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Linear Quadratic Mean Field Games – Part I: The Asymptotic Solvability Problem

Minyi Huang Mengjie Zhou

Abstract— **This paper investigates the so-called asymptotic solvability problem in linear quadratic (LQ) mean field games. The model has asymptotic solvability if for all sufficiently large population sizes, the corresponding game has a set of feedback Nash strategies subject to a mild regularity requirement. We provide a necessary and sufficient condition and show that in this case the solution converges to a mean field limit. This is accomplished by developing a re-scaling method to derive a low dimensional ordinary differential equation (ODE) system, where a non-symmetric Riccati ODE has a central role.**

Index Terms— **Asymptotic solvability, mean field game, re-scaling, Riccati equation.**

I. INTRODUCTION

Mean field game theory has undergone a phenomenal growth. It provides a powerful methodology for handling complexity in noncooperative mean field decision problems. The readers are referred to [2], [3] for an overview of the theory and applications. The analysis in the LQ setting has attracted substantial interest due to its appealing analytical structure [8], [14], [16]. Specifically, the strategy of an individual player can be determined in a feedback form using its own state.

Two important methodologies called the top-down approach and bottom-up approaches [3], respectively, have been widely used in the analysis of mean field games. By the top-down approach [8], [9], one determines the best response of a representative agent to a mean field of an infinite population, and next all the agents's best responses should regenerate that mean field. This procedure formalizes a fixed point problem which can be solved and further used to design decentralized strategies. The bottom-up approach (also called the direct approach [11]) starts by formally solving an N-player game to obtain a large coupled solution equation system. The next step is to derive a simple liming equation system by taking $N \to \infty$ [13]; also see [12] for a probabilistic framework.

This paper considers the LQ mean field game and addresses the so-called asymptotic solvability. We start with an entirely conventional solution of the game by dynamic programming and derive a set of coupled Riccati ODEs. This method may be viewed as an instance of the bottom-up approach. Our objective is to find a necessary and sufficient condition for the sequence of games to be appropriately solvable. It turns out that such a condition is completely determined by a single low dimensional non-symmetric Riccati ODE. The derivation of this condition involves a novel re-scaling method for large-scale coupled equations with two-scales of interactions. We further determine the mean field limit of the individual strategies. Our approach has connection with an early model of mean field social optimization, which studies a high dimensional algebraic Riccati equation and uses symmetry for dimension reduction [6, Sec. 6.3]. The methodology of extracting a low dimensional structure, here as a non-symmetric Riccati ODE, to capture essential information of the large scale decision model shares similarity to identifying low dimensional dynamics of coupled oscillators in the physics literature [15], [17], [19]. Other related works include [5], [18], [21]. An optimal control problem for a set of agents with mean field coupling is solved in [5] by a large-scale Riccati ODE, where a mean field limit is derived for the Riccati equation using the scalar state and symmetry. An LQ Nash game of infinite horizon is analyzed in [18] where the number of players increases to infinity. The method is to postulate the strategies of all players and examine the control problem of a fixed player subject to the mean field dynamics. Then a family of low dimensional control problems and the parameterized algebraic Riccati equations can be solved by an implicit function theorem. Sufficient conditions are obtained for solvability when the population size is large. The solvability of LQ games with increasing population sizes in the set-up of [13] is studied in [21] analyzing $2N$-coupled steady-state Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck-Kolmogorov (FPK) equations, where each player's control is restricted to be local state feedback from the beginning. Some algebraic conditions are obtained. However, it requires some restrictions on the model parameters, including symmetric state coefficients in dynamics.

Note that the top-down approach can also be used to solve the LQ mean field game [3]. In part II [11] of this paper, we investigate the relation between the top-down approach and the bottom-up approach as developed in this paper. A surprising finding is that they are not always equivalent.

The organization of the paper is as follows. Section II describes the LQ Nash game together with its solution via dynamic programming and Riccati equations. Section III presents the necessary and sufficient condition for asymptotic solvability, for which we give the proof in Section IV. Section V presents further mean field limits related to the dynamic programming equation and derives decentralized strategies. An illustrative example is provided in Section VI.

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M. Huang is with the School of Mathematics and Statistics, Carleton University, Ottawa, K1S 5B6 ON, Canada (mhuang@math.carleton.ca).

M. Zhou is with the School of Mathematics and Statistics, Carleton University, Ottawa, K1S 5B6 ON, Canada (mengjiezhou@cmail.carleton.ca).

Section VII concludes the paper.

II. The LQ Nash Game

Consider a population of N players (or agents) denoted by $\mathscr{A}_i$, $1 \le i \le N$. The state process $X_i(t)$ of $\mathscr{A}_i$ satisfies the following stochastic differential equation (SDE)

$$dX_i(t) = \big(AX_i(t) + Bu_i(t) + GX^{(N)}(t)\big)dt + DdW_i(t), \quad (1)$$

where the state $X_i \in \mathbb{R}^n$, control $u_i \in \mathbb{R}^{n_1}$, and $X^{(N)} = \frac{1}{N}\sum_{k=1}^N X_k$. The initial states $\{X_i(0), 1 \le i \le N\}$ are independent with $EX_i(0) = x_i(0)$ and finite second moment. The N standard n_2-dimensional Brownian motions $\{W_i, 1 \le i \le N\}$ are independent and also independent of the initial states. For symmetric matrix $S \ge 0$, we may write $x^T Sx = |x|_S^2$. The cost of player $\mathscr{A}_i$ is given by

$$\begin{aligned} J_i = E\int_0^T \Big(|X_i(t) - \Gamma X^{(N)}(t) - \eta|_Q^2 + u_i^T(t)Ru_i(t)\Big)dt \\ + E|X_i(T) - \Gamma_f X^{(N)}(T) - \eta_f|_{Q_f}^2. \end{aligned} \quad (2)$$

The constant matrices (or vectors) A, B, G, D, Γ, Q, R, Γ_f, Q_f, η, η_f above have compatible dimensions, and $Q \ge 0$, $R > 0$, $Q_f \ge 0$. For notational simplicity, we only consider constant parameters for the model. Our analysis and results can be easily extended to the case of time-dependent parameters.

