variables x_i and λ_i with the estimation $\hat{x}_i$ obtained by implementing the distributed observer through the local information interaction.

B. Stability analysis

In this subsection, using the time-scale separation approach, we prove that the seeking algorithm (9) guarantees the convergence of the strategy profile to an arbitrarily small neighborhood of its generalized Nash equilibrium given in Definition 1. The underlying idea behind the time-scale separation approach lies in that the distributed observer (8) and seeking algorithm (9) are considered as a composite system with slow dynamics (9) and fast dynamics (8). By following this idea, the corresponding analysis is divided into two parts. The first one focuses on the initialization time interval $[0, T_1]$. Proposition 1 shows that for arbitrarily small T_1, a sufficiently large σ can be chosen for the distributed observer (8) such that $\|\hat{x}_i - x\| \leq \mu$, $i \in \mathcal{N}$, where μ is an arbitrarily small constant. With this result, Theorem 1 next shows that the strategy profile converges to a neighborhood of the generalized Nash equilibrium.

Proposition 1. *Consider the distributed observer* (8)*. Suppose that Assumptions 1-3 hold. For any $\mu > 0$ and $T_1 > 0$, there always exists a sufficiently large parameter σ such that each estimation error $\hat{x}_{ij} - x_j$, $i, j \in \mathcal{N}$ converges to the μ-neighborhood of the origin in finite time T_1, and then maintains within it afterwards.*

Proof. First, the derivative of the positive definite function $V_{ij} = \frac{1}{2}|\hat{x}_{ij} - x_j|^2$, $i, j \in \mathcal{N}$ satisfies

$$\dot{V}_{ij} = (\hat{x}_{ij} - x_j)(\dot{\hat{x}}_{ij} - \dot{x}_j). \tag{10}$$

Consider the distributed observer (8). If $\hat{x}_{ij} = r$ with $\rho_{ij} \geq 0$, then $\dot{\hat{x}}_{ij} = 0$. Due to the fact that $x_j \in \bar{\Omega}_r$ ensured by the seeking algorithm (9), this implies that

$$\dot{V}_{ij} = -(r - x_j)\dot{x}_j \leq (r - x_j)(\rho_{ij} - \dot{x}_j). \tag{11}$$

In the same manner, if $\hat{x}_{ij} = -r$ with $\rho_{ij} \leq 0$, we can also show that (11) holds. Therefore, the distributed observer (8) guarantees that

$$\dot{V}_{ij} \leq (\hat{x}_{ij} - x_j)(\rho_{ij} - \dot{x}_j). \tag{12}$$

Next, assign a Lyapunov function $V = \sum_{i=1}^n \sum_{j=1}^n V_{ij}$. In terms of (8b) and (12), its derivative can be derived to satisfy

$$\begin{aligned}\dot{V} \leq & \sum_{i=1}^n \sum_{j=1}^n (\hat{x}_{ij} - x_j)(\rho_{ij} - \dot{x}_j) \\ = & \sum_{i=1}^n \sum_{j=1}^n \sigma \left(-\sum_{k=1}^n a_{ik}(\hat{x}_{ij} - \hat{x}_{kj}) - a_{ij}(\hat{x}_{ij} - x_j) \right) \\ & - \sum_{i=1}^n \sum_{j=1}^n (\hat{x}_{ij} - x_j)\dot{x}_j \\ = & -\sigma \tilde{x}^T (L \otimes I_n + \bar{L})\tilde{x} - \tilde{x}^T(\dot{x} \otimes \mathbf{1_n}),\end{aligned} \tag{13}$$

where $\tilde{x} = \mathrm{col}(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ with $\tilde{x}_i = \hat{x}_i - x$, $i \in \mathcal{N}$, L is the Laplacian matrix associated with the underlying graph $\mathcal{G}$ and $\bar{L} = \mathrm{diag}\{a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{n1}, a_{n2}, \cdots, a_{nn}\}$. According to [1], matrix $(L \otimes I_n + \bar{L})$ is positive definite under Assumption 1. In addition, by considering Assumption 2, it follows from (8) and (9) that $\nabla_{x_i} f_i(\hat{x}_i)$, $\nabla_{x_i} g_i(\hat{x}_i)$ and λ_i are bounded with respect to $\hat{x}_i \in \Omega$. This implies that there exists a positive constant β such that $\|\dot{x} \otimes \mathbf{1_n}\| \leq \beta$. Based on the above argument, $\dot{V}$ further satisfies

