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Cooperative Linear-Quadratic Mean Field Control and Its Hamiltonian Matrix Analysis

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***Index Terms*— Hamiltonian matrix, linear quadratic (LQ), mean field control, mean field game, social optimization.**

I. Extended Abstract

Mean field game (MFG) theory studies stochastic decision making problems involving a large number of noncooperative and individually insignificant agents, and provides a powerful methodology to reduce the complexity in designing strategies [13]. For an overview of the theory and applications, the readers are referred to [4], [7], [11], [14], [16], [19] and references therein.

There is a parallel development on mean field social optimization where a large number of agents cooperatively minimize a social cost as the sum of individual costs. Different from mean field games, the individual strategy selection of an agent is not selfish and should take into account both self improvement and the aggregate impact on other agents' costs. Mean field social optimization problems have been studied in multi-agent collective motion [1], [25], social consensus control [23], economic theory [24]. Other related literature includes Markov decision processes using aggregate statistics and their mean field limit [10], LQ mean field teams [2], LQ social optimization with a major plyaer [17], mean field teams with Markov jumps [27], social optimization with nonlinear diffusion dynamics [26], and cooperative stochastic differential games [29].

In this paper, we consider social optimization in an LQ model of uniform agents. The dynamics of agent i are given by the stochastic differential equation (SDE):

$$dx_i = (Ax_i + Bu_i)dt + DdW_i, \quad t \geq 0, \quad 1 \leq i \leq N. \tag{1}$$

We use $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ to denote an underlying filtered probability space. The state x_i and the control u_i are n and n_1 dimensional vectors respectively. The initial states $\{x_i(0),\ 1 \leq i \leq N\}$ are independent. The noise processes $\{W_i,\ 1 \leq i \leq N\}$ are n_2 dimensional independent standard Brownian motions adapted to $\mathscr{F}_t$, which are also independent of $\{x_j(0),\ 1 \leq j \leq N\}$. The constant matrices A, B and D have compatible dimensions. Given a symmetric matrix $M \geq 0$, the quadratic form $z^T Mz$ may be denoted as $|z|_M^2$. Denote $u := (u_1, \cdots, u_N)$.

The individual cost for agent i is given by

$$J_i(u(\cdot)) = E\int_0^\infty e^{-\rho t}[|x_i - \Phi(x^{(N)})|_Q^2 + u_i^T Ru_i]dt, \tag{2}$$

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where $\Phi(x^{(N)}) = \Gamma x^{(N)} + \eta$ and $x^{(N)} := (1/N)\sum x_i$ is the mean field coupling term. The constant matrices or vectors Γ, $Q \geq 0$, $R > 0$ and η have compatible dimensions, and $\rho > 0$ is a discount factor. The social cost is defined as

$$J_{\text{soc}}^{(N)}(u(\cdot)) = \sum_{i=1}^N J_i(u(\cdot)). \tag{3}$$

The minimization of the social cost is a standard optimal control problem. However, the exact solution requires centralized information for each agent. So a solution of practical interest is to find a set of decentralized strategies which has negligible optimality loss in minimizing $J_{\text{soc}}^{(N)}(u(\cdot))$ for large N and the solution method has been developed in [15]. We make the standing assumption for this paper:

(A1) (A,B) is stabilizable and $(A, Q^{\frac{1}{2}})$ is detectable.

Under (A1), there exists a unique solution $\Pi \geq 0$ to the algebraic Riccati equation (ARE):

$$\rho\Pi = \Pi A + A^T\Pi - \Pi BR^{-1}B^T\Pi + Q. \tag{4}$$

Definition 1 *For integer $k \geq 1$ and real number $r > 0$, $C_r([0,\infty), \mathbb{R}^k)$ consists of all functions $f \in C([0,\infty), \mathbb{R}^k)$ such that $\sup_{t\geq 0}|f(t)|e^{-r't} < \infty$, for some $0 < r' < r$. Here r' may depend on f.*