Define

$$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{Nn}, \quad W(t) = \begin{bmatrix} W_1(t) \\ \vdots \\ W_N(t) \end{bmatrix} \in \mathbb{R}^{Nn_2},$$

$$\widehat{\boldsymbol{A}} = \mathrm{diag}[A, \cdots, A] + \mathbf{1}_{n\times n} \otimes \frac{G}{N} \in \mathbb{R}^{Nn\times Nn},$$

$$\widehat{\boldsymbol{D}} = \mathrm{diag}[D, \cdots, D] \in \mathbb{R}^{Nn\times Nn_2},$$

$$\boldsymbol{B}_k = e_k^N \otimes B \in \mathbb{R}^{Nn\times n_1}, \quad 1 \le k \le N.$$

We denote by $\mathbf{1}_{k\times l}$ a $k \times l$ matrix with all entries equal to 1, by $\otimes$ the Kronecker product, and by the column vectors $\{e_1^k, \ldots, e_k^k\}$ the canonical basis of $\mathbb{R}^k$. We may use a subscript n to indicate the identity matrix I_n to be $n \times n$.

Now we write (1) in the form

$$dX(t) = \Big(\widehat{\boldsymbol{A}}X(t) + \sum_{k=1}^N \boldsymbol{B}_k u_k(t)\Big)dt + \widehat{\boldsymbol{D}}dW(t). \quad (3)$$

Under closed-loop state information, we denote the value function of $\mathscr{A}_i$ by $V_i(t, \boldsymbol{x})$, $1 \le i \le N$, which corresponds to the initial condition $X(t) = \boldsymbol{x} = (x_1^T, \ldots, x_N^T)^T$. The set of value functions is determined by the system of HJB equations

$$\begin{aligned} 0 = \frac{\partial V_i}{\partial t} + \min_{u_i\in\mathbb{R}^{n_1}} \Bigg(\frac{\partial^T V_i}{\partial \boldsymbol{x}}\big(\widehat{\boldsymbol{A}}\boldsymbol{x} + \sum_{k=1}^N \boldsymbol{B}_k u_k\big) + u_i^T Ru_i \\ + |x_i - \Gamma x^{(N)} - \eta|_Q^2 + \frac{1}{2}\mathrm{Tr}\Big(\widehat{\boldsymbol{D}}^T (V_i)_{\boldsymbol{xx}} \widehat{\boldsymbol{D}}\Big)\Bigg), \end{aligned} \quad (4)$$

$$V_i(T, \boldsymbol{x}) = |x_i - \Gamma_f x^{(N)} - \eta_f|_{Q_f}^2, \quad 1 \le i \le N,$$

where $x^{(N)} = (1/N)\sum_{k=1}^N x_k$ and the minimizer is

$$u_i = -\frac{1}{2}R^{-1}\boldsymbol{B}_i^T \frac{\partial V_i}{\partial \boldsymbol{x}}.$$

Next we substitute u_i into (4):

$$\begin{aligned} 0 =& \frac{\partial V_i}{\partial t} + \frac{\partial^T V_i}{\partial \boldsymbol{x}}\Big(\widehat{\boldsymbol{A}}\boldsymbol{x} - \sum_{k=1}^N \frac{1}{2}\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T \frac{\partial V_k}{\partial \boldsymbol{x}}\Big) \\ &+ |x_i - \Gamma x^{(N)} - \eta|_Q^2 \\ &+ \frac{1}{4}\frac{\partial^T V_i}{\partial \boldsymbol{x}}\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T \frac{\partial V_i}{\partial \boldsymbol{x}} + \frac{1}{2}\mathrm{Tr}\Big(\widehat{\boldsymbol{D}}^T (V_i)_{\boldsymbol{xx}}\widehat{\boldsymbol{D}}\Big). \end{aligned} \quad (5)$$

Denote

$$\boldsymbol{K}_i = [0, \cdots, 0, I_n, 0, \cdots, 0] - \frac{1}{N}[\Gamma, \Gamma, \cdots, \Gamma],$$

$$\boldsymbol{K}_{if} = [0, \cdots, 0, I_n, 0, \cdots, 0] - \frac{1}{N}[\Gamma_f, \Gamma_f, \cdots, \Gamma_f],$$

$$\boldsymbol{Q}_i = \boldsymbol{K}_i^T Q\boldsymbol{K}_i, \quad \boldsymbol{Q}_{if} = \boldsymbol{K}_{if}^T Q_f \boldsymbol{K}_{if},$$

where I_n is the ith submatrix. We write

$$|x_i - \Gamma x^{(N)} - \eta|_Q^2 = \boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{K}_i^T Q\eta + \eta^T Q\eta,$$

and we can write $|x_i - \Gamma_f x^{(N)} - \eta|_{Q_f}^2$ in a similar form.

Suppose $V_i(t, \boldsymbol{x})$ has the following form

$$V_i(t, \boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{P}_i(t)\boldsymbol{x} + 2\boldsymbol{S}_i^T(t)\boldsymbol{x} + \boldsymbol{r}_i(t), \quad (6)$$

where $\boldsymbol{P}_i$ is symmetric. Then

$$\frac{\partial V_i}{\partial \boldsymbol{x}} = 2\boldsymbol{P}_i(t)\boldsymbol{x} + 2\boldsymbol{S}_i(t), \quad \frac{\partial^2 V_i}{\partial \boldsymbol{x}^2} = 2\boldsymbol{P}_i(t). \quad (7)$$