$$\dot{V} \leq -\sigma\alpha\|\tilde{x}\|^2 + \beta\|\tilde{x}\|, \tag{14}$$

where α is the smallest eigenvalue of $(L \otimes I_n + \bar{L})$. Then, take $W = \sqrt{2V}$. When $V \neq 0$, it follows from (14) that the derivative of W satisfies

$$\dot{W} = -\sigma\alpha W + \beta. \tag{15}$$

When $V = 0$, the Dini derivative of W satisfies $D^+W \leq \beta$. Hence, D^+W satisfies (15) all the time. According to Comparison Principal [17], it follows that

$$W(t) \leq e^{-\sigma\alpha t}W(0) + (1 - e^{-\sigma\alpha t})\frac{\beta}{\sigma\alpha}, \ \forall t \geq 0. \tag{16}$$

Given any μ and T_1, if σ is chosen such that

$$\sigma > \begin{cases} \frac{\beta}{\alpha(\mu - W(0))}, & \text{if } W(0) < \mu, \\ \max\{\frac{2\beta}{\mu\alpha}, \frac{1}{\alpha T_1} \ln \frac{2W(0)}{\mu}\}, & \text{if } W(0) \geq \mu, \end{cases} \tag{17}$$

then $W(t) \leq \mu$ for $t \in [T_1, \infty)$. This implies that $|\tilde{x}_{ij}(t)| \leq \mu$, $t \in [T_1, \infty)$ for $i, j \in \mathcal{N}$. ☐

Remark 2. *Without considering the finite time constraint, it follows from* (16) *that each estimation error $x_i - x$, $i \in \mathcal{N}$ is always bounded and ultimately converges to set $\mathcal{Z}_i = \{(x_i - x) \in \mathbb{R}^n | \|x_i - x\| \leq \frac{\sqrt{n}\beta}{\alpha\sigma}\}$. It can be observed from this convergent set that increasing parameter σ reduces the ultimate convergent bound, effectively improving the estimation accuracy.*

According to the seeking algorithm (9), each decision variable x_i, $i \in \mathcal{N}$ cannot escape from the domain of definition $\bar{\Omega}_r$ r during the time interval $[0, T_1]$. We next focus on the convergence performance of the strategy profile driven by the seeking algorithm (9) for the time interval $[T_1, \infty)$. The main result is summarized in Theorem 1.

Theorem 1. *Consider the distributed observer* (8) *and seeking algorithm* (9)*. Suppose that Assumptions 1-3 hold. There exist sufficiently large parameters $\lambda_{\max}$ and σ such that the strategy profile x converges to a small neighborhood of the generalized Nash equilibrium.*

Proof. Assign a Lyapunov function as follows:

$$U = \frac{1}{2}\sum_{i=1}^n (x_i - \bar{x}_i)^2 + \frac{1}{2}\sum_{i=1}^n (\lambda_i - \bar{\lambda}_i)^2. \tag{18}$$

where $\bar{x}_i$ and $\bar{\lambda}_i$ are defined in Lemma 3. Its derivative satisfies

$$\dot{U} = -\sum_{i=1}^n (x_i - \bar{x}_i)\dot{x}_i + \sum_{i=1}^n (\lambda_i - \bar{\lambda}_i)\dot{\lambda}_i. \tag{19}$$

Recall (9a). If $x_i = r$ with $u_i > 0$, then

$$(x_i - \bar{x}_i)\dot{x}_i = 0 \leq (x_i - \bar{x}_i)u_i. \tag{20}$$

Similarly, if $x_i = -r$ with $u_i < 0$, (20) also holds. Choose $\lambda_{\max}$ such that $\lambda_{\max} \geq \max_{i \in \mathcal{N}} \bar{\lambda}_i$. In such a case, with the same argument as above, it also follows from (9c) that

$$(\lambda_i - \bar{\lambda}_i)\dot{\lambda}_i \leq (\lambda_i - \bar{\lambda}_i)g(\hat{x}_i). \tag{21}$$