Denote $Q_\Gamma = \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma$ and $\eta_\Gamma = (I - \Gamma^T)Q\eta$. We introduce the Social Certainty Equivalence (SCE) equation system:

$$\frac{d\bar{x}}{dt} = (A - BR^{-1}B^T\Pi)\bar{x} - BR^{-1}B^T s, \tag{5}$$

$$\frac{ds}{dt} = Q_\Gamma\bar{x} + (\rho I - A^T + \Pi BR^{-1}B^T)s + \eta_\Gamma, \tag{6}$$

where $\bar{x}(0) = x_0$, and we look for $(\bar{x}, s) \in C_{\rho/2}([0,\infty), \mathbb{R}^{2n})$. If a finite time horizon $[0,T]$ is considered for (2), s will have a terminal condition $s(T)$ and Π will depend on time. This results in a standard two point boundary value (TPBV) problem for linear ordinary differential equations (ODEs). Given the infinite horizon, the system (5)-(6) may be viewed as a partial boundary value (PBV) problem where s satisfies a growth condition instead of a terminal condition. Since the initial condition s_0 is unspecified, a key part of finding a solution is to determine s_0.

The key result in [15] is that if (5)-(6) has a unique solution, the set of decentralized strategies

$$\hat{u}_i = -R^{-1}B^T(\Pi x_i + s), \quad 1 \leq i \leq N, \tag{7}$$

has asymptotic social optimality. In other words, centralized strategies can further reduce the cost $J_{\text{soc}}^{(N)}(u(\cdot))$ by at most $O(1)$.

To give the reader insights, a briefly review of the procedure of constructing (5)-(6) via person-by-person optimality [12] and consistent mean field approximations is provided in Appendix A of [9]. In fact, [15] constructed a more general version of (5)-(6) where the parameter A is randomized over the population and accordingly $\bar{x}$ in the equation of s is replaced by a mean field averaging over the nonuniform population.

A. Our approach and contributions

After some transformation, the coefficient matrix

$$\begin{bmatrix} A-BR^{-1}B^T\Pi & -BR^{-1}B^T \\ Q_\Gamma & \rho I - A^T + \Pi BR^{-1}B^T \end{bmatrix}$$

of (5)-(6) reduces to a Hamiltonian matrix which can be associated with an LQ optimal control problem with state weight matrix $-Q_\Gamma$. The connection to such an LQ control problem is remarkable since its state weight may not be positive semi-definite [28], [30] although $Q \geq 0$ in $J_{\rm soc}^{(N)}$. When $Q_\Gamma \leq 0$, existence and uniqueness of the solution has been proved [15, Theorem 4.3] by a standard Riccati equation approach. On the other hand, due to the intrinsic optimal control nature of the social optimization problem, one expects to obtain solvability of the SCE equation system under much more general conditions, which is the focus of this work.

We develop a new approach to prove existence and uniqueness of the solution of (5)-(6) for a general Q_Γ by exploiting a Hamiltonian matrix structure and the well known invariant subspace method [6]. Specifically, aided by the solution of a continuous-time algebraic Riccati equation (CARE) with possibly indefinite state weight, we decompose the Hamiltonian matrix into a block-wise triangular form where the stable eigenvalues are separated from the unstable ones. To numerically solve the Riccati equation, we extend the classic Schur method [20] to the present case with possibly indefinite state weight. The approach of decomposing the stable invariant subspace is further extended to solve LQ mean field games; see [3], [5], [21], [22] for related literature on LQ mean field games. The main results of this paper have been reported in [8], [9].

B. The SCE equation system

Definition 2 *[6] A matrix $K \in \mathbb{R}^{2n\times 2n}$ is called a Hamiltonian matrix if JK is symmetric, where*

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

We can transform the coefficient matrix in (5)-(6) to a Hamiltonian matrix by defining

$$\tilde{x} = e^{-\rho t/2}\bar{x}, \quad \tilde{s} = e^{-\rho t/2}s,$$

and we obtain

$$\begin{bmatrix} \frac{d\tilde{x}}{dt} \\ \frac{d\tilde{s}}{dt} \end{bmatrix} = H \begin{bmatrix} \tilde{x} \\ \tilde{s} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\eta}_\Gamma \end{bmatrix}, \tag{8}$$

where $\tilde{x}(0) = x_0$, $\tilde{\eta}_\Gamma = e^{-\frac{\rho}{2}t}\eta_\Gamma$, and

$$H = \begin{bmatrix} \mathscr{A} & -BR^{-1}B^T \\ Q_\Gamma & -\mathscr{A}^T \end{bmatrix}, \quad \mathscr{A} = A - BR^{-1}B^T\Pi - \frac{\rho}{2}I. \tag{9}$$