We substitute (6) and (7) into (5) and derive the equation systems:

$$\begin{cases} \dot{\boldsymbol{P}}_i(t) = -\Big(\boldsymbol{P}_i(t)\widehat{\boldsymbol{A}} + \widehat{\boldsymbol{A}}^T\boldsymbol{P}_i(t)\Big) + \\ \qquad \Big(\boldsymbol{P}_i(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}{\boldsymbol{B}_k}^T\boldsymbol{P}_k(t) \\ \qquad + \sum_{k=1}^N \boldsymbol{P}_k(t)\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{P}_i(t)\Big) \\ \qquad - \boldsymbol{P}_i(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_i(t) - \boldsymbol{Q}_i, \\ \boldsymbol{P}_i(T) = \boldsymbol{Q}_{if}, \end{cases} \quad (8)$$

$$\begin{cases} \dot{\boldsymbol{S}}_i(t) = -\widehat{\boldsymbol{A}}^T\boldsymbol{S}_i(t) - \boldsymbol{P}_i(t)\boldsymbol{B}_i R^{-1}{\boldsymbol{B}_i}^T\boldsymbol{S}_i(t) \\ \qquad + \boldsymbol{P}_i(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k(t) \\ \qquad + \sum_{k=1}^N \boldsymbol{P}_k(t)\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_i(t) \\ \qquad + \boldsymbol{K}_i^T Q\eta, \\ \boldsymbol{S}_i(T) = -\boldsymbol{K}_{if}^T Q_f\eta_f, \end{cases} \quad (9)$$

$$\begin{cases} \dot{\boldsymbol{r}}_i(t) = 2{\boldsymbol{S}_i}^T(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k(t) \\ \qquad - \boldsymbol{S}_i^T(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T\boldsymbol{S}_i(t) \\ \qquad - \eta^T Q\eta - \mathrm{Tr}\big(\widehat{\boldsymbol{D}}^T\boldsymbol{P}_i(t)\widehat{\boldsymbol{D}}\big), \\ \boldsymbol{r}_i(T) = \eta_f^T Q_f\eta_f. \end{cases} \quad (10)$$

Remark 1: If (8) has a solution $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ on $[\tau, T] \subseteq [0, T]$, such a solution is unique due to the local Lipschitz continuity of the vector field [4]. Taking transpose on both sides of (8) gives an ODE system for $\boldsymbol{P}_i^T$, $1 \le i \le N$, which shows that $(\boldsymbol{P}_1^T, \cdots, \boldsymbol{P}_N^T)$ still satisfies (8). So the ODE system (8) guarantees each $\boldsymbol{P}_i$ to be symmetric

Remark 2: If (8) has a unique solution $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ on $[0,T]$, then we can uniquely solve $(\boldsymbol{S}_1, \cdots, \boldsymbol{S}_N)$ and $(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_N)$ by using linear ODEs.

For the N-player game, we consider closed-loop perfect state information, so that the state vector $X(t)$ is available to each player.

Theorem 1: Suppose that (8) has a unique solution $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ on $[0,T]$. Then we can uniquely solve (9), (10), and the game of N players has a set of feedback Nash strategies given by

$$u_i = -R^{-1}\boldsymbol{B}_i^T(\boldsymbol{P}_i X(t) + \boldsymbol{S}_i), \quad 1 \le i \le N.$$

Proof: This theorem follows the standard results in [1, Theorem 6.16, Corollary 6.5] . ■

By Theorem 1, the solution of the feedback Nash strategies with closed-loop perfect state information completely reduces to the study of (8). For this reason, our subsequent analysis starts by analyzing (8).

III. Asymptotic Solvability

For an $l \times m$ real matrix $Z = (z_{ij})_{i \le l, j \le m}$, denote the l_1-norm $\|Z\|_{l_1} = \sum_{i,j} |z_{ij}|$.

Definition 2: The sequence of Nash games (1)-(2) has asymptotic solvability if there exists N_0 such that for all $N \ge N_0$, $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ in (8) has a solution on $[0,T]$ and

$$\sup_{N \ge N_0} \sup_{1 \le i \le N, 0 \le t \le T} \|\boldsymbol{P}_i(t)\|_{l_1} < \infty. \tag{11}$$

Definition 2 only involves the Riccati equations. This is sufficient due to Remark 2. The boundedness condition (11) is to impose certain regularity of the solutions, which is necessary for studying the asymptotic behavior of the system when $N \to \infty$.

Let the $Nn \times Nn$ identity matrix be partitioned in the form:

$$I_{Nn} = \begin{bmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & I_n \end{bmatrix}.$$

For $1 \le i \ne j \le N$, exchanging the ith and jth rows of submatrices in I_{Nn}, let J_{ij} denote the resulting matrix. For instance, we have

$$J_{12} = \begin{bmatrix} 0 & I_n & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & I_n \end{bmatrix}.$$

It is easy to check that $J_{ij}^T = J_{ij}^{-1} = J_{ij}$.

Theorem 3: We assume that (8) has a solution $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ on $[0,T]$. Then the following holds.

i) $\boldsymbol{P}_1(t)$ has the representation

$$\boldsymbol{P}_1(t) = \begin{bmatrix} \Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \end{bmatrix}, \tag{12}$$

where Π_1 and Π_3 are $n \times n$ symmetric matrices.

ii) For $i > 1$, $\boldsymbol{P}_i(t) = J_{1i}^T \boldsymbol{P}_1(t) J_{1i}$.

Proof: See Appendix A. ■

By Theorem 3, (11) is equivalent to the following condition:

$$\sup_{N \ge N_0, 0 \le t \le T} \left[|\Pi_1(t)| + N|\Pi_2(t)| + N^2|\Pi_3(t)| \right] < \infty. \tag{13}$$

We present some continuous dependence result of parameterized ODEs. This will play a key role in establishing Theorem 5 below. Consider

$$\dot{x} = f(t,x), \quad x(0) = z \in \mathbb{R}^K, \tag{14}$$
$$\dot{y} = f(t,y) + g(\varepsilon,t,y), \tag{15}$$

where $y(0) = z_\varepsilon \in \mathbb{R}^K$, $0 < \varepsilon \le 1$.

Let $\phi(t,x) = f(t,x)$, or $f(t,x) + g(\varepsilon,t,x)$.

A1) $\sup_{\varepsilon, 0 \le t \le T} |f(t,0)| + |g(\varepsilon,t,0)| \le C_1$.

A2) $\phi(\cdot,x)$ is Lebesgue measurable for each fixed $x \in \mathbb{R}^K$.