Thus, $\dot{U}$ satisfies

$$\begin{aligned}\dot{U} \leq &-\sum_{i=1}^{n}(x_i-\bar{x}_i)(\nabla_{x_i}f_i(\hat{x}_i)+\lambda_i\nabla_{x_i}g(\hat{x}_i))\\ &+\sum_{i=1}^{n}(\lambda_i-\bar{\lambda}_i)g(\hat{x}_i)\\ =&-\sum_{i=1}^{n}(x_i-\bar{x}_i)(\nabla_{x_i}f_i(\bar{x})+\bar{\lambda}_i\nabla_{x_i}g(\bar{x}))\\ &-(x-\bar{x})^T(F(x)-F(\bar{x}))\\ &+\sum_{i=1}^{n}(x_i-\bar{x}_i)(\nabla_{x_i}f_i(x)-\nabla_{x_i}f_i(\hat{x}_i))\\ &-\sum_{i=1}^{n}(x_i-\bar{x}_i)(\lambda_i\nabla_{x_i}g(x)-\bar{\lambda}_i\nabla_{x_i}g(\bar{x}))\\ &+\sum_{i=1}^{n}\lambda_i(x_i-\bar{x}_i)(\nabla_{x_i}g(x)-\nabla_{x_i}g(\hat{x}_i))\\ &+\sum_{i=1}^{n}(\lambda_i-\bar{\lambda}_i)g(x)-\sum_{i=1}^{n}(\lambda_i-\bar{\lambda}_i)(g(x)-g(\hat{x}_i)).\end{aligned}$$

Lemma 3 has guaranteed that $\nabla_{x_i}f_i(\bar{x})+\bar{\lambda}_i\nabla_{x_i}g(\bar{x})=0$, $i\in\mathcal{N}$. Then, under Assumption 3, it follows that

$$-(x-\bar{x})^T(F(x)-F(\bar{x}))\leq-\omega\|x-\bar{x}\|^2. \quad (22)$$

In addition, considering Assumption 3 and the fact that $(\lambda_i-\bar{\lambda}_i)g(\bar{x})\leq 0$, we have that

$$\begin{aligned}&-\sum_{i=1}^{n}\big((x_i-\bar{x}_i)(\lambda_i\nabla_{x_i}g(x)-\bar{\lambda}_i\nabla_{x_i}g(\bar{x}))-(\lambda_i-\bar{\lambda}_i)g(x)\big)\\ =&-\sum_{i=1}^{n}\big(\lambda_i(g(\bar{x})-g(x)-\nabla_{x_i}g(x)(\bar{x}_i-x_i))\\ &+\bar{\lambda}_i(g(x)-g(\bar{x})-\nabla_{x_i}g(\bar{x})(x_i-\bar{x}_i))-(\lambda_i-\bar{\lambda}_i)g(\bar{x})\big)\\ \leq&0,\quad i\in\mathcal{N}. \quad (23)\end{aligned}$$

It then follows that $\dot{U}$ satisfies

$$\begin{aligned}\dot{U}\leq&-\omega\|x-\bar{x}\|^2+\sum_{i=1}^{n}(x_i-\bar{x}_i)(\nabla_{x_i}f_i(x)-\nabla_{x_i}f_i(\hat{x}_i))\\ &+\sum_{i=1}^{n}\lambda_i(x_i-\bar{x}_i)(\nabla_{x_i}g(x)-\nabla_{x_i}g(\hat{x}))\\ &-\sum_{i=1}^{n}(\lambda_i-\bar{\lambda}_i)(g(x)-g(\hat{x})). \quad (24)\end{aligned}$$

Next, according to Assumption 2 and the fact that $\lambda_i,\bar{\lambda}_i\in[0,\lambda_{\max}]$, it follows that for $t\in[T_1,\infty)$,

$$\dot{U}\leq-\omega\|x-\bar{x}\|^2+\theta(1+2\lambda_{\max})\|x-\bar{x}\|\|x-\hat{x}\|. \quad (25)$$

Proposition 1 has shown that for arbitrary $\mu>0$ and $T_1>0$, a sufficiently large σ can guarantee that $\|x(t)-\hat{x}(t)\|\leq\mu$ for $t\in[T_1,\infty)$. With this result, $\dot{U}(t)$ satisfies

$$\dot{U}\leq-\omega\|x-\bar{x}\|^2+\theta\mu(1+2\lambda_{\max})\|x-\bar{x}\|, \quad (26)$$

for $t\in[T_1,\infty)$, which implies that $\dot{U}$ is negative definite given that $\|x-\bar{x}\|>\frac{\theta\mu(1+2\lambda_{\max})}{\omega}$. Therefore, $x-\bar{x}$ is bounded and ultimately converges to set $\mathcal{Z}=\{(x-\bar{x})\in\mathbb{R}^n|\|x-\bar{x}\|\leq\frac{\theta\mu(1+2\lambda_{\max})}{\omega}\}$. □

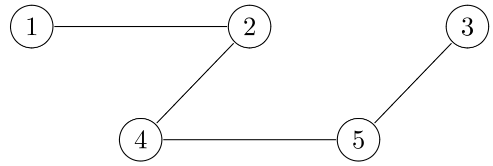

Fig. 1. Network graph among players.