Note that $\tilde{\eta}_\Gamma$ in (8) is a function of t. Since Q_Γ is symmetric, H is a Hamiltonian matrix. By exploiting the properties of H, the ODE (8) can be analyzed.

Theorem 3 *Assume that the pair (A,B) is stabilizable and the Hamiltonian matrix H in (9) has no eigenvalues on the imaginary axis. Then there exists a unique initial condition s_0 such that (5)-(6) has a solution $(\bar{x}, s) \in C_{\rho/2}([0,\infty), \mathbb{R}^{2n})$.*

For the special case $Q_\Gamma \leq 0$, since $\mathscr{A}$ is stable, by [18, Theorem 9.3.3.], we can show that H necessarily has no eigenvalues with zero real parts.

C. Extension to mean field games

We consider the Nash game of N players with dynamics and costs given by (1)-(2). According to mean field game theory [13], [14], [15], the decentralized strategies for the above N player game can be designed by using the following ODE system:

$$\begin{cases} \dfrac{d\bar{x}}{dt} = (A - BR^{-1}B^T\Pi)\bar{x} - BR^{-1}B^T s, & (10) \\ \dfrac{ds}{dt} = Q\Gamma\bar{x} + (\rho I - A^T + \Pi BR^{-1}B^T)s + Q\eta, & (11) \end{cases}$$

where $\bar{x}(0) = x_0$ is given. Letting $\tilde{x} = e^{-\rho t/2}\bar{x}$ and $\tilde{s} = e^{-\rho t/2}s$, we obtain

$$\begin{cases} \dfrac{d\tilde{x}}{dt} = \mathscr{A}\tilde{x} - BR^{-1}B^T\tilde{s}, \\ \dfrac{d\tilde{s}}{dt} = Q\Gamma\tilde{x} - \mathscr{A}^T s + \tilde{\eta}_\Gamma, \end{cases}$$

where $\tilde{x}(0) = x_0$, $\mathscr{A} = A - BR^{-1}B^T\Pi - \frac{\rho}{2}I$ and $\tilde{\eta}_\Gamma = e^{-\rho t/2}Q\eta$.

Notice that $Q\Gamma$ is generally asymmetric and the coefficient matrix (see (12)) does not have a Hamiltonian structure. However, we can adapt the invariant subspace method to find a solution $(\bar{x}, s) \in C_{\rho/2}([0,\infty), \mathbb{R}^{2n})$. Let

$$M = \begin{bmatrix} \mathscr{A} & -BR^{-1}B^T \\ Q\Gamma & -\mathscr{A}^T \end{bmatrix}. \tag{12}$$

Theorem 4 *Suppose M in (12) has n stable eigenvalues and n eigenvalues with strictly positive real parts. Assume there exists an invertible matrix $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, where U_{11} is invertible, such that*

$$U^{-1}MU = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

where M_{11} has n stable eigenvalues and M_{22} has n eigenvalues with strictly positive real parts. Then for any fixed initial condition x_0 of (10)-(11), there exists a unique s_0 given by

$$\begin{aligned} s_0 =& U_{21}U_{11}^{-1}\bar{x}_0 \\ &+ (U_{21}U_{11}^{-1}U_{12} - U_{22})\int_0^{+\infty} e^{-M_{22}\tau}V_{22}Q\eta e^{-\rho\tau/2}d\tau, \end{aligned}$$

where $V = U^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$ *such that (10)-(11) has a solution* $(\bar{x}, s) \in C_{\rho/2}([0,\infty), \mathbb{R}^{2n})$.

The existence of such a matrix U is related to the existence of so-called graph invariant subspace for M; see e.g. [6].

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