A3) For each $t \in [0,T]$, $\phi(t,x): \mathbb{R}^K \to \mathbb{R}^K$ is locally Lipschitz continuous in x, uniformly with respect to (t,ε), i.e., for any fixed $r > 0$, and $x,y \in B_r(0)$ which is the open ball of radius r centering 0,

$$|\phi(t,x) - \phi(t,y)| \le \mathrm{Lip}(r)|x-y|,$$

where $\mathrm{Lip}(r)$ depends only on r, not on $\varepsilon, t \in [0,T]$.

A4) For each fixed $r > 0$,

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T, y \in B_r(0)} |g(\varepsilon,t,y)| = 0, \quad \lim_{\varepsilon \to 0} |z_\varepsilon - z| = 0.$$

If the solutions to (14) and (15), denoted by $x^z(t)$ and $y^\varepsilon(t)$, exist on $[0,T]$, they are unique by the local Lipschitz condition; for (14) in this case denote $\delta_\varepsilon = \int_0^T |g(\varepsilon,\tau,x^z(\tau))| d\tau$, which converges to 0 as $\varepsilon \to 0$ due to A4).

Theorem 4: i) If (14) has a solution $x^z(\cdot)$ on $[0,T]$, then there exists $0 < \bar{\varepsilon} \le 1$ such that for all $0 < \varepsilon \le \bar{\varepsilon}$, (15) has a solution $y^\varepsilon(\cdot)$ on $[0,T]$ and

$$\sup_{0 \le t \le T} |y^\varepsilon(t) - x^z(t)| = O(|z_\varepsilon - z| + \delta_\varepsilon). \tag{16}$$

ii) Suppose there exists a sequence $\{\varepsilon_i, i \ge 1\}$ where $0 < \varepsilon_i \le 1$ and $\lim_{i \to \infty} \varepsilon_i = 0$ such that (15) with $\varepsilon = \varepsilon_i$ has a solution y^{ε_i} on $[0,T]$ and $\sup_{i \ge 1, 0 \le t \le T} |y^{\varepsilon_i}(t)| \le C_2$ for some constant C_2. Then (14) has a solution on $[0,T]$.

Proof: See Appendix B. ■

Remark 3: If (14) and (15) are replaced by matrix ODEs and (or) a terminal condition at T is used in each equation, the results in Theorem 4 still hold.

Let

$$M = BR^{-1}B^T.$$

Before presenting further results, we introduce two Riccati ODEs:

$$\begin{cases} \dot{\Lambda}_1 = \Lambda_1 M \Lambda_1 - (\Lambda_1 A + A^T \Lambda_1) - Q, \\ \Lambda_1(T) = Q_f, \end{cases} \tag{17}$$

and

$$\begin{cases} \dot{\Lambda}_2 = \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\ \qquad -(\Lambda_1 G + \Lambda_2(A+G) + A^T \Lambda_2) + Q\Gamma, \\ \Lambda_2(T) = -Q_f \Gamma_f. \end{cases} \tag{18}$$

Note that (17) is the standard Riccati ODE in LQ optimal control and has a unique solution Λ_1 on $[0,T]$. Equation (18) is a non-symmetric Riccati ODE where Λ_1 is now treated as a known function. We state the main theorem on asymptotic solvability. The proof is postponed to Section IV.

Theorem 5: The sequence of games in (1)-(2) has asymptotic solvability if and only if (18) has a unique solution on $[0,T]$. ■

Our method of proving Theorem 5 is to re-scale by defining

$$\Lambda_1^N = \Pi_1(t),\ \Lambda_2^N = N\Pi_2(t),\ \Lambda_3^N = N^2\Pi_3(t), \tag{19}$$

and examine their ODE system. We introduce the additional equation

$$\begin{cases} \dot{\Lambda}_3 = \Lambda_2^T M \Lambda_2 + \Lambda_3 M \Lambda_1 + \Lambda_1 M \Lambda_3 + \Lambda_3 M \Lambda_2 + \Lambda_2^T M \Lambda_3 \\ \qquad -\left(\Lambda_2^T G + G^T \Lambda_2 + \Lambda_3(A+G) + (A^T+G^T)\Lambda_3\right) \\ \qquad -\Gamma^T Q \Gamma, \\ \Lambda_3(T) = \Gamma_f^T Q_f \Gamma_f. \end{cases} \tag{20}$$

Note that after (17) and (18) are solved on $[0,T]$ or otherwise on a maximal existence interval for the latter, (20) becomes a linear ODE.

Theorem 6: Suppose (18) has a solution on $[0,T]$. Then we have

$$\sup_{0\le t\le T} (|\Pi_1 - \Lambda_1| + |N\Pi_2 - \Lambda_2| + |N^2\Pi_3 - \Lambda_3|) = O(1/N).$$

Proof: The bound follows from Theorem 4 i) by use of g_1, g_2, g_3 and the initial conditions which appear in the equations of Λ_1^N, Λ_2^N, Λ_3^N. ■

IV. Proof of Theorem 5

Note that $\Pi_3 = \Pi_4$. We rewrite the system of (A.3), (A.4) and (A.5) by use of a set of new variables

$$\Lambda_1^N = \Pi_1(t),\ \Lambda_2^N = N\Pi_2(t),\ \Lambda_3^N = N^2\Pi_3(t).$$