Remark 3. *Note from the convergent set $\mathcal{Z}$ in the analysis of Theorem 1 that, the convergent accuracy is concerned with parameter μ, which reflects the estimation accuracy using the distributed observer* (8) *on the strategy profile x for each player. As shown in Proposition 1, increasing parameter σ is an effective way to decrease μ, which further improves the convergent accuracy of the strategy profile x to its generalized Nash equilibrium.*

V. SIMULATIONS

In this section, an example is given to verify our seeking algorithm. Suppose that there is a non-cooperative game composed by five players. The cost function $f_i(x)$ assigned to each player is $f_i(x)=m_if(x)$, where $m_1=1$, $m_2=5$, $m_3=2$, $m_4=3$, $m_5=2$ and

$$\begin{aligned}f(x)=&5x_1^2+2x_1x_2+5x_2^2+x_2x_3+x_2x_5+\frac{5}{2}x_3^2+x_3x_4\\ &+x_4^2+2x_4x_5+3x_5^2-2x_1+3x_2-8x_3-6x_4+5x_5,\end{aligned}$$

with $x_i\in\bar{\Omega}_{10}$, $i\in\{1,2,\cdots,5\}$. Note that each $f_i(x)$ is coupled by others' decision variables. The network graph among these five players is illustrated in Fig. 1, which can be examined to be connected. The parameters in the seeking algorithm are chosen as $\sigma=100$ and $\lambda_{\max}=10$. It can be calculated that the Nash equilibrium of this game is $[0.2829,-0.4143,1.555,0.64,-0.9776]$. Fig. 2 exhibits the convergence trajectory of each decision variable to the Nash equilibrium. In addition, consider that the strategy profile suffers from the constraint: $\sum_{i=1}^{5}x_i^2\leq 1.5^2$. In such a case, the generalized Nash equilibrium can be calculated as $[0.1942,-0.3507,1.119,0.5597,-0.7244]$. Fig. 3 shows that each decision variable subject to the inequality constraint converges to the neighborhood of the generalized Nash equilibrium. This verifies the convergence performance of the proposed seeking algorithm.

VI. CONCLUSION

This paper studies the distributed Nash equilibrium seeking problem for non-cooperative games subject to a coupled inequality constraint. All the cost functions and the constrained function are coupled by the decision variables. A distributed observer with the projection property is first developed such that each player obtains all the others' decision variables. Using these estimations and the projection operation, we then propose a distributed Nash equilibrium seeking scheme. By using the time-scale separation approach, it is proven that

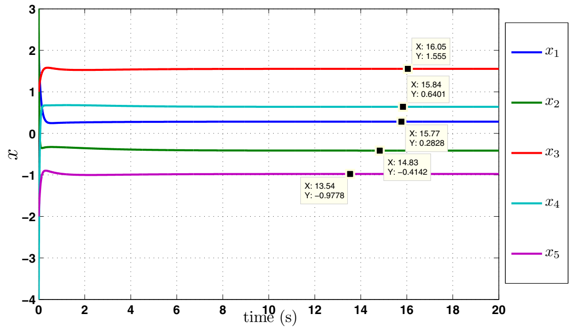

Fig. 2. Trajectories of decision variables without constraint (example 1).

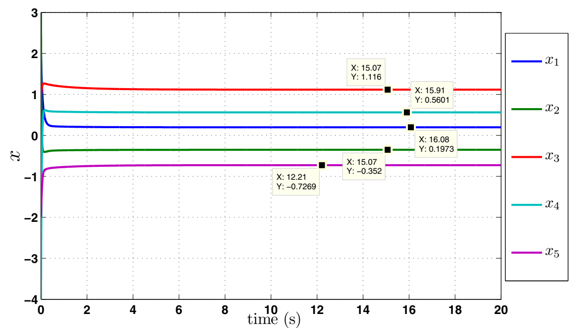

Fig. 3. Trajectories of decision variables with constraint (example 1).

the proposed seeking algorithm guarantees the convergence of the strategy profile to an arbitrarily small neighborhood of the generalized Nash equilibrium.

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