Here and hereafter N is used as a superscript in various places. This should be clear from the context. We can determine functions g_k, $1 \le k \le 3$, and obtain

$$\begin{aligned} \dot{\Lambda}_1^N &= \Lambda_1^N M \Lambda_1^N - (\Lambda_1^N A + A^T \Lambda_1^N) - Q \\ &\quad + g_1(1/N, \Lambda_1^N, \Lambda_2^N), \qquad (21) \\ \Lambda_1^N(T) &= (I - \frac{\Gamma_f^T}{N}) Q_f (I - \frac{\Gamma_f}{N}), \end{aligned}$$

$$\begin{aligned} \dot{\Lambda}_2^N &= \Lambda_1^N M \Lambda_2^N + \Lambda_2^N M \Lambda_1^N + \Lambda_2^N M \Lambda_2^N \\ &\quad - (\Lambda_1^N G + \Lambda_2^N (G+A) + A^T \Lambda_2^N) + Q\Gamma \\ &\quad + g_2(1/N, \Lambda_2^N, \Lambda_3^N), \qquad (22) \\ \Lambda_2^N(T) &= -(I - \frac{\Gamma_f^T}{N}) Q_f \Gamma_f, \end{aligned}$$

$$\begin{aligned} \dot{\Lambda}_3^N &= (\Lambda_2^N)^T M \Lambda_2^N + \Lambda_3^N M \Lambda_1^N + \Lambda_1^N M \Lambda_3^N \\ &\quad + \Lambda_4^N M \Lambda_2^N + (\Lambda_2^N)^T M \Lambda_4^N \\ &\quad - \Big((\Lambda_2^N)^T G + G^T \Lambda_2^N + \Lambda_4^N G + G^T \Lambda_4 + \Lambda_3^N A + A^T \Lambda_3^N \Big) \\ &\quad - \Gamma^T Q \Gamma \\ &\quad + g_3(1/N, \Lambda_2^N, \Lambda_3^N), \qquad (23) \\ \Lambda_3^N(T) &= \Gamma_f^T Q_f \Gamma_f. \end{aligned}$$

In particular, we can determine

$$\begin{aligned} g_1 =& \frac{1}{N}(1 - \frac{1}{N})(\Lambda_2^N M \Lambda_2^N + (\Lambda_2^N)^T M (\Lambda_2^N)^T) \\ & - \frac{1}{N}(\Lambda_1^N G + G^T \Lambda_1^N) \\ & - \frac{1}{N}(1 - \frac{1}{N})(\Lambda_2^N G + G^T (\Lambda_2^N)^T) \\ & + \frac{1}{N}(\Gamma^T Q + Q\Gamma) - \frac{1}{N^2}\Gamma^T Q \Gamma. \end{aligned}$$

The expressions of g_2, g_3 can be determined in a similar way and the detail is omitted here.

Note that if (18) has a unique solution on $[0,T]$, we can uniquely solve Λ_3. In view of g_1, g_2, g_3 and the terminal conditions in (21)-(23), by Theorem 4 and Remark 3, we obtain the desired result. ■

V. Decentralized Control

Proposition 7: Suppose that (8) has a solution $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ on $[0,T]$. Then $\boldsymbol{S}_i(t)$ in (9) has the form

$$\boldsymbol{S}_i(t) = [\theta_2^T(t), \cdots, \theta_1^T(t), \cdots, \theta_2^T(t)]^T, \tag{24}$$

in which the ith sub-vector is $\theta_1(t) \in \mathbb{R}^n$ and the remaining sub-vectors are $\theta_2(t) \in \mathbb{R}^n$. Moreover, $\boldsymbol{r}_1 = \cdots = \boldsymbol{r}_N$.

Proof: We can show that

$$(J_{23}^T \boldsymbol{S}_1,\ J_{23}^T \boldsymbol{S}_3,\ J_{23}^T \boldsymbol{S}_2,\ J_{23}^T \boldsymbol{S}_4, \cdots, J_{23}^T \boldsymbol{S}_N)$$

satisfies (9) as $(\boldsymbol{S}_1, \cdots, \boldsymbol{S}_N)$ does. Hence $\boldsymbol{S}_1 = J_{23}^T \boldsymbol{S}_1$. We can further show $\boldsymbol{S}_1 = J_{12}^T \boldsymbol{S}_2$. Following the argument in the proof of Lemma A.1, we obtain the representation (24).

By (24), we further obtain

$$\begin{aligned} \dot{\boldsymbol{r}}_i(t) &= \theta_1^T M \theta_1 + 2(N-1)\theta_2^T M \theta_1 - \mathrm{Tr}(D^T \Pi_1 D) \\ &\quad - (N-1)\mathrm{Tr}(D^T \Pi_3 D) - \eta^T Q \eta \qquad (25) \\ \boldsymbol{r}_i(T) &= \eta_f^T Q_f \eta_f, \end{aligned}$$

for all i, so that $\boldsymbol{r}_1 = \cdots = \boldsymbol{r}_N$, ■

Recalling $M = BR^{-1}B^T$, we derive

$$\begin{aligned}\dot{\theta}_1(t) &= \Pi_1 M\theta_1 + (N-1)(\Pi_2 M\theta_1 + \Pi_2^T M\theta_2)\\ &\quad - \big(A^T + \frac{G^T}{N}\big)\theta_1 - \frac{N-1}{N}G^T\theta_2\\ &\quad + \big(I - \frac{\Gamma^T}{N}\big)Q\eta, \qquad (26)\\ \theta_1(T) &= -\big(I - \frac{\Gamma_f^T}{N}\big)Q_f\eta_f,\\ \dot{\theta}_2(t) &= \big(\Pi_2^T + (N-1)\Pi_3\big)M\theta_1\\ &\quad + \big(\Pi_1 + (N-2)\Pi_2^T\big)M\theta_2 - \frac{1}{N}G^T\theta_1\\ &\quad - \big(A^T + \frac{N-1}{N}G^T\big)\theta_2 - \frac{1}{N}\Gamma^T Q\eta, \qquad (27)\\ \theta_2(T) &= \frac{1}{N}\Gamma_f^T Q_f\eta_f.\end{aligned}$$

Let

$$\chi_1^N(t) = \theta_1(t), \quad \chi_2^N(t) = N\theta_2(t).$$

We may write the ODEs of $\chi_1^N(t)$ and $\chi_2^N(t)$, which have the limiting form:

$$\begin{aligned}\dot{\chi}_1(t) &= (\Lambda_1 M + \Lambda_2 M - A^T)\chi_1 + Q\eta, \qquad (28)\\ \chi_1(T) &= -Q_f\eta_f,\end{aligned}$$

and

$$\begin{aligned}\dot{\chi}_2(t) &= ((\Lambda_2 + \Lambda_4)M - G^T)\chi_1\\ &\quad + ((\Lambda_1 + \Lambda_2)M - (A^T + G^T))\chi_2 - \Gamma^T Q\eta, \qquad (29)\\ \chi_2(T) &= \Gamma_f^T Q_f\eta_f.\end{aligned}$$

For (25), we have the limiting form

$$\begin{aligned}\dot{r}(t) &= \chi_1^T M\chi_1 + 2\chi_2^T M\chi_1 - \mathrm{Tr}(D^T\Lambda_1 D) - \eta^T Q\eta,\\ r(T) &= \eta_f^T Q_f\eta_f.\end{aligned}$$

Proposition 8: If asymptotic solvability holds,

$$\sup_{0\le t\le T}(|\theta_1(t) - \chi_1(t)| + |N\theta_2(t) - \chi_2(t)|) = O(1/N). \qquad (30)$$

Proof: Under asymptotic solvability, we uniquely solve $(\Lambda_1, \Lambda_2, \Lambda_3, \chi_1, \chi_2)$ on $[0,T]$. We obtain (30) by writing the ODE system of $(\Lambda_1^N, \Lambda_2^N, \Lambda_3^N, \chi_1^N, \chi_2^N)$ and next applying Theorem 4. ∎

A. Decentralized control and mean field dynamics

By Theorem 1, the strategy of player $\mathcal{A}_i$ is

$$u_i = -R^{-1}B^T\Big(\Pi_1(t)X_i + \Pi_2(t)\sum_{j\neq i}X_j + \theta_1(t)\Big). \qquad (31)$$

The closed-loop equation of X_i is now given by

$$\begin{aligned}dX_i(t) = \Big(AX_i - M\Big(\Pi_1(t)X_i + \Pi_2(t)\sum_{j\neq i}X_j\\ + \theta_1(t)\Big) + GX^{(N)}\Big)dt + DdW_i. \qquad (32)\end{aligned}$$

We introduce

$$\frac{d\bar{X}}{dt} = (A - M(\Lambda_1 + \Lambda_2) + G)\bar{X} - M\chi_1(t),$$

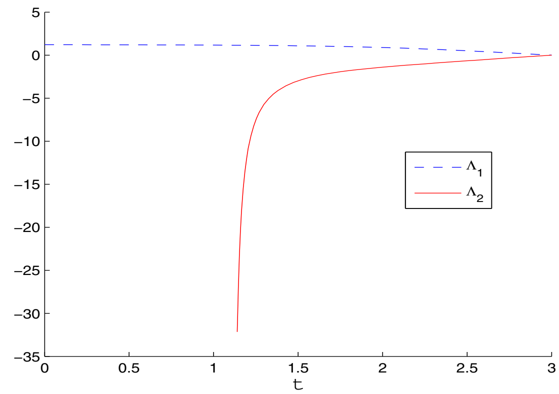

Fig. 1. Λ_2 has a maximal existence interval small than $[0,T]$

where $\bar{X}(0) = x_0$, and further approximate $X^{(N)}$ by $\bar{X}$. When $N \to \infty$, we obtain the decentralized control law

$$u_i^d = -R^{-1}B^T(\Lambda_1(t)X_i + \Lambda_2(t)\bar{X} + \chi_1(t)).$$

The next lemma provides an error estimate for the mean field approximation.

Proposition 9: Suppose $E\sup_{i\ge 1}|X_i(0)|^2 \le C$ for some fixed C and $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N EX_i(0) = x_0$. Then

$$\sup_{0\le t\le T} E|X^{(N)}(t) - \bar{X}(t)|^2 = O(|\frac{1}{N}\sum_{i=1}^N EX_i(0) - x_0|^2 + 1/N).$$

Proof: We first write the SDE for $X^{(N)}$ and find the explicit expression of $X^{(N)}(t) - \bar{X}(t)$. The proposition follows from elementary estimates by use of Theorem 6 and Proposition 8. ∎

VI. An Example

Example Take $A = 0.2$, $B = G = Q = R = 1$, $\Gamma = 1.2$, $\Gamma_f = 0$, $Q_f = 0$, $T = 3$. Consider the equation system (17) and (18):

$$\begin{aligned}\dot{\Lambda}_1 &= \Lambda_1^2 - 0.4\Lambda_1 - 1, \quad \Lambda_1(T) = 0,\\ \dot{\Lambda}_2 &= 2\Lambda_1\Lambda_2 + \Lambda_2^2 - (\Lambda_1 + 1.4\Lambda_2) + 1.2, \quad \Lambda_2(T) = 0.\end{aligned}$$

Λ_1 can be solved explicitly. It is numerically illustrated in Fig. 1 that Λ_2 does not have a solution on the whole interval $[0,T]$, implying no asymptotic solvability.

VII. Conclusion

This paper studies the asymptotic solvability problem for LQ mean field games and obtains a necessary and sufficient condition via a non-asymmetric Riccati ODE. The re-scaling technique used in this paper can be extended to more general models in terms of dynamics and interaction patterns [7], [10]. This will be reported in our future work.

Appendix A: Proof of Theorem 3

We prove the following lemma first.

Lemma A.1: We assume that (8) has a solution $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ on $[0,T]$. Then the following holds.

i) $\boldsymbol{P}_1(t)$ has the representation

$$\boldsymbol{P}_1(t) = \begin{bmatrix} \Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_4(t) & \cdots & \Pi_4(t) \\ \Pi_2^T(t) & \Pi_4(t) & \Pi_3(t) & \cdots & \Pi_4(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^T(t) & \Pi_4(t) & \Pi_4(t) & \cdots & \Pi_3(t) \end{bmatrix} \quad \text{(A.1)}$$

where Π_1, Π_3 and Π_4 are $n \times n$ symmetric matrices.

ii) For $i > 1$, $\boldsymbol{P}_i(t) = J_{1i}^T \boldsymbol{P}_1(t) J_{1i}$.

Proof: Step 1. For $1 \le i \le N$, denote

$$\boldsymbol{P}_i = (\boldsymbol{P}_i^{jk})_{1 \le j,k \le N},$$

where each $\boldsymbol{P}_i^{jk}$ is an $n \times n$ matrix. Define the new functions $J_{23}^T \boldsymbol{P}_i J_{23}$, $i = 1, \ldots, N$. By elementary calculations, we see that

$$(J_{23}^T \boldsymbol{P}_1 J_{23},\ J_{23}^T \boldsymbol{P}_3 J_{23},\ J_{23}^T \boldsymbol{P}_2 J_{23},\ J_{23}^T \boldsymbol{P}_4 J_{23}, \cdots, J_{23}^T \boldsymbol{P}_N J_{23})$$

satisfies (8) together with its terminal condition as $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ does. Hence $\boldsymbol{P}_1 = J_{23}^T \boldsymbol{P}_1 J_{23}$, which implies

$$\boldsymbol{P}_1^{12} = \boldsymbol{P}_1^{13}, \quad \boldsymbol{P}_1^{22} = P_1^{33}, \quad \boldsymbol{P}_1^{23} = \boldsymbol{P}_1^{32}. \quad \text{(A.2)}$$

Repeating the above by using J_{2k}, $k \ge 4$, in place of J_{23}, we obtain

$$\boldsymbol{P}_1^{12} = \boldsymbol{P}_1^{13} = \cdots = \boldsymbol{P}_1^{1N}, \qquad \boldsymbol{P}_1^{22} = \boldsymbol{P}_1^{33} = \cdots = \boldsymbol{P}_1^{NN}.$$

We similarly obtain $\boldsymbol{P}_1 = J_{24}^T \boldsymbol{P}_1 J_{24}$, and this gives

$$\boldsymbol{P}_1^{23} = \boldsymbol{P}_1^{24}.$$

Repeating the similar argument, we can check all other remaining off-diagonal submatrices. Since $\boldsymbol{P}_1$ is symmetric (also see Remark 1), $(\boldsymbol{P}_1^{23})^T = \boldsymbol{P}_1^{32}$, $\boldsymbol{P}_1^{23}$ is symmetric by (A.2). By the above method we can show that all off-diagonal submatrices on neither the first row nor the first column are identical and symmetric. Therefore we obtain the representation of $\boldsymbol{P}_1$.

Step 2. We can verify that both

$$(J_{12}^T \boldsymbol{P}_2 J_{12},\ J_{12}^T \boldsymbol{P}_1 J_{12},\ J_{12}^T \boldsymbol{P}_3 J_{12}, \cdots, J_{12}^T \boldsymbol{P}_N J_{12})$$

and $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ satisfy (8). Hence $\boldsymbol{P}_2 = J_{12}^T \boldsymbol{P}_1 J_{12}$. All other cases can be similarly checked. ■

Proof of Theorem 3: By Lemma A.1, we have

$$\begin{aligned} \dot{\Pi}_1(t) &= \Pi_1 M \Pi_1 + (N-1)(\Pi_2 M \Pi_2 + \Pi_2^T M \Pi_2^T) \\ &\quad - \Big(\Pi_1 (A + \frac{G}{N}) + (A^T + \frac{G^T}{N})\Pi_1\Big) \\ &\quad - (1 - \frac{1}{N})(\Pi_2 G + G^T \Pi_2^T) \\ &\quad - (I - \frac{\Gamma^T}{N}) Q (I - \frac{\Gamma}{N}), \qquad \text{(A.3)} \\ \Pi_1(T) &= (I - \frac{\Gamma_f^T}{N}) Q_f (I - \frac{\Gamma_f}{N}), \end{aligned}$$

and

$$\begin{aligned} \dot{\Pi}_2(t) &= \Pi_1 M \Pi_2 + \Pi_2 M \Pi_1 + \Pi_2^T M \Pi_3 \\ &\quad + (N-2)(\Pi_2 M \Pi_2 + \Pi_2^T M \Pi_4) \\ &\quad - \Big(\Pi_1 \frac{G}{N} + \frac{G^T}{N}\Pi_3 + \frac{N-2}{N} G^T \Pi_4 \\ &\quad + \Pi_2 (A + \frac{N-1}{N} G) + (A^T + \frac{G^T}{N})\Pi_2\Big) \\ &\quad + (I - \frac{\Gamma^T}{N}) Q \frac{\Gamma}{N}, \qquad \text{(A.4)} \\ \Pi_2(T) &= -(I - \frac{\Gamma_f^T}{N}) Q_f \frac{\Gamma_f}{N}, \end{aligned}$$

and

$$\begin{aligned} \dot{\Pi}_3(t) &= \Pi_2^T M \Pi_2 + \Pi_3 M \Pi_1 + \Pi_1 M \Pi_3 \\ &\quad + (N-2)(\Pi_4 M \Pi_2 + \Pi_2^T M \Pi_4) \\ &\quad - \Big(\frac{1}{N}(\Pi_2^T G + G^T \Pi_2) \\ &\quad + \Pi_3 (A + \frac{G}{N}) + (A^T + \frac{G^T}{N})\Pi_3 \\ &\quad + \frac{N-2}{N}(\Pi_4 G + G^T \Pi_4)\Big) \\ &\quad - \frac{\Gamma^T}{N} Q \frac{\Gamma}{N}, \qquad \text{(A.5)} \\ \Pi_3(T) &= \frac{\Gamma_f^T}{N} Q_f \frac{\Gamma_f}{N}, \end{aligned}$$

and

$$\begin{aligned} \dot{\Pi}_4(t) &= \Pi_2^T M \Pi_2 + \Pi_4 M \Pi_1 + \Pi_1 M \Pi_4 \\ &\quad + \Pi_3 M \Pi_2 + \Pi_2^T M \Pi_3 \\ &\quad + (N-3)(\Pi_4 M \Pi_2 + \Pi_2^T M \Pi_4) \\ &\quad - \Big(\frac{1}{N}(\Pi_2^T G + G^T \Pi_2 + \Pi_3 G + G^T \Pi_3) \\ &\quad + \Pi_4 (A + \frac{N-2}{N} G) + (A^T + \frac{N-2}{N} G^T)\Pi_4\Big) \\ &\quad - \frac{\Gamma^T}{N} Q \frac{\Gamma}{N}, \qquad \text{(A.6)} \\ \Pi_4(T) &= \frac{\Gamma_f^T}{N} Q_f \frac{\Gamma_f}{N}. \end{aligned}$$

Then we can further show that $\Pi_3 - \Pi_4$ satisfies a linear ODE when Π_1 and Π_2 are fixed and that $\Pi_3(T) - \Pi_4(T) = 0$. This gives $\Pi_3 = \Pi_4$ on $[0,T]$. ■

Appendix B: Proof of Theorem 4

Proof: i) Let $x^z(t)$ be the solution of (14) on $[0,T]$, and we can find a constant C_z such that $\sup_{0 \le t \le T} |x^z(t)| \le C_z$, and $\sup_{0 < \varepsilon \le 1} |z_\varepsilon| \le C_z$. Fix the open ball $B_{2C_z}(0)$. For $x, y \in B_{2C_z}(0)$ and $t \in [0,T]$, we have

$$|\phi(t,x) - \phi(t,y)| \le \mathrm{Lip}(2C_z)|x - y|.$$

For each $\varepsilon \le 1$, by A1)-A3), (15) has a solution $y^\varepsilon(t)$ defined either (a) for all $t \in [0,T]$ or (b) on a maximal interval $[0, t_{\max})$ for some $0 < t_{\max} < T$.

Below we show that for all small ε, (b) does not occur. We prove by contradiction. Suppose for any small $\varepsilon_0 > 0$, there exists $0 < \varepsilon < \varepsilon_0$ such that (b) occurs with the corresponding $0 < t_{\max} < T$. Since $[0, t_{\max})$ is the maximal existence interval, we have $\lim_{t \uparrow t_{\max}} |y^\varepsilon(t)| = \infty$ [4]. Therefore for some $0 < t_m < t_{\max}$,

$$y^\varepsilon(t_m) \in \partial B_{2C_z}(0), \tag{B.1}$$

and

$$y^\varepsilon(t) \in B_{2C_z}(0), \quad \forall 0 \le t < t_m. \tag{B.2}$$

For $t < t_{\max}$, we have

$$\begin{aligned} &y^\varepsilon(t) - x^z(t) = z_\varepsilon - z \\ &+ \int_0^t \Big[f(\tau, y^\varepsilon(\tau)) + g(\varepsilon, \tau, y^\varepsilon(\tau)) - f(\tau, x^z(\tau)) \Big] d\tau. \end{aligned}$$

Denote $\zeta(\tau) = f(\tau, y^\varepsilon(\tau)) + g(\varepsilon, \tau, y^\varepsilon(\tau)) - f(\tau, x^z(\tau))$ and it follows that

$$\begin{aligned} |\zeta(\tau)| &= |\zeta(\tau) - g(\varepsilon, \tau, x^z(\tau)) + g(\varepsilon, \tau, x^z(\tau))| \\ &\le \mathrm{Lip}(2C_z)|y^\varepsilon(\tau) - x^z(\tau)| + |g(\varepsilon, \tau, x^z(\tau))|. \end{aligned}$$

Now for $0 \le t < t_m$,

$$\begin{aligned} |y^\varepsilon(t) - x^z(t)| \le\ & |z_\varepsilon - z| + \delta_\varepsilon \\ & + \int_0^t \mathrm{Lip}(2C_z)|y^\varepsilon(\tau) - x^z(\tau)| d\tau. \end{aligned}$$

Note that $\delta_\varepsilon = \int_0^T |g(\varepsilon, \tau, x^z(\tau))| d\tau \to 0$ as $\varepsilon \to 0$. By Gronwall's lemma,

$$|y^\varepsilon(t) - x^z(t)| \le (\delta_\varepsilon + |z_\varepsilon - z|) e^{\mathrm{Lip}(2C_z) t}$$

for all $t \le t_m$. We can find $\bar\varepsilon > 0$ such that for all $\varepsilon \le \bar\varepsilon$,

$$(\delta_\varepsilon + |z_\varepsilon - z|) e^{\mathrm{Lip}(2C_z) T} < \frac{C_z}{3}.$$

Then for all $0 \le t \le t_m$, $y^\varepsilon(t) \in B_{3C_z/2}(0)$, which is a contradiction to (B.1). We conclude for all $0 < \varepsilon \le \bar\varepsilon$, y^ε is defined on $[0, T]$. Next, (16) follows readily.

ii) We have

$$y^{\varepsilon_i}(t) = z_{\varepsilon_i} + \int_0^t \Big[f\big(\tau, y^{\varepsilon_i}(\tau)\big) + g\big(\varepsilon, \tau, y^{\varepsilon_i}(\tau)\big) \Big] d\tau, \tag{B.3}$$

and

$$\begin{aligned} &|f\big(\tau, y^{\varepsilon_i}(\tau)\big) + g\big(\varepsilon, \tau, y^{\varepsilon_i}(\tau)\big)| \\ &\le \mathrm{Lip}(C_2)|y^{\varepsilon_i}(\tau)| + |f(\tau, 0) + g(\varepsilon, \tau, 0)| \\ &\le \mathrm{Lip}(C_2)|y^{\varepsilon_i}(\tau)| + C_1 \\ &\le \mathrm{Lip}(C_2) C_2 + C_1, \end{aligned} \tag{B.4}$$

where C_1 is given in A1).

By (B.3)-(B.4), the functions $\{y^{\varepsilon_i}(\cdot), i \ge 1\}$ are uniformly bounded and equicontinuous. By Ascoli's lemma, there exists a subsequence $\{y^{\varepsilon_{i_j}}(\cdot), j = 1, 2, 3, \cdots\}$ such that $y^{\varepsilon_{i_j}}$ converges to $y^* \in C\big([0, T], \mathbb{R}^K\big)$ uniformly on $[0, T]$. Hence,

$$y^*(t) = z + \int_0^t f\big(\tau, y^*(\tau)\big) + g\big(\varepsilon, \tau, y^*(\tau)\big) d\tau$$

for all $t \in [0, T]$. So (14) has a solution. ■

The proof in part i) follows the method in [20, sec. 2.4] and [22, pp. 486].